Local limit theorems for complex valued sequences: Old & New

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Abstract

In probability theory, local limit theorems provide an asymptotic expansion of the convolution powers of a probability distribution supported on $\mathbb{Z}$ with uniform bounds on the remainders. In this review, we present some recent results for the iterated convolution of complex valued integrable sequences in one space dimension. In the so-called parabolic case, we give a complete expansion, at any accuracy order, for these convolution powers and we provide sharp, pointwise, generalized Gaussian bounds for the remainders. We also present an extension of our main result to the semi-discrete setting (time-continuous convolution problems), and discuss several natural perspectives.

1 Introduction

For a given integrable complex valued sequence $\boldsymbol{a} \in \ell^1(\mathbb{Z};\mathbb{C})$, we define iteratively:

$$\forall\, n \in \mathbb{N}^*\,, \quad \boldsymbol{a}^{(n+1)} \,:=\, \boldsymbol{a}^{(n)} \star \boldsymbol{a}\,,$$

with the initialization $\boldsymbol{a}^{(1)} := \boldsymbol{a}$. Here, the notation $\star$ stands for the convolution between sequences. More precisely, for any $\boldsymbol{a} = (a_\ell)_{\ell\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{C})$ and $\boldsymbol{b} = (b_\ell)_{\ell\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{C})$, the convolution $\boldsymbol{a} \star \boldsymbol{b}$ is given by

$$\forall\, \ell \in \mathbb{Z}\,, \quad (\boldsymbol{a} \star \boldsymbol{b})_\ell := \sum_{\ell' \in \mathbb{Z}} a_{\ell-\ell'}\, b_{\ell'}\,.$$

The celebrated Young's inequality ensures that the above convolution $\boldsymbol{a} \star \boldsymbol{b}$ is well defined for any $\boldsymbol{a} \in \ell^1(\mathbb{Z};\mathbb{C})$ and $\boldsymbol{b} \in \ell^1(\mathbb{Z};\mathbb{C})$, and also belongs to $\ell^1(\mathbb{Z};\mathbb{C})$, which endows this space with a Banach algebra structure. Our aim is to present a brief survey on recent results on the study of some classes of geometric sequences in this algebra and, more specifically, develop on those results which provide sharp pointwise estimates on the sequence $\boldsymbol{a}^{(n)}$ for all $n \in \mathbb{N}^*$. As we shall see later on, this problem is connected with the large time behavior of finite difference approximations for evolutionary partial differential equations, which is our main motivation for studying this problem.

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When the sequence $\boldsymbol{a}$ is a finitely supported probability distribution, that is, when it is real non negative, satisfies the normalization condition

$$\sum_{\ell\in\mathbb{Z}} a_\ell \,=\, 1,$$

and only finitely many a_ℓ's are nonzero, the asymptotic behavior of $\boldsymbol{a}^{(n)}$ for large values of n is well-known, and is described by the so-called local limit theorem in probability theory [25, Chapter VII]. Indeed, in that setting, for each $\ell \in \mathbb{Z}$, the coefficient a_ℓ corresponds to the probability $\mathbb{P}(X = \ell)$ where X is a given random variable with values in $\mathbb{Z}$. If $X_1, ..., X_n, ...$ are independent, identically distributed, random variables following the same law as X with distribution given by $\boldsymbol{a}$, then the probability that the random walk $X_1 + \cdots + X_n$ is at lattice site ℓ corresponds to the value at the index ℓ of the iterated convolution $(n-1)$-times of the sequence $\boldsymbol{a}$. That is we have the relation

$$\boldsymbol{a}_\ell^{(n)} = \mathbb{P}(X_1 + \cdots + X_n = \ell),$$

for all $n \in \mathbb{N}^*$ and $\ell \in \mathbb{Z}$. Assuming that the sequence $\boldsymbol{a}$ possesses at least two nonzero elements and is aperiodic[1], the local limit theorem provides the existence of a sequence of real valued functions $Q_m : \mathbb{R} \to \mathbb{R}$ indexed by $m \in \mathbb{N}^*$ such that for all $M \in \mathbb{N}^*$:

$$\boldsymbol{a}_\ell^{(n)} - \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{x_{n,\ell}^2}{2}\right) - \sum_{m=1}^{M} \frac{Q_m(x_{n,\ell})}{n^{(m+1)/2}} \underset{n\to+\infty}{=} o\left(\frac{1}{n^{(M+1)/2}}\right), \quad x_{n,\ell} := \frac{\ell - \alpha n}{\sigma\sqrt{n}}, \tag{1}$$

where $\alpha := \sum_{\ell\in\mathbb{Z}} \ell a_\ell$ and $\sigma^2 := \sum_{\ell\in\mathbb{Z}} \ell^2 a_\ell - \alpha^2 > 0$ are respectively the mean and the variance of the random variable X_1 with probability distribution $\boldsymbol{a}$. In the above asymptotic expansion, the error term is understood to be uniform with respect to $\ell \in \mathbb{Z}$. In fact, each term Q_m appearing in the expansion can be explicitly computed as a linear combination of derivatives of the Gaussian function $x \mapsto \exp\left(-\frac{x^2}{2}\right)$ and can thus be expressed by using Hermite polynomials. We refer to [25, Chapter VII] for more details.

Over the past decades, there has been a long series of work that have studied the generalization of the local limit theorem to the case where the sequence $\boldsymbol{a}$ is complex valued [8, 11–14, 17, 19, 26–28, 31, 32] and thus dropping the positivity assumption of the probabilistic framework. Beyond its own analytical interest, this problem is particularly relevant for instance when one studies the large time behavior of finite difference approximation of evolution equations [13, 19, 32] or data smoothing problems [17, 31]. We also point to the recent article [14] which gives a large overview of examples where this issue is meaningful. In numerical analysis, when one discretizes by means of a finite difference scheme an evolutionary linear partial differential equation[2] set on the real line $\mathbb{R}$, one is let to study problems of the form

$$\forall n \in \mathbb{N}, \quad \boldsymbol{u}^{n+1} = \boldsymbol{a} \star \boldsymbol{u}^n,$$

for a given initial sequence $\boldsymbol{u}^0$, and $\boldsymbol{a}$ that now encodes the properties of the finite difference scheme. Assuming that $\boldsymbol{a} \in \ell^1(\mathbb{Z};\mathbb{C})$, which is typically satisfied when the stencil of the scheme is finite, one can define the linear convolution operator $\mathscr{L}_{\boldsymbol{a}} : \boldsymbol{u} \mapsto \boldsymbol{a} \star \boldsymbol{u}$ which acts boundedly on $\ell^q(\mathbb{Z};\mathbb{C})$ for any $q \in [1, +\infty]$. Using the morphism property $\mathscr{L}_{\boldsymbol{a}} \circ \mathscr{L}_{\boldsymbol{b}} = \mathscr{L}_{\boldsymbol{a}\star\boldsymbol{b}}$, one has $(\mathscr{L}_{\boldsymbol{a}})^n = \mathscr{L}_{\boldsymbol{a}^{(n)}}$ for all $n \in \mathbb{N}^*$, such that Young's inequality gives

$$\|\boldsymbol{u}^n\|_{\ell^\infty} = \left\|\mathscr{L}_{\boldsymbol{a}^{(n)}} \boldsymbol{u}^0\right\|_{\ell^\infty} = \left\|\boldsymbol{a}^{(n)} \star \boldsymbol{u}^0\right\|_{\ell^\infty} \leq \left\|\boldsymbol{a}^{(n)}\right\|_{\ell^1} \left\|\boldsymbol{u}^0\right\|_{\ell^\infty}.$$

[1] We refer to [21] for the definition of aperiodicity.

[2] The transport equation or the heat equation are typical examples.

One then deduces that a sufficient condition[3] for the stability of the numerical scheme in the maximum norm, which reads as

$$\sup_{n\in\mathbb{N}} |||\mathscr{L}_{\boldsymbol{a}^{(n)}}|||_{\ell^\infty\to\ell^\infty} < +\infty,$$

is given by the boundedness of the sequence $\left(\left\|\boldsymbol{a}^{(n)}\right\|_{\ell^1}\right)_{n\in\mathbb{N}}$. Proving such boundedness is typically achieved by obtaining pointwise bounds on the iterates $\boldsymbol{a}^{(n)}$, see [11, 32].

