FORMAL MODULI PROBLEMS AND DERIVED LIE ALGEBRAS

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Abstract. We show that derived Lie algebras and restricted Lie algebras classify formal moduli problems on Artin divided power algebras and truncated algebras, respectively.

1. Introduction

Given a moduli space over a field k and a k-rational point $x \in \mathcal{M}(k)$, derived deformation theory aims to describe the formal neighbourhood of the point x in terms of the tangent complex $T_x\mathcal{M}$. As a classical example, $H^0(T_x\mathcal{M})$ can be identified with the set of first order infinitesimal paths at x, and the obstruction to lifting an n-th order path to an $(n+1)$-st order path is given by a specific class in $H^1(T_x\mathcal{M})$. The relation between formal neighbourhoods and their tangent complexes has been made more precise in terms of derived geometry. Indeed, the formal neighbourhood of a (sufficiently well-behaved) derived k-stack around a k-point is encoded by a *formal moduli problem*, that is, a functor

$$X\colon \mathrm{Art}_k \longrightarrow \mathcal{S} \tag{1.1}$$

from the ∞-category of Artin simplicial k-algebras with residue field k, satisfying a derived version of the Schlessinger condition. When k is a field of characteristic zero, a fundamental result of Lurie and Pridham states that taking the tangent complex refines to an equivalence between the ∞-category of such formal moduli problems and that of (shifted) dg-Lie algebras.

This has recently been generalised to arbitrary fields by Brantner and Mathew [5], who show that taking tangent complexes refines to an equivalence between the ∞-categories of formal moduli problems (1.1) and *partition Lie algebras*. These partition Lie algebras are homotopy-theoretic refinements of shifted dg-Lie algebras, and carry an algebraic structure controlled by the Σ_r-equivariant topology of (suspensions of) the partition complexes T_r.

The aim of this note is to describe two, arguably more classical, variants of this result by Brantner–Mathew. First, we show that formal moduli problems on Artin simplicial k-algebras *with divided powers* are classified by derived Lie algebras (Example 2.3). Here we assume Artin divided power algebras to also be nilpotent with respect to their divided power structure; our methods do not apply to the (geometrically more interesting) case of formal moduli problems defined on divided power algebras that are only nilpotent as k-algebras.

Second, we show that derived restricted Lie algebras are equivalent to certain *truncated* formal moduli problems. Over a perfect field, these can also be understood more geometrically as describing formal moduli problems equipped with a trivialisation of the Frobenius, i.e. a factorisation of the Frobenius over the basepoint (Section 5).

The main idea behind these results is that the structure of shifted derived (restricted) Lie algebras is also encoded by the partition complexes T_r. To make this more precise, we will make use of the operadic formalism developed in [4]: the cochains on the partitions complexes can be organised into a derived operad $\mathrm{Lie}^{\pi}_{\Delta}$, which simultaneously controls the three types of Lie algebras mentioned above by varying its action on the ∞-category of k-modules.

Notation and conventions. Throughout, let k be a field of characteristic $p > 0$. The unbounded derived ∞-category of k will be denoted Mod_k; we will refer to its objects as *k-modules.* We will write $\mathrm{Vect}^{\omega}_k \subseteq \mathrm{Vect}_k \subset \mathrm{Mod}_k$ for the full subcategories of (finite dimensional) k-vector spaces, concentrated in degree 0. Furthermore, we let $\mathrm{Mod}_k^{\mathrm{ft,cn}}$ and $\mathrm{Mod}_k^{\mathrm{ft,ccn}}$ denote the full subcategories of Mod_k spanned by the connective, resp. coconnective k-modules M such that each $\pi_i(M) \in \mathrm{Vect}_k^{\omega}$. Equivalently, these subcategories are spanned by the geometric realisations of simplicial objects, resp. totalisations of cosimplicial objects in Vect_k^{ω}. Finally, all operads and symmetric sequences are assumed to be concentrated in arity ≥ 1.