As expected (see [8, 11, 12, 27] for many illustrations), a much larger variety of possible behaviors is obtained by dropping the positivity assumption which correspond, in the language of partial differential equations, either to *parabolic* or *dispersive* behaviors. In this review, we shall only focus on the parabolic case, which is referred to as the *stable* case in [32]. This is precisely the situation where the iterated convolutions will be bounded in the ℓ^1 norm. A fundamental result obtained by Randles and Saloff-Coste [27] is a generalization of the local limit theorem for a large class of complex valued sequences with finite support and the identification of the leading order term of an asymptotic expansion similar to (1), which is referred to as the *attractor* in [27]. Very recently, for the same class of finitely supported complex valued sequences, Coeuret [8] has proved a generalization of the asymptotic expansion (1) together with the derivation of a sharp rate of convergence in the form a generalized Gaussian bound for the remainder of this new-found asymptotic expansion. In the probability framework discussed above, the main result of [8] ensures that for any sequence $\boldsymbol{a}$ which is a finitely supported probability distribution with at least two nonzero elements and aperiodic, for any $M \in \mathbb{N}^*$ there exist two positive real constants $C_M > 0$ and $c_M > 0$ such that

$$\forall (n,\ell) \in \mathbb{N}^* \times \mathbb{Z}\,, \quad \left| \boldsymbol{a}_\ell^{(n)} - \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{x_{n,\ell}^2}{2}\right) - \sum_{m=1}^{M} \frac{Q_m(x_{n,\ell})}{n^{(m+1)/2}} \right| \leq \frac{C_M}{n^{(M+2)/2}} \exp\left(-c_M \frac{x_{n,\ell}^2}{2}\right).$$

Such a quantified asymptotic expansion already proves very useful in probability theory since it allows to very easily retrieve (a non-optimal version of) the well-known Berry-Esseen inequality [3, 16].

All, the results in the above mentioned references [8, 14, 27] contain either technical restrictions on the class of complex sequences $\boldsymbol{a}$ considered, typically on the Fourier transform[4] of $\boldsymbol{a}$, and/or did not provide sharp enough estimates for the remainders. In our recent contribution [12], we have managed to drop all previous technical restrictions and derive an asymptotic expansion up to any order with a sharp, generalized Gaussian estimate for the remainders. More precisely, we shall consider, from now on, complex valued sequences which satisfy the following assumption.

Assumption 1 (Holomorphy). *The sequence* $\boldsymbol{a} = (a_\ell)_{\ell\in\mathbb{Z}}$ *belongs to* $\ell^1(\mathbb{Z};\mathbb{C})$ *and its associated Fourier series:*

$$F_{\boldsymbol{a}} \quad : \quad \zeta \in \mathbb{C} \longmapsto \sum_{\ell\in\mathbb{Z}} a_\ell\, \zeta^\ell\,,$$

defines a holomorphic function on an annulus $\{\zeta \in \mathbb{C} \,|\, 1-\varepsilon < |\zeta| < 1+\varepsilon\}$ *for some* $\varepsilon > 0$. *Furthermore, there holds:*

$$\sup_{\kappa\in\mathbb{S}^1} |F_{\boldsymbol{a}}(\kappa)| \,=\, 1\,.$$

In numerical analysis, the latter normalization for the maximum of $|F_{\boldsymbol{a}}|$ on the unit circle corresponds to the so-called *von Neumann stability condition* [23], and it is made in order to avoid introducing

[3] It is actually a necessary and sufficient condition, see [32].

[4] This function is referred to as the *characteristic function* in probability theory, see [25].

additional terms in the main result below. Up to multiplying the sequence $\boldsymbol{a}$ by some positive number, one can always fix the maximum to 1. Lets us also note that thanks to Cauchy's formula [30], the holomorphy of $F_{\boldsymbol{a}}$ on an annulus that contains the unit circle $\mathbb{S}^1$ is equivalent to the exponential localization of the sequence $\boldsymbol{a}$, that is the existence of a positive constant c such that:

$$\sup_{\ell \in \mathbb{Z}} \mathrm{e}^{c\,|\ell|}\,|a_\ell| < +\infty\,.$$

Given a sequence $\boldsymbol{a}$ that satisfies Assumption 1, it is well-known [6, 11] that one of the following two alternatives is satisfied. Either $F_{\boldsymbol{a}}(\kappa)$ has modulus 1 for any $\kappa \in \mathbb{S}^1$ (e.g. $F_{\boldsymbol{a}}$ is a Blaschke product [30]). Or, there exists a finite set of pairwise distinct points on the unit circle $\mathbb{S}^1$ such that $F_{\boldsymbol{a}}$ has modulus 1 precisely at these points. The reader interested in the first alternative may consult [19] and [6, Theorem 3.1]. From now on, we shall place ourselves in the second case and make the following assumption.

Assumption 2 (Tangency)**.** *Let the sequence* $\boldsymbol{a}$ *satisfy Assumption 1. We assume that there exists a finite set of pairwise distinct points* $\{\underline{\kappa}_1,\dots,\underline{\kappa}_K\}$*,* $K \ge 1$*, in* $\mathbb{S}^1$ *such that* $F_{\boldsymbol{a}}(\underline{\kappa}_k)$ *has modulus* 1 *for any* $k \in \{1,\dots,K\}$ *and:*

$$\forall\,\kappa \in \mathbb{S}^1 \setminus \big\{\underline{\kappa}_1,\dots,\underline{\kappa}_K\big\}\,, \quad \big|F_{\boldsymbol{a}}(\kappa)\big| \,<\, 1\,.$$

The points $F_{\boldsymbol{a}}(\underline{\kappa}_k)$ will be referred to the tangency points since these are the points where the curve[5] $\{F_{\boldsymbol{a}}(\kappa)\,|\,\kappa \in \mathbb{S}^1\}$ meets the unit circle $\mathbb{S}^1$. Our third and last assumption describes the behavior of the asymptotic expansion of $F_{\boldsymbol{a}}$ near the tangency points $\underline{\kappa}_k$, and it will be in this assumption that will be encoded the *parabolic* nature of the sequence $\boldsymbol{a}$ already mentioned above.

Assumption 3 (Parabolicity)**.** *Let the sequence* $\boldsymbol{a}$ *satisfy Assumption 1 and Assumption 2. Moreover, at any point* $\underline{\kappa}_k \in \mathbb{S}^1$*,* $k \in \{1,\dots,K\}$*, where the modulus of* $F_{\boldsymbol{a}}$ *attains the value* 1*, there exists a real number* α_k*, a complex number* β_k *with positive real part and a nonzero integer* $\mu_k \in \mathbb{N}^*$ *such that, as the complex number* ξ *tends to zero, there holds:*

$$F_{\boldsymbol{a}}\Big(\underline{\kappa}_k\,\mathrm{e}^{\mathbf{i}\,\xi}\Big) \;=\; F_{\boldsymbol{a}}(\underline{\kappa}_k)\,\exp\Big(\mathbf{i}\,\alpha_k\,\xi - \beta_k\,\xi^{2\,\mu_k} + O(\xi^{2\,\mu_k+1})\Big)\,. \tag{2}$$

Using Assumptions 1, 2 and 3, we consider a point $\underline{\kappa}_k \in \mathbb{S}^1$ where $|F_{\boldsymbol{a}}(\underline{\kappa}_k)| = 1$. For any sufficiently small $\xi \in \mathbb{C}$, we can write $F_{\boldsymbol{a}}\left(\underline{\kappa}_k\,\mathrm{e}^{\mathbf{i}\,\xi}\right)$ as the following convergent power series:

$$F_{\boldsymbol{a}}\Big(\underline{\kappa}_k\,\mathrm{e}^{\mathbf{i}\,\xi}\Big) \;=\; F_{\boldsymbol{a}}(\underline{\kappa}_k)\,\exp\left(\mathbf{i}\,\alpha_k\,\xi - \beta_k\,\xi^{2\,\mu_k} + \sum_{\nu \ge 2\,\mu_k+1} \frac{\gamma_{k,\nu}}{\nu\,!}\,(\mathbf{i}\,\xi)^\nu\right)\,, \tag{3}$$

where the coefficients $\gamma_{k,\nu}$ play the role of *cumulants* in probability theory. Following Petrov [25], we expand a power series in two variables (Y,Z) as follows:

$$\exp\left(\sum_{\nu \ge 1} \frac{\gamma_{k,2\,\mu_k+\nu}}{(2\,\mu_k+\nu)\,!}\,Y^{2\,\mu_k+\nu}\,Z^\nu\right) \;=\; 1 + \sum_{m \ge 1} P_{k,m}(Y)\,Z^m\,, \tag{4}$$

where the $P_{k,m}$'s are polynomials with complex coefficients that depend on the cumulants $\gamma_{k,\nu}$ (see several formulas below based on the Faà di Bruno formula [9]).

[5]Because of Assumption 1, this curve is located inside the closed unit disk.

In order to state the main result of [12], we also need to define the aforementioned *attractors* of [27]. For any nonzero integer $\mu \in \mathbb{N}^*$ and for any complex number β with positive real part, we introduce the function:

$$H_{2\mu}^{\beta} \, : \, x \in \mathbb{R} \longmapsto \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathbf{i}\, x\, \theta} \, \mathrm{e}^{-\beta\, \theta^{2\mu}} \, \mathrm{d}\theta \, . \tag{5}$$

The only properties that we shall need on these attractors is the fact for any β with positive real part and $\mu \in \mathbb{N}^*$, the function $H_{2\mu}^{\beta}$ has super-exponential decay at infinity as well as its derivatives:

$$\forall N \in \mathbb{N}, \quad \exists C > 0, \quad \forall x \in \mathbb{R}, \quad \left| H_{2\mu}^{\beta}(x) \right| + \cdots + \left| (H_{2\mu}^{\beta})^{(N)}(x) \right| \le C \, \exp\left(-\frac{1}{C} \, |x|^{\frac{2\mu}{2\mu-1}} \right) .$$

With the above notations, our main result reads as follows.

Theorem 1 (Local limit theorem from [12])**.** *Let the sequence $\boldsymbol{a}$ satisfy Assumptions 1, 2 and 3. Then there exist an integer $L \in \mathbb{N}^*$ and some positive constant $c_0 > 0$ such that for any $n \in \mathbb{N}^*$ and $\ell \in \mathbb{Z}$ with $|\ell| > L\, n$, there holds:*

$$\left| \boldsymbol{a}_{\ell}^{(n)} \right| \le \exp(-c_0\, n - c_0\, |\ell|) \, . \tag{6}$$

Moreover, for any integer $M \in \mathbb{N}$, there exist some positive constants C_M and c_M (that depend on M and $\boldsymbol{a}$) such that the following holds: for any $n \in \mathbb{N}^*$ and $\ell \in \mathbb{Z}$ with $|\ell| \le L\, n$, there holds:

$$\left| \boldsymbol{a}_{\ell}^{(n)} - \sum_{k=1}^{K} \frac{\underline{\kappa_k}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa_k})^n}{n^{1/(2\mu_k)}} \, H_{2\mu_k}^{\beta_k} \left(\frac{\ell - \alpha_k n}{n^{1/(2\mu_k)}} \right) - \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{\underline{\kappa_k}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa_k})^n}{n^{(m+1)/(2\mu_k)}} \left(P_{k,m}(-\mathrm{d}/\mathrm{d}x) H_{2\mu_k}^{\beta_k} \right) \left(\frac{\ell - \alpha_k n}{n^{1/(2\mu_k)}} \right) \right|$$
$$\le C_M \sum_{k=1}^{K} \frac{1}{n^{(M+2)/(2\mu_k)}} \, \exp\left(-c_M \, \left(\frac{|\ell - \alpha_k\, n|}{n^{1/(2\mu_k)}} \right)^{\frac{2\mu_k}{2\mu_k-1}} \right) , \tag{7}$$

where the polynomials $P_{k,m}$ are defined in (4).

We already make a first comment regarding the statement of Theorem 1. Since by Assumption 1 the Fourier series $F_{\boldsymbol{a}}$ is only assumed to be holomorphic on a given annulus which contains the unit circle $\mathbb{S}^1$, which we recall equivalently means that the sequence $\boldsymbol{a}$ is exponentially localized, then, for a fixed n, the decay at infinity of $\boldsymbol{a}_{\ell}^{(n)}$ is at best exponential. This is exactly the result of our first estimate (6). Nevertheless, if one further assumes that $\boldsymbol{a}$ is finitely supported, then $F_{\boldsymbol{a}}$ is a trigonometric polynomial on $\mathbb{S}^1$, and in that case, one can prove (see [8]), that our generalized Gaussian bound (7) holds not only in the large sector $\{\ell \in \mathbb{Z} \mid |\ell| \le Ln\}$ but for all $\ell \in \mathbb{Z}$.

Let us come back to our initial example where the sequence $\boldsymbol{a}$ is a finitely supported probability distribution with a least two nonzero elements and aperiodic. Since the sequence is finitely supported, has only positive elements which sum to 1, Assumption 1 is therefore satisfied. Furthermore, since $\boldsymbol{a}$ has at least two nonzero elements, $F_{\boldsymbol{a}}(\kappa)$ cannot have modulus 1 for any $\kappa \in \mathbb{S}^1$ for otherwise the sequence $\boldsymbol{a}$ would have a single nonzero element equal to 1. As a consequence, Assumption 2 is also satisfied. The aperiodicity of $\boldsymbol{a}$ implies that $F_{\boldsymbol{a}}$ has modulus 1 only at $\kappa = 1$ for $\kappa \in \mathbb{S}^1$, that is $K = 1$ in the notation of Assumption 2. The asymptotic expansion (2) is verified with $\alpha_1 = \alpha$ the mean of the random variable X, $\mu_1 = 1$ and $2\beta_1 = \sigma^2$ with $\sigma^2 > 0$ the variance of X, namely one has

$$F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\xi}\right) \underset{\xi \to 0}{=} \exp\left(\mathbf{i}\, \alpha\, \xi - \frac{\sigma^2}{2}\, \xi^2 + O(\xi^3) \right) .$$

From the definition (5), we compute:

$$\forall\, x \in \mathbb{R}\,, \quad H_2^{\frac{\sigma^2}{2}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\left(-\frac{x^2}{2\,\sigma^2}\right)\,.$$

As a consequence, the asymptotic expansion provided by Theorem 1 reads:

$$\boldsymbol{a}_\ell^{(n)} \sim \frac{1}{\sqrt{2\pi\sigma^2 n}}\,\exp\left(-\frac{(\ell-\alpha n)^2}{2\sigma^2 n}\right) + \sum_{m\geq 1} \frac{1}{n^{(m+1)/2}}\,\left(P_{1,m}(-\mathrm{d}/\mathrm{d}x) H_2^{\frac{\sigma^2}{2}}\right)\left(\frac{\ell-\alpha n}{\sqrt{n}}\right)\,,$$

which precisely coincides with (1), since $\left(P_{1,m}(-\mathrm{d}/\mathrm{d}x) H_2^{\frac{\sigma^2}{2}}\right)\left(\frac{\ell-\alpha n}{\sqrt{n}}\right)$ can be shown to be equal to $Q_m(x_{n,\ell})$ given in [25]. For instance, we shall see below that the polynomial $P_{1,1}$ is given by:

$$P_{1,1}(X) \;=\; \frac{\gamma_{1,3}}{3\,!}\,X^3\,,$$

where $\gamma_{1,3}$ is the cumulant of order 3 at the zero frequency, see (3). This means that the two first terms ($M = 1$) in the expansion (7) are:

$$\frac{1}{\sqrt{2\pi\sigma^2 n}}\,\exp\left(-\frac{(\ell-\alpha n)^2}{2\sigma^2 n}\right) - \frac{\gamma_{1,3}}{3\,!\,n}\,\left(H_2^{\frac{\sigma^2}{2}}\right)'''\left(\frac{\ell-\alpha n}{\sqrt{n}}\right)\,,$$

and these two first terms can be rewritten as:

$$\frac{1}{\sqrt{2\,\pi\sigma^2 n}}\,\exp\left(-\frac{x_{n,\ell}^2}{2}\right) - \frac{\gamma_{1,3}}{3\,!\,\sigma^3\,n}\,(x_{n,\ell}^3 - 3x_{n,\ell})\,\frac{1}{\sqrt{2\,\pi\sigma^2}}\,\exp\left(-\frac{x_{n,\ell}^2}{2}\right)\,, \quad \text{with} \quad x_{n,\ell} = \frac{\ell-\alpha\,n}{\sigma\,\sqrt{n}}\,,$$

which is with the expression in [25, Theorem 13, page 205].

Next, coming back to the numerical analysis framework, for a given sequence $\boldsymbol{a}$ satisfying Assumptions 1, 2 and 3, an immediate consequence of Theorem 1 is the following explicit expression for the large time asymptotic of the iterates $\mathscr{L}_{\boldsymbol{a}^{(n)}}\boldsymbol{u}^0 = \boldsymbol{a}^{(n)} \star \boldsymbol{u}^0$ of the numerical scheme for any initial condition $\boldsymbol{u}^0 \in \ell^q(\mathbb{Z};\mathbb{C})$ with $q \in [1,+\infty]$. More specifically, for any integer $M \in \mathbb{N}$, there exists some positive constant C_M such that for any $\boldsymbol{u}^0 \in \ell^q(\mathbb{Z};\mathbb{C})$ there holds:

$$\begin{aligned}\Bigg\|\boldsymbol{a}^{(n)} \star \boldsymbol{u}^0 &- \sum_{k=1}^{K} \frac{\underline{\kappa}_k^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa}_k)^n}{n^{1/(2\,\mu_k)}}\,H_{2\mu_k}^{\beta_k}\left(\frac{\cdot-\alpha_k n}{n^{1/(2\,\mu_k)}}\right)\star \boldsymbol{u}^0 \\ &- \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{\underline{\kappa}_k^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa}_k)^n}{n^{(m+1)/(2\,\mu_k)}}\left(P_{k,m}(-\mathrm{d}/\mathrm{d}x) H_{2\mu_k}^{\beta_k}\right)\left(\frac{\cdot-\alpha_k n}{n^{1/(2\,\mu_k)}}\right)\star \boldsymbol{u}^0\Bigg\|_{\ell^q} \leq \frac{C_M\,\|\boldsymbol{u}^0\|_{\ell^q}}{n^{(M+1)/(2\,\mu)}}\,,\end{aligned}$$

with $\mu := \max_k \mu_k$. The generalized Gaussian bound on the remainder terms obtained in Theorem 1 is thus crucial in the derivation of the above estimate. Indeed, uniform bounds such as the one previously derived in the literature [25, 27] (see (1) in the probability case) would not have allowed to obtain such an estimate.