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2. Recollections on derived algebras

Throughout this paper, we will freely make use of the formalism of (non-abelian) derived functors developed in [5], see also [12]. In this section, we will provide a brief reminder on how this can be used to define and study various types of algebras in the unbounded derived ∞-category Mod_k. To this end, let us use introduce the following terminology:

Definition 2.1. Let $\mathcal{D}$ be an ∞-category with small limits and colimits and let $F\colon \mathrm{Mod}_k \longrightarrow \mathcal{D}$ be a functor. We will say that F is a (non-abelian) *derived functor* if it preserves all sifted colimits, as well as totalisations of cosimplicial diagrams in Vect_k^{ω} that are m-coskeletal for some m.

Every derived functor can be recovered from its restriction to Vect_k^{ω} by a combination of left and right Kan extension [5, Construction 3.22] . On the other hand, a functor $F\colon \mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k$ preserving filtered colimits extends uniquely to a derived functor $\mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$ as soon as F can be written as a filtered colimit of functors of finite degree [5, Theorem 3.27]. In this case, the derived functor preserves totalisations of all cosimplicial diagrams in Vect_k^{ω} [4, Remark 2.45]. Taking derived functors is compatible with composition, so that the derived functor of a monad $T\colon \mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k$ gives rise to a monad $T\colon \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$; we will refer to its algebras as *derived T-algebras.*

Example 2.2. Taking augmentation ideals defines a functor $\mathrm{Ring}_k^{\mathrm{aug}} \longrightarrow \mathrm{Vect}_k$ from the (ordinary) category of augmented commutative k-algebras to vector spaces. This is a monadic functor, with left adjoint sending $V \mapsto \mathrm{Sym}(V)$. The resulting monad on Vect_k is a sum of finite degree functors and hence yields a derived monad on Mod_k. We will denote its ∞-category of algebras simply by DAlg_k and refer to its objects as *derived augmented (commutative) k-algebras.*

One can apply the same argument to augmented k-algebras that are truncated ($x^p = 0$ for every x in the augmentation ideal) or come with a divided power structure on the augmentation ideal. Taking the augmentation ideal is then a monadic right adjoint, with left adjoint sending a vector space V to the truncated symmetric algebra $\mathrm{Sym}^{\mathrm{tr}}(V)$ and divided power algebra $\Gamma(V)$. The corresponding monads can be derived, so that we obtain ∞-categories of *derived truncated algebras* and *derived divided power algebras* over k, related by forgetful functors

$$\mathrm{DAlg}_k^{\mathrm{pd}} \xrightarrow{\text{forget}} \mathrm{DAlg}_k^{\mathrm{tr}} \xrightarrow{\text{forget}} \mathrm{DAlg}_k.$$

Example 2.3. Consider the monads Lie, $\mathrm{Lie}^{\mathrm{res}}\colon \mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k$ taking the free Lie algebra (with $[x, x] = 0$) and restricted Lie algebra. Both monads preserve filtered colimits and are given by sums of finite degree functors, and hence yield derived monads on Mod_k. We will refer to algebras over these monads as *derived (restricted) Lie algebras* and denote their ∞-categories by

$$\mathrm{DLie}_k \qquad \text{and} \qquad \mathrm{DLie}_k^{\mathrm{res}}.$$

The theory of operads provides another way to define and study algebras in Mod_k. Recall that classically, a (k-linear) operad is a symmetric sequence of k-vector spaces, endowed with an associative algebra structure with respect to the composition product. The category of symmetric sequences acts on the category of k-vector spaces, so that every operad $\mathcal{P}$ gives rise to a monad and hence to a notion of $\mathcal{P}$-algebra. All of this can be extended to the level of the derived ∞-category Mod_k by taking derived functors. More precisely, the category of symmetric sequences has a certain derived analogue that fits into a sequence of inclusions

$$k[\mathcal{O}_\Sigma] \hookrightarrow \mathrm{sSeq}_k^{\mathrm{gen,cn}} \hookrightarrow \mathrm{sSeq}_k^{\mathrm{gen},\vee}. \tag{2.4}$$