As already emphasized, some analogues of Theorem 1 have been proved in [8, 11, 27], but with either some restrictions on the number of tangency points and/or the drifts, and/or the number of terms in the expansion (7). To our best knowledge, it seems that our framework is the most general so far when it comes to consider sequences $\boldsymbol{a}$ whose Fourier series $F_{\boldsymbol{a}}$ enjoys an asymptotic expansion of the form (3) at each tangency point. Maybe, the main restriction that we manage to lift is the fact that we do not necessarily consider sequences with finite support. Let us remark that sequences satisfying Assumptions 1, 2 and 3 of Theorem 1 with infinite support naturally arise in the study of *implicit schemes* in numerical analysis. An implicit discretization of an evolutionary linear partial differential equation by means of finite differences in space yields recurrence relations of the form

$$\forall n \in \mathbb{N}\,, \quad \boldsymbol{b} \star \boldsymbol{u}^{n+1} = \boldsymbol{c} \star \boldsymbol{u}^{n},$$

where $\boldsymbol{b}$ and $\boldsymbol{c}$ are two finitely supported sequences, and we assume that the sequence $\boldsymbol{b}$ contains at least two nonzero elements. Upon assuming that $\boldsymbol{b}$ is invertible on $\ell^1(\mathbb{Z};\mathbb{C})$ for the convolution, which thanks to the Wiener-Levy theorem [24] is equivalent to the fact that

$$\forall \kappa \in \mathbb{S}^1\,, \quad F_{\boldsymbol{b}}(\kappa) = \sum_{\ell\in\mathbb{Z}} b_\ell \kappa^\ell \neq 0,$$

then one can rewrite the numerical scheme in the form $\boldsymbol{u}^{n+1} = \boldsymbol{a} \star \boldsymbol{u}^n$ with $\boldsymbol{a} := \boldsymbol{b}^{-1} \star \boldsymbol{c} \in \ell^1(\mathbb{Z};\mathbb{C})$ which has an infinite support. We refer the interested reader to [11, Section 4.2] for an example in such a setting.

In the following Section 2, we present the main ideas towards the proof of Theorem 1. Finally, we conclude in Section 3 by presenting some natural extensions and perspectives related to this work.

2 Sketch of the proof of Theorem 1

The proof of Theorem 1 naturally splits into two parts corresponding to the two regimes leading to estimates (6) and (7). Nevertheless, the starting point of the analysis in both cases is the following key expression for the coefficient $\boldsymbol{a}_\ell^{(n)}$ which is obtained by inverse Fourier transform

$$\forall (n,\ell) \in \mathbb{N}^* \times \mathbb{Z}\,, \quad \boldsymbol{a}_\ell^{(n)} = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\, F_{\boldsymbol{a}} \left(\mathrm{e}^{\mathbf{i}\,\theta}\right)^n \,\mathrm{d}\theta\,. \tag{8}$$

In the above expression, we will make use of the crucial holomorphy hypothesis of Assumption 1 to deform appropriately the integral in the complex plane (within the domain of holomorphy of $F_{\boldsymbol{a}}$ which is a given annulus around the unit circle $\mathbb{S}^1$). The type of deformations will typically depend on the regime considered for (n,ℓ), and more importantly, we may allow the contours to depend on (n,ℓ).

In the far field regime, that is when the ratio $|\ell|/n$ is large, the idea is to integrate along a circle with radius either strictly larger or smaller than 1 depending on the sign of ℓ, and that remains in the annulus $\{\zeta \in \mathbb{C} \,|\, 1-\varepsilon < |\zeta| < 1+\varepsilon\}$ for some $\varepsilon > 0$ provided by Assumption 1. Assume for example that $\ell > 0$. We set $\varrho := \ln(1+\varepsilon/2)$, with $\varepsilon > 0$, and Cauchy's formula readily gives

$$\boldsymbol{a}_\ell^{(n)} = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\ell\,\varrho} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\, F_{\boldsymbol{a}} \left(\mathrm{e}^{\mathbf{i}\,(\theta-\mathbf{i}\,\varrho)}\right)^n \,\mathrm{d}\theta\,.$$

If $C_\varrho > 0$ denotes the following constant

$$C_\varrho := \sup_{\theta\in[-\pi,\pi]} \left| F_{\boldsymbol{a}} \left(\mathrm{e}^{\mathbf{i}\,(\theta-\mathbf{i}\,\varrho)}\right) \right|,$$

then one gets the bound

$$\forall (n,\ell) \in \mathbb{N}^* \times \mathbb{Z}\,, \quad \left|\boldsymbol{a}_\ell^{(n)}\right| \leq \exp\left(-\delta\ell + C_\varrho n\right).$$

Upon denoting by $\lfloor\cdot\rfloor$ the greatest integer function, we let $L := \left\lfloor \frac{2C_\varrho}{\varrho} \right\rfloor$, and for all $\ell \geq Ln$, we thus have

$$\left|\boldsymbol{a}_\ell^{(n)}\right| \leq \exp\left(-\frac{\varrho}{2}\ell\right) \leq \exp\left(-\frac{\varrho}{4}\ell - \frac{\varrho L}{4}n\right),$$

which proves (6) for $\ell > 0$ (the case $\ell < 0$ is handled similarly). So from now on we only consider those values of $\ell \in \mathbb{Z}$ for which $|\ell| \leq Ln$ with $n \in \mathbb{N}^*$.

One of the main task in proving estimate (7) of Theorem 1 is to make appear each term of the expansion involving the attractors and the polynomials $P_{k,m}$. In order to lighten the presentation and the notations, we shall only consider the case where there is a single tangency point for $F_{\boldsymbol{a}}$, that is $K = 1$ in Assumption 2, and drop the subscript 1 in our notation. Thus, we have a unique $\underline{\kappa} = \exp(\mathbf{i}\underline{\theta})$ with $\underline{\theta} \in [0, 2\pi)$ such that $|F_{\boldsymbol{a}}(\underline{\kappa})| = 1$, and for all sufficiently small $\xi \in \mathbb{C}$, we can write the convergent power series

$$F_{\boldsymbol{a}}\left(\underline{\kappa}\,\mathrm{e}^{\mathbf{i}\,\xi}\right) = F_{\boldsymbol{a}}(\underline{\kappa})\,\exp\left(\mathbf{i}\,\alpha\,\xi - \beta\,\xi^{2\,\mu} + \sum_{\nu\geq 2\mu+1} \frac{\gamma_\nu}{\nu!}(\mathbf{i}\xi)^\nu\right),$$

with $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$, $\mu \geq 1$ an integer and cumulants $\gamma_\nu \in \mathbb{C}$. From (4), we get the existence of complex polynomials P_m defined via

$$\exp\left(\sum_{\nu\geq 1} \frac{\gamma_{2\,\mu+\nu}}{(2\,\mu+\nu)\,!}\,Y^{2\,\mu+\nu}\,Z^\nu\right) = 1 + \sum_{m\geq 1} P_m(Y)\,Z^m\,.$$

With these notations at hand, we let $\delta > 0$ be a small positive real number that shall be fixed at some point in the proof, and $\theta_0 \in \mathbb{R}$ such that $\exp(\mathbf{i}\theta_0)$ does not belong to the arc $\left\{\underline{\kappa}\mathrm{e}^{\mathbf{i}\theta} \mid \theta \in [-\delta, \delta]\right\}$ of the unit circle and $\underline{\theta} \in [\theta_0, \theta_0 + 2\pi]$. We start once again with (8) and write

$$\boldsymbol{a}_\ell^{(n)} = \frac{1}{2\,\pi}\int_{\theta_0}^{\theta_0+2\pi} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\,F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\theta}\right)^n\,\mathrm{d}\theta\,,$$

and then we further split the integral as

$$\boldsymbol{a}_\ell^{(n)} = \frac{1}{2\,\pi}\int_{\theta_0}^{\underline{\theta}-\delta} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\,F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\theta}\right)^n\,\mathrm{d}\theta + \frac{1}{2\,\pi}\int_{\underline{\theta}-\delta}^{\underline{\theta}+\delta} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\,F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\theta}\right)^n\,\mathrm{d}\theta + \frac{1}{2\,\pi}\int_{\underline{\theta}+\delta}^{\theta_0+2\pi} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\,F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\theta}\right)^n\,\mathrm{d}\theta.$$

From Assumption 2, we note that

$$\forall \theta \in [\theta_0, \underline{\theta}-\delta] \cup [\underline{\theta}+\delta, \theta_0+2\pi]\,, \quad \left|F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\theta}\right)\right| < 1,$$

such that there exists $c > 0$ so that we get

$$\left|\boldsymbol{a}_\ell^{(n)} - \frac{\underline{\kappa}^{-\ell}F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi}\int_{-\delta}^{\delta} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\,\left(F_{\boldsymbol{a}}(\underline{\kappa})^{-1}F_{\boldsymbol{a}}\left(\underline{\kappa}\mathrm{e}^{\mathbf{i}\,\theta}\right)\right)^n\,\mathrm{d}\theta\right| \leq \mathrm{e}^{-cn}\,,$$

where we have performed a change of variables to shift the interval in θ in the remaining integral. We can then substitute the above convergent power series into the expression to obtain

$$\left|\boldsymbol{a}_\ell^{(n)} - \frac{\underline{\kappa}^{-\ell}F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi}\int_{-\delta}^{\delta} \mathrm{e}^{-\mathbf{i}\,(\ell-n\alpha)\,\theta}\,\mathrm{e}^{-n\beta\theta^{2\mu}}\,\exp\left(n\sum_{\nu\geq 2\mu+1}\frac{\gamma_\nu}{\nu!}(\mathbf{i}\theta)^\nu\right)\,\mathrm{d}\theta\right| \leq \mathrm{e}^{-cn}\,.$$

Next, we perform a *parabolic* rescaling in the integral via $\theta \to \theta/n^{\frac{1}{2\mu}}$, so that we get

$$\left| \boldsymbol{a}_\ell^{(n)} - \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi\, n^{\frac{1}{2\mu}}} \int_{-\delta/n^{\frac{1}{2\mu}}}^{\delta/n^{\frac{1}{2\mu}}} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\theta^{2\mu}} \, \exp\left((\mathbf{i}\theta)^{2\mu} \sum_{\nu\geq 1} \frac{\gamma_{2\mu+\nu}}{(2\mu+\nu)!} \left(\frac{\mathbf{i}\theta}{n^{\frac{1}{2\mu}}} \right)^\nu \right) \mathrm{d}\theta \right| \leq \mathrm{e}^{-cn}\,,$$

$$\text{with} \quad \omega := \frac{\ell - \alpha n}{n^{\frac{1}{2\mu}}}\,.$$

The above form of the integral motivates the introduction of the following complex valued function of two arguments:

$$g(w,z) := \exp\left(w^{2\mu} \sum_{\nu\geq 1} \frac{\gamma_{2\mu+\nu}}{(2\mu+\nu)!} z^\nu \right),$$

where we recognize that

$$\exp\left((\mathbf{i}\theta)^{2\mu} \sum_{\nu\geq 1} \frac{\gamma_{2\mu+\nu}}{(2\mu+\nu)!} \left(\frac{\mathbf{i}\theta}{n^{\frac{1}{2\mu}}} \right)^\nu \right) = g\left(\mathbf{i}\theta, \frac{\mathbf{i}\theta}{n^{\frac{1}{2\mu}}} \right).$$

Let us already remark that thanks to the holomorphy of $F_{\boldsymbol{a}}$ on the annulus $\{\zeta \in \mathbb{C} \,|\, 1-\varepsilon < |\zeta| < 1+\varepsilon\}$, there exists $\delta_0 > 0$ small enough such that g is holomorphic on $\mathbb{C} \times B(0,\delta_0)$ where $B(0,\delta_0)$ denotes the open disk in the complex plane centered at the origin and of radius $\delta_0 > 0$. The next step will now be to approximate g by its Taylor expansion. More precisely, upon assuming that δ at least satisfies $\delta \in (0,\delta_0)$, for any $M \in \mathbb{N}^*$, we obtain

$$\left| \boldsymbol{a}_\ell^{(n)} - \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi\, n^{\frac{1}{2\mu}}} \int_{-\delta/n^{\frac{1}{2\mu}}}^{\delta/n^{\frac{1}{2\mu}}} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\theta^{2\mu}} \sum_{m=0}^{M} \frac{(\mathbf{i}\theta)^m}{n^{m/2\mu}\, m!} \frac{\partial^m g}{\partial z^m}(\mathbf{i}\theta, 0)\, \mathrm{d}\theta \right| \leq \mathrm{e}^{-cn} + \left| \mathscr{E}_{n,\ell}^1 \right|,$$

where the error term is defined as

$$\mathscr{E}_{n,\ell}^1 := \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi\, n^{\frac{1}{2\mu}}} \int_{-\delta/n^{\frac{1}{2\mu}}}^{\delta/n^{\frac{1}{2\mu}}} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\theta^{2\mu}} \times \left(g\left(\mathbf{i}\theta, \frac{\mathbf{i}\theta}{n^{\frac{1}{2\mu}}} \right) - \sum_{m=0}^{M} \frac{(\mathbf{i}\theta)^m}{n^{m/2\mu}\, m!} \frac{\partial^m g}{\partial z^m}(\mathbf{i}\theta, 0) \right) \mathrm{d}\theta.$$

We shall come back to the estimate of the error term $\mathscr{E}_{n,\ell}^1$ later on, and we first finish to obtain our complete expansion which involves the attractors $H_{2\mu}^\beta$ and their derivatives.

Inspecting at the definition (5), it is then natural to approximate the integral on the large segment $[-\delta/n^{\frac{1}{2\mu}}, \delta/n^{\frac{1}{2\mu}}]$ by an integral over the whole real line $\mathbb{R}$. Using once more the triangle inequality yields

$$\left| \boldsymbol{a}_\ell^{(n)} - \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi\, n^{\frac{1}{2\mu}}} \int_{\mathbb{R}} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\theta^{2\mu}} \sum_{m=0}^{M} \frac{(\mathbf{i}\theta)^m}{n^{m/2\mu}\, m!} \frac{\partial^m g}{\partial z^m}(\mathbf{i}\theta, 0)\, \mathrm{d}\theta \right| \leq \mathrm{e}^{-cn} + \left| \mathscr{E}_{n,\ell}^1 \right| + \left| \mathscr{E}_{n,\ell}^2 \right|,$$

where the new error term is given by

$$\mathscr{E}_{n,\ell}^2 := \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{2\,\pi\, n^{\frac{1}{2\mu}}} \int_{\mathbb{R} \setminus [-\delta/n^{\frac{1}{2\mu}}, \delta/n^{\frac{1}{2\mu}}]} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\theta^{2\mu}} \sum_{m=0}^{M} \frac{(\mathbf{i}\theta)^m}{n^{m/2\mu}\, m!} \frac{\partial^m g}{\partial z^m}(\mathbf{i}\theta, 0)\, \mathrm{d}\theta\,.$$

Delaying once again the derivation of an estimate for $\mathscr{E}^2_{n,\ell}$, it only remains to link the polynomials P_m to the function g and its partial derivatives. The key observation is the following identity

$$\forall m \geq 1\,, \quad \forall w \in \mathbb{C}\,, \quad P_m(w) \;=\; \frac{w^m}{m!}\frac{\partial^m g}{\partial z^m}(w,0)\,, \tag{9}$$

which can be easily verified from the definition (4) of P_m. If $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ denotes a finitely supported integer valued sequence, we shall introduce the following notations:

$$\langle \boldsymbol{\nu} \rangle := \sum_{\ell \geq 1} \ell\, \nu_\ell\,, \quad |\boldsymbol{\nu}| := \sum_{\ell \geq 1} \nu_\ell\,, \quad \boldsymbol{\nu}\,! := \prod_{\ell \geq 1} \nu_\ell\,!\,,$$

which all make sense for finitely supported sequences as we consider here. Then, combining the expression (9) obtained on the polynomials P_m and the Faà di Bruno formula, we obtain:

$$\forall m \in \mathbb{N}\,, \quad P_m(Y) = Y^m \sum_{\langle \boldsymbol{\nu} \rangle = m} \frac{Y^{2\,\mu\,|\boldsymbol{\nu}|}}{\boldsymbol{\nu}\,!} \prod_{\ell \geq 1} \left(\frac{\gamma_{2\,\mu+\ell}}{(2\,\mu+\ell)\,!} \right)^{\nu_\ell}\,, \tag{10}$$

which is the same expression as in [25, Chapter VII]. For instance, we have in particular $P_0(X) = 1$ and

$$P_1(X) = \frac{\gamma_{2\,\mu+1}}{(2\,\mu+1)\,!}\, X^{2\,\mu+1}\,,$$

which for $\mu = 1$ justifies the formula we had already postulated for our probability example. So, using (10) into our last estimate of $\boldsymbol{a}^{(n)}_\ell$ together with the properties of the Fourier transform, we have finally obtained

$$\left| \boldsymbol{a}^{(n)}_\ell - \sum_{m=0}^{M} \frac{\underline{\kappa}^{-\ell} F_{\boldsymbol{a}}(\underline{\kappa})^n}{n^{(m+1)/2\mu}} \left(P_m\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right) H^\beta_{2\mu} \right)(\omega) \right| \leq \mathrm{e}^{-cn} \,+\, \left|\mathscr{E}^1_{n,\ell}\right| \,+\, \left|\mathscr{E}^2_{n,\ell}\right|\,,$$

which gives precisely the terms of the asymptotic expansion (7). To conclude the proof it thus remains to obtain sharp generalized Gaussian estimates for the two error terms.

Let us first notice that the second error term $\mathscr{E}^2_{n,\ell}$ is easily handled by observing that

$$\left|\mathscr{E}^2_{n,\ell}\right| \leq \sum_{m=0}^{M} \frac{1}{2\,\pi\, n^{(m+1)/2\mu}} \int_{\mathbb{R}\setminus[-\delta/n^{\frac{1}{2\mu}},\delta/n^{\frac{1}{2\mu}}]} \mathrm{e}^{-\mathrm{Re}(\beta)\theta^{2\mu}}\, |P_m(\mathbf{i}\theta)|\; \mathrm{d}\theta \leq C\mathrm{e}^{-cn},$$

for some positive constants $C > 0$ and $c > 0$ that do not depend on n and ℓ.

Before handling the error term $\mathscr{E}^1_{n,\ell}$, let us first note that the holomorphy of g ensures the existence of two constants $C > 0$ and $C_0 > 0$[6] together with $\widehat{\delta} \in (0, \delta_0)$ such that there holds

$$\left| g(w,z) - \sum_{m=0}^{M} \frac{\partial^m g}{\partial z^m}(w,0)\, \frac{z^m}{m\,!} \right| \leq C\,|z|^{M+1}\, \exp\left(\frac{\mathrm{Re}(\beta)}{2}\, (\mathrm{Re}\, w)^{2\,\mu} + C_0\, (\mathrm{Im}\, w)^{2\,\mu} \right)\,, \tag{11}$$

for all $(w,z) \in \mathbb{C} \times \overline{\mathcal{C}(0,\widehat{\delta})}$ where $\mathcal{C}(0,\widehat{\delta}) = \Big\{ z \in \mathbb{C} \;|\; \max(|\mathrm{Re}(z)|, \mathrm{Im}(z)) < \widehat{\delta} \Big\}$. And so, we can now fix once for all δ as $\delta = \widehat{\delta}/2$ which justifies all our previous computations.

[6]The constant $C_0 > 0$ can be chosen such that for all $u \in \mathbb{C}$, there also holds $\mathrm{Re}\left(\beta u^{2\mu}\right) \geq \dfrac{\mathrm{Re}(\beta)}{2}(\mathrm{Re} u)^{2\mu} - C_0(\mathrm{Im} u)^{2\mu}$.

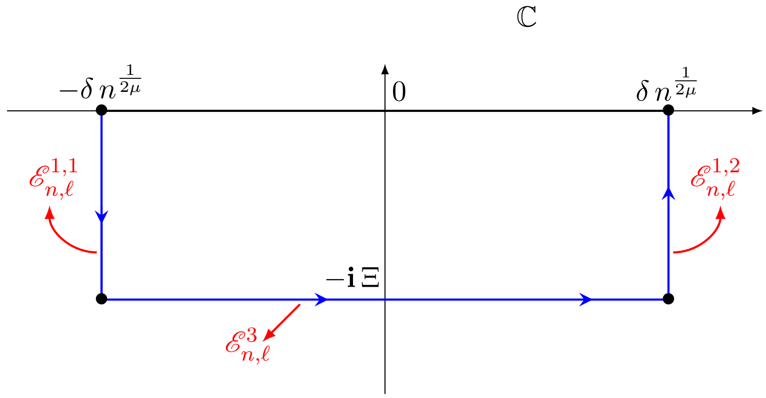

Figure 1: *The integration contour in the case $\omega \geq 0$ (in blue). The bullets correspond to the endpoints of the three segments that define the new contour. The initial contour is depicted in black. Each new integral appears in red.*

The final strategy is to use a well-chosen contour in order to derive a sharp bound for $\mathscr{E}^1_{n,\ell}$. Without loss of generality, we may assume that $\omega \geq 0$, the argument being similar when it is negative, and we consider the contour depicted in Figure 1 where the constant Ξ appearing there is defined as

$$\Xi := \begin{cases} \left(\dfrac{\omega}{4\,\mu\, C_0}\right)^{\frac{1}{2\,\mu-1}}, & \text{if } \dfrac{\omega}{4\,\mu\, C_0} \leq \delta^{2\,\mu-1}\, n^{\frac{2\,\mu-1}{2\,\mu}}, \\[2ex] \delta\, n^{1/(2\,\mu)}, & \text{if } \dfrac{\omega}{4\,\mu\, C_0} \geq \delta^{2\,\mu-1}\, n^{\frac{2\,\mu-1}{2\,\mu}}. \end{cases} \tag{12}$$

Consequently, with the above definition, for any z on the contour that is depicted in blue in Figure 1, we have $\max(|\mathrm{Re}\ z|, |\mathrm{Im}\ z|)/n^{\frac{1}{2\mu}} \leq \delta$ and we shall therefore be able to apply Cauchy's formula for holomorphic functions and also use the estimate (11). And Cauchy's formula gives:

$$\int_{-\delta\, n^{\frac{1}{2\mu}}}^{\delta\, n^{\frac{1}{2\mu}}} \mathrm{e}^{-\mathbf{i}\,\omega\,\theta}\, \mathrm{e}^{-\beta\,\theta^{2\,\mu}} \left(g\left(\mathbf{i}\,\theta, \frac{\mathbf{i}\,\theta}{n^{\frac{1}{2\mu}}}\right) - \sum_{m=0}^{M} \frac{(\mathbf{i}\,\theta)^m}{m\,!\, n^{\frac{m}{2\mu}}} \frac{\partial^m g}{\partial z^m}(\mathbf{i}\,\theta, 0) \right) \mathrm{d}\theta = \mathscr{E}^{1,1}_{n,\ell} + \mathscr{E}^{1,2}_{n,\ell} + \mathscr{E}^{3}_{n,\ell}\,,$$

where $\mathscr{E}^{1,1}_{n,\ell}$, resp. $\mathscr{E}^{1,2}_{n,\ell}$, corresponds to the integral on the left, resp. right, vertical segment, and $\mathscr{E}^{3}_{n,\ell}$ corresponds to the integral on the horizontal segment (see Figure 1). Both integrals along the vertical segments contribute to exponentially decaying terms as it can be noticed by direct computations of the form

$$\left|\mathscr{E}^{1,1}_{n,\ell}\right| + \left|\mathscr{E}^{1,2}_{n,\ell}\right| \leq C\, \mathrm{e}^{-cn} \int_0^{\Xi} \exp\left(-\frac{2\mu-1}{2\mu}\,\omega\, u\right) \mathrm{d}u \leq C\,\Xi\, \mathrm{e}^{-cn} \leq \widetilde{C}\mathrm{e}^{-\widetilde{c}n},$$

since $\Xi \leq \delta n^{\frac{1}{2\mu}}$. For the last contribution, using the definition of Ξ, we obtain the bound

$$\begin{aligned} \left|\mathscr{E}^{3}_{n,\ell}\right| &\leq C\, \exp\left(-\omega\,\Xi + 2C_0\,\Xi^{2\,\mu}\right) \int_{-\delta\, n^{1/2\,\mu}}^{\delta\, n^{1/2\,\mu}} \mathrm{e}^{-\frac{\mathrm{Re}(\beta)}{2}\,\theta^{2\,\mu}} \frac{|\theta - \mathbf{i}\,\Xi|^{M+1}}{n^{(M+1)/2\mu}}\, \mathrm{d}\theta \\ &\leq C \frac{\left(1+\Xi^{M+1}\right)}{n^{(M+1)/2\mu}} \exp\left(-\frac{2\mu-1}{2\mu}\,\omega\,\Xi\right). \end{aligned}$$

In the first regime of (12), we readily get our desired generalized Gaussian estimate:

$$\left|\mathscr{E}^3_{n,\ell}\right| \leq \frac{C}{n^{(M+1)/2\mu}} \exp\left(-c\,\omega^{\frac{2\mu}{2\mu-1}}\right),$$

while in second regime we simply get an exponential bound

$$\left|\mathscr{E}^3_{n,\ell}\right| \leq C\mathrm{e}^{-cn},$$

which implies the estimate

$$\left|\mathscr{E}^3_{n,\ell}\right| \leq C\mathrm{e}^{-cn} + \frac{C}{n^{(M+1)/2\mu}} \exp\left(-c\left(\frac{|\ell-\alpha n|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

that holds for any $(n,\ell) \in \mathbb{N}^* \times \mathbb{Z}$. To conclude the proof it only remains to check that the exponentially small terms in n can be absorbed into the generalized bound. This can always be achieved when $|\ell| \leq Ln$, and the proof of Theorem 1 is thus complete.

3 Some extensions and perspectives

We now discuss some extensions and possible perspectives related to Theorem 1.

3.1 The semi-discrete case

For a given sequence $\boldsymbol{b} \in \ell^1(\mathbb{Z};\mathbb{C})$, we consider linear evolution problems of the form

$$\forall t > 0\,, \quad \boldsymbol{u}'(t) \;=\; \boldsymbol{b} \star \boldsymbol{u}(t)\,, \tag{13}$$

where $\boldsymbol{u}'(t)$ stand for the time derivate of the sequence valued function $\boldsymbol{u} : t \mapsto \boldsymbol{u}(t) = (u_\ell(t))_{\ell\in\mathbb{Z}}$, that is

$$\forall (t,\ell) \in \mathbb{R}^*_+ \times \mathbb{Z}\,, \quad \big(\boldsymbol{u}'(t)\big)_\ell = \frac{\mathrm{d}u_\ell}{\mathrm{d}t}(t)\,.$$

Such a class of evolution problems naturally arise in numerical analysis as the semi-discretization in space of linear partial differential equations by means of finite differences schemes [1, 2], in probability for the study of continuous-time random walks [21] or in some applications in biology (e.g. spreading of agents on a lattice [4, 20]). Since the operator $\mathscr{L}_{\boldsymbol{b}} : \boldsymbol{u} \mapsto \boldsymbol{b} \star \boldsymbol{u}$ is a bounded operator on any $\ell^q(\mathbb{Z};\mathbb{C})$ with $q \in [1,+\infty]$, it is the infinitesimal generator of a uniformly continuous semigroup acting on any $\ell^q(\mathbb{Z};\mathbb{C})$ that we shall denote by $(\mathscr{S}_{\boldsymbol{b}}(t))_{t>0}$ and whose action on $\ell^q(\mathbb{Z};\mathbb{C})$ can be expressed as (see [15]):

$$\forall t > 0\,, \quad \mathscr{S}_{\boldsymbol{b}}(t) : \boldsymbol{u} \mapsto \mathscr{S}_{\boldsymbol{b}}(t)\boldsymbol{u} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \boldsymbol{b}^{(n)} \star \boldsymbol{u}.$$

The above series is indeed uniformly convergent for any $\boldsymbol{u} \in \ell^q(\mathbb{Z};\mathbb{C})$, and for all $t > 0$ we have the crude uniform bound

$$|||\mathscr{S}_{\boldsymbol{b}}(t)|||_{\ell^q\to\ell^q} \leq \mathrm{e}^{t\|\boldsymbol{b}\|_{\ell^1}}.$$

We are interested in deriving a precise pointwise asymptotic expansion of the Green's function associated to the semi-group $(\mathscr{S}_{\boldsymbol{b}}(t))_{t>0}$ which is defined as follows. For any sequence $\boldsymbol{u}^0 \in \ell^q(\mathbb{Z};\mathbb{C})$ with $q \in [1,+\infty]$, there exists a unique global solution $\boldsymbol{u} \in \mathscr{C}^0([0,+\infty);\ell^q(\mathbb{Z};\mathbb{C})) \cap \mathscr{C}^1((0,+\infty);\ell^q(\mathbb{Z};\mathbb{C}))$ of (13) which initially satisfies $\boldsymbol{u}(0) = \boldsymbol{u}^0$. This solution is explicitly given through the following representation formula

$$\forall t > 0\,, \quad \boldsymbol{u}(t) = \mathscr{S}_{\boldsymbol{b}}(t)\boldsymbol{u}^0 = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\, \boldsymbol{b}^{(n)} \star \boldsymbol{u}^0.$$

The Green's function $\boldsymbol{G}^{\boldsymbol{b}}$ is then the unique solution associated to the initial condition $\boldsymbol{u}^0 = \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ stands for the Dirac delta sequence which satisfies $\boldsymbol{\delta}_\ell = 1$ if $\ell = 0$ and $\boldsymbol{\delta}_\ell = 0$ otherwise. We thus have

$$\forall t > 0\,, \quad \boldsymbol{G}^{\boldsymbol{b}}(t) = \mathscr{S}_{\boldsymbol{b}}(t)\boldsymbol{\delta} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\, \boldsymbol{b}^{(n)} \star \boldsymbol{\delta} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\, \boldsymbol{b}^{(n)},$$

since $\boldsymbol{\delta}$ is a unitary element for the convolution, that is $\boldsymbol{b} \star \boldsymbol{\delta} = \boldsymbol{b}$. As a consequence, solutions to (13) starting from $\boldsymbol{u}(0) = \boldsymbol{u}^0 \in \ell^q(\mathbb{Z};\mathbb{C})$ simply write

$$\forall t > 0\,, \quad \boldsymbol{u}(t) = \mathscr{S}_{\boldsymbol{b}}(t)\boldsymbol{u}^0 = \boldsymbol{G}^{\boldsymbol{b}}(t) \star \boldsymbol{u}^0.$$

We now present the main assumptions in the sequence $\boldsymbol{b}$ considered in this semi-discrete case.

Assumption 4 (Semi-discrete setting)**.** *The sequence $\boldsymbol{b} = (b_\ell)_{\ell\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z};\mathbb{C})$ and its associated Fourier series:*

$$\nu_{\boldsymbol{b}} \quad : \quad \zeta \in \mathbb{C} \longmapsto \sum_{\ell\in\mathbb{Z}} b_\ell\, \zeta^\ell\,,$$

defines a holomorphic function on an annulus $\{\zeta \in \mathbb{C} \,|\, 1-\varepsilon < |\zeta| < 1+\varepsilon\}$ for some $\varepsilon > 0$. Furthermore, there holds:

$$\sup_{\kappa\in\mathbb{S}^1} \mathrm{Re}(\nu_{\boldsymbol{b}}(\kappa)) \;=\; 0\,.$$

We also assume that there exists a finite set of pairwise distinct points $\{\underline{\kappa}_1,\dots,\underline{\kappa}_K\}$, $K \geq 1$, in $\mathbb{S}^1$ such that the real part of $\nu_{\boldsymbol{b}}(\underline{\kappa}_k)$ is 0 for any $k \in \{1,\dots,K\}$ and:

$$\forall\, \kappa \in \mathbb{S}^1 \setminus \left\{\underline{\kappa}_1,\dots,\underline{\kappa}_K\right\}, \quad \mathrm{Re}\,(\nu_{\boldsymbol{b}}(\kappa)) \;<\; 0\,.$$

Moreover, at any point $\underline{\kappa}_k \in \mathbb{S}^1$, $k \in \{1,\dots,K\}$, where the the real part of $\nu_{\boldsymbol{b}}(\underline{\kappa}_k)$ vanishes, that is $\nu_{\boldsymbol{b}}(\underline{\kappa}_k) = \mathbf{i}\underline{z}_k$ for some $\underline{z}_k \in \mathbb{R}$, there exists a real number α_k, a complex number β_k with positive real part and a nonzero integer $\mu_k \in \mathbb{N}^*$ such that, as the complex number ξ tends to zero, there holds:

$$\nu_{\boldsymbol{b}}\left(\underline{\kappa}_k\, \mathrm{e}^{\mathbf{i}\,\xi}\right) \;=\; \mathbf{i}\underline{z}_k \;+\; \mathbf{i}\,\alpha_k\,\xi - \beta_k\,\xi^{2\,\mu_k} + O(\xi^{2\,\mu_k+1})\,. \tag{14}$$

Under the above assumption on the sequence $\boldsymbol{b}$, we also have the following convergent power series:

$$\nu_{\boldsymbol{b}}\left(\underline{\kappa}_k\, \mathrm{e}^{\mathbf{i}\,\xi}\right) \;=\; \mathbf{i}\underline{z}_k \;+\; \mathbf{i}\,\alpha_k\,\xi - \beta_k\,\xi^{2\,\mu_k} + \sum_{\nu\geq 2\,\mu_k+1} \frac{\gamma_{k,\nu}}{\nu\,!}\,(\mathbf{i}\,\xi)^\nu\,, \tag{15}$$

for some complex coefficients $\gamma_{k,\nu} \in \mathbb{C}$ which allow to define the polynomials $P_{k,m}$ exactly as in (4). An analogue of Theorem 1 in this semi-discrete setting reads as follows.

Theorem 2 (Semi-discrete local limit theorem)**.** *Let the sequence $\boldsymbol{b}$ satisfy Assumption 4. Then there exist a positive real number $L > 0$ and some positive constant $c_0 > 0$ such that for any $t \geq 1$ and $\ell \in \mathbb{Z}$ with $|\ell| > L\,t$, there holds:*

$$\left| \boldsymbol{G}_\ell^{\boldsymbol{b}}(t) \right| \leq \exp(-c_0\,t - c_0\,|\ell|)\,. \tag{16}$$

Moreover, for any integer $M \in \mathbb{N}$, there exist some positive constants C_M and c_M (that depend on M and $\boldsymbol{a}$) such that the following holds: for any $t \geq 1$ and $\ell \in \mathbb{Z}$ with $|\ell| \leq L\,t$, there holds:

$$\left| \boldsymbol{G}_\ell^{\boldsymbol{b}}(t) - \sum_{k=1}^{K} \frac{\kappa_k^{-\ell} \mathrm{e}^{\mathbf{i}\,t\,z_k}}{t^{1/(2\,\mu_k)}}\, H_{2\,\mu_k}^{\beta_k} \left(\frac{\ell - \alpha_k n}{t^{1/(2\,\mu_k)}} \right) - \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{\kappa_k^{-\ell} \mathrm{e}^{t\,\mathbf{i}\,z_k}}{t^{(m+1)/(2\,\mu_k)}} \left(P_{k,m}(-\mathrm{d}/\mathrm{d}x) H_{2\,\mu_k}^{\beta_k} \right) \left(\frac{\ell - \alpha_k n}{t^{1/(2\,\mu_k)}} \right) \right| \\ \leq C_M \sum_{k=1}^{K} \frac{1}{t^{(M+2)/(2\,\mu_k)}} \exp\left(-c_M \left(\frac{|\ell - \alpha_k\,n|}{t^{1/(2\,\mu_k)}} \right)^{\frac{2\,\mu_k}{2\,\mu_k - 1}} \right). \tag{17}$$

It is important to remark that estimates (16) and (17) are only valid for large times, here taken as $t \geq 1$. Indeed, for small times $0 < t < 1$, we do not expect the validity of such an estimate, and one needs to be careful with the singularity at $t = 0$. Theorem 2 was already proved in [4] in a specific case at order $M = 1$ with a different method of proof. Instead of using Fourier series on $\mathbb{Z}$, [4] uses Laplace transform in time to derive an alternate representation formula for the Green's function $\boldsymbol{G}^{\boldsymbol{b}}(t)$. Here, for any $t > 0$ and $\ell \in \mathbb{Z}$, we have the following analogue of (8) that is

$$\boldsymbol{G}_\ell^{\boldsymbol{b}}(t) \;=\; \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathbf{i}\,\ell\,\theta}\, \mathrm{e}^{t\,\nu_{\boldsymbol{b}}(\mathrm{e}^{\mathbf{i}\theta})}\, \mathrm{d}\theta\,.$$

With this representation formula, we observe that the proof of Theorem 1 readily applies to the continuous setting without any difficulty. Coming back to the example addressed in [4], it corresponds in our setting to the finitely supported sequence $\boldsymbol{b}$ defined as

$$b_{-1} = \frac{1}{\chi}, \quad b_0 = -\chi - \frac{1}{\chi}, \quad b_1 = \chi, \quad \text{with} \quad b_\ell = 0 \text{ for } |\ell| \geq 2,$$

for some $\chi > 1$. In that case, the associated amplification factor reads

$$\nu_{\boldsymbol{b}}(\mathrm{e}^{\mathbf{i}\theta}) = \frac{1}{\chi} e^{-\mathbf{i}\theta} - \chi - \frac{1}{\chi} + \chi e^{\mathbf{i}\theta} = -\left(\chi + \frac{1}{\chi}\right)(1 - \cos(\theta)) + \mathbf{i}\left(\chi - \frac{1}{\chi}\right)\sin(\theta), \quad \theta \in [0, 2\pi),$$

such that there is a unique tangency point at the origin for $\theta = 0$ with asymptotic expansion

$$\nu_{\boldsymbol{b}}(\mathrm{e}^{\mathbf{i}\xi}) = \mathbf{i}\left(\chi - \frac{1}{\chi}\right)\xi - \left(\chi + \frac{1}{\chi}\right)\frac{\xi^2}{2} - \left(\chi - \frac{1}{\chi}\right)\frac{(\mathbf{i}\xi)^3}{3!} + O(\xi^4),$$

as $\xi \to 0$. This implies that $\alpha = \chi - \frac{1}{\chi} > 0$, $\beta = \frac{\chi}{2} + \frac{1}{2\chi}$, $\mu = 1$ and we can even deduce the first cumulant $\gamma_3 = -\alpha = \chi - \frac{1}{\chi}$. As a consequence, Theorem 2 applied to this case with $M = 1$ gives for all $t \geq 1$ and $|\ell| \leq Lt$ for some $L > 0$:

$$\left| \boldsymbol{G}_\ell^{\boldsymbol{b}}(t) - \frac{1}{\sqrt{4\pi\beta t}} \exp\left(-\frac{x_\ell(t)^2}{2}\right) - \frac{\alpha}{24\beta^2\sqrt{2\pi}t}\left(x_\ell(t)^3 - 3x_\ell(t)\right)\exp\left(-\frac{x_\ell(t)^2}{2}\right) \right| \leq \frac{C}{t^{3/2}} \exp\left(-c\,x_\ell(t)^2\right),$$

for two positives constants $C, c > 0$, and where we have set

$$x_\ell(t) := \frac{\ell - \alpha t}{\sqrt{2\beta t}}.$$

This above expansion[7] is precisely the one derived in [4, Proposition 4.1] .

3.2 The dispersive case

As we already explained in the introduction, the parabolicity hypothesis of Assumption 3 is one of two possible behaviors referenced by Thomée [32], the other behavior being the *dispersive* case which is also referred to as the *unstable* case since it yields unboundedness of the sequence $\left(\left\|\boldsymbol{a}^{(n)}\right\|_{\ell^1}\right)_{n\in\mathbb{N}}$, see [19, 32]. Let $\boldsymbol{a}$ be a given sequence that satisfies Assumption 1 and Assumption 2, a tangency point $\underline{\kappa}_k$ is said to be *dispersive* if the asymptotic expansion of the modulus of the amplification factor function $F_{\boldsymbol{a}}$ near this point satisfies the following property. There exist a real number α_k, a complex number β_k with positive real part, two nonzero integers $\nu_k, \mu_k \in \mathbb{N}$ verifying $1 < \nu_k < 2\mu_k$ and a real polynomial p_k with $p_k(0) \neq 0$ such that

$$F_{\boldsymbol{a}}\left(\underline{\kappa}_k\, \mathrm{e}^{\mathbf{i}\,\xi}\right) \;=\; F_{\boldsymbol{a}}(\underline{\kappa}_k)\, \exp\left(\mathbf{i}\,\alpha_k\,\xi + \mathbf{i}\xi^{\nu_k} p_k(\xi) - \beta_k\,\xi^{2\,\mu_k} + O\left(\xi^{2\,\mu_k+1}\right)\right)\,, \tag{18}$$

for all complex number ξ that tends to zero. We refer to [5, 10, 19, 27] for examples of sequences yielding dispersive tangency points. In numerical analysis, a celebrated example where the above expansion holds is given by the so-called Lax-Wendroff scheme for the transport equation [18, 22] and is characterized by a sequence $\boldsymbol{a}$ taking values

$$a_{-1} = \frac{-\lambda + \lambda^2}{2}, \quad a_0 = 1 - \lambda^2, \quad a_1 = \frac{\lambda + \lambda^2}{2}, \qquad \text{with} \qquad a_\ell = 0 \text{ for } |\ell| \geq 2$$

for some $\lambda \in (-1, 1)$. A straightforward computation yields

$$\left|F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\theta}\right)\right| = 1 - 4\lambda^2(1-\lambda^2)\left(\sin\frac{\theta}{2}\right)^4, \quad \theta \in \mathbb{R},$$

which implies that the modulus of $F_{\boldsymbol{a}}$ has a unique tangency point $\underline{\kappa} = 1$ with $F_{\boldsymbol{a}}(1) = 1$ together with asymptotic expansion

$$F_{\boldsymbol{a}}\left(\mathrm{e}^{\mathbf{i}\,\xi}\right) \underset{\xi\to 0}{=} \exp\left(\mathbf{i}\,\lambda\,\xi - \mathbf{i}\frac{\lambda(1-\lambda^2)}{6}\xi^3 - \frac{\lambda^2(1-\lambda^2)}{8}\xi^4 + O\left(\xi^5\right)\right),$$

which is of the form of (18) with $\nu = 3$ and $2\mu = 4$. To our best knowledge, it is still an open problem to prove an analogue of Theorem 1 for dispersive tangency points. The more advanced result available in the literature is [27, Theorem 1.2] which provides the leading order term of $\boldsymbol{a}^{(n)}$. Specified to the Lax-Wendroff example, [27, Theorem 1.2] shows that

$$\boldsymbol{a}_\ell^{(n)} = \frac{1}{n^{1/3}}\, A_3^{\mathbf{i}\varpi}\left(\frac{\ell - \lambda n}{n^{1/3}}\right) + o(n^{-1/3}),$$

uniformly with respect to $\ell \in \mathbb{Z}$, where $\varpi = \frac{\lambda(1-\lambda^2)}{6}$ and the function $A_3^{\mathbf{i}\varpi}$ is defined as the oscillatory integral[8]:

$$\forall x \in \mathbb{R} \quad A_3^{\mathbf{i}\varpi}(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathbf{i}xu}\, \mathrm{e}^{-\mathbf{i}\varpi u^3}\mathrm{d}u\,.$$

We shall deal with the local limit theorem (up to any order) in the dispersive case in a subsequent work.

[7] Borrowing the notation from [4], one has $\alpha = c_*$ and $\beta = \cosh(\lambda_*)$.

[8] The function $A_3^{\mathbf{i}/3}$ is the classical Airy function.

3.3 The multi-dimensional case

Another very interesting perspective is the generalization of Theorem 1 in the multi-dimensional setting where the sequence $\boldsymbol{a}$ is now indexed on the d-dimensional lattice $\mathbb{Z}^d$, with some integer $d \geq 2$. We refer to the fascinating recent developments [7, 26, 28, 29] on the subject. As expected, the multi-dimensional setting presents a much richer classification of the tangency points compared to the one-dimensional case which has essentially two cases (parabolic and dispersive)[9]. Borrowing the terminology used in the aforementioned references, the multi-dimensional analogue of the parabolic case studied here would correspond to the case where all tangency points of $F_{\boldsymbol{a}}$ are of positive homogeneous type, see [28, Definition 1.3.]. In that case, one can hope to prove a result analogous to Theorem 1, and we leave it for future work.

[9]We do not address here the more degenerate case where $F_{\boldsymbol{a}}(\kappa)$ has modulus 1 for any $\kappa \in \mathbb{S}^1$. We refer to [5] for some examples in that case and formal results.

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