Here the (ordinary) category $k[\mathcal{O}_\Sigma]$ is the smallest full additive subcategory of the category of symmetric sequences of k-vector spaces that contains all symmetric sequences $k[\Sigma_r/H]$ spanned by a Σ_r-orbit (for any arity $r \geq 1$). We write $\mathrm{sSeq}_k^{\mathrm{gen,cn}} = \mathcal{P}_\Sigma(k[\mathcal{O}_\Sigma])$ for its completion under sifted colimits and refer to its objects as (connective) *derived symmetric sequences.*

Note that $\mathrm{sSeq}_k^{\mathrm{gen,cn}}$ decomposes into a product of ∞-categories, one for each arity $r \geq 1$. For each derived symmetric sequence X, its arity r part $X(r)$ is the restriction of $X\colon k[\mathcal{O}_\Sigma] \longrightarrow \mathcal{S}$ to the full additive subcategory generated by the $k[\Sigma_r/H]$ for a fixed arity r. Informally, one can think of the value of X on $k[\Sigma_r/H]$ as encoding the genuine fixed points $X(r)^H$.

Example 2.5. There is a left adjoint functor $k[-]\colon \prod_{r\geq 1} \mathcal{S}^{\Sigma_r} \to \mathrm{sSeq}_k^{\mathrm{gen}}$ sending each sequence of genuine Σ_r-spaces X to its *k-linearisation* $k[X]$. This functor sends each orbit Σ_r/H to the corresponding object in $k[\mathcal{O}_\Sigma]$ and is extended by colimits.

For example, the k-linearisation of the terminal object of $\prod \mathcal{S}^{\Sigma_r}$ defines a derived symmetric sequence Com with $\mathrm{Com}(r)^H = k$ for all $H < \Sigma_r$. Note that this differs from the usual ("Borel") symmetric sequence $\mathbb{E}_\infty$ with $\mathbb{E}_\infty(r) = k$ and $\mathbb{E}_\infty(r)^H = 0$ for all $\{e\} \neq H < \Sigma_r$.

The ∞-category $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ appearing in (2.4) is a certain compactly generated stable ∞-category, equipped with a left complete t-structure whose connective part is $\mathrm{sSeq}_k^{\mathrm{gen,cn}}$. We refer to [4, Definition 3.72] for a precise definition of $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ and content ourselves with recalling its main features. Most importantly, the formalism of derived functors extends to $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$:

Construction 2.6. Let us say that $F\colon \mathrm{sSeq}_k^{\mathrm{gen},\vee} \longrightarrow \mathcal{D}$ is a *derived functor* if it preserves sifted colimits and totalisations of cosimplicial diagrams in $k[\mathcal{O}_\Sigma]$. Every derived functor can be obtained from its restriction to $k[\mathcal{O}_\Sigma]$ by a combination of left and right Kan extension [4, Construction 2.44].

Conversely, suppose that $\mathcal{D}$ is a sufficiently nice stable ∞-category, such as Mod_k or $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$, and that $F\colon k[\mathcal{O}_\Sigma] \longrightarrow \mathcal{D}$ is a filtered colimit of finite degree functors. Then F admits a (unique) derived functor $F\colon \mathrm{sSeq}^{\mathrm{gen},\vee} \longrightarrow \mathcal{D}$ [4, Proposition 2.46]. A similar construction applies to functors in multiple variables [4, Theorem 2.52].

Using this, all the standard operations on symmetric sequences have natural extensions to $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$. In particular, one can endow the ∞-category $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ with a monoidal structure

$$\circ\colon \mathrm{sSeq}_k^{\mathrm{gen},\vee} \times \mathrm{sSeq}_k^{\mathrm{gen},\vee} \longrightarrow \mathrm{sSeq}_k^{\mathrm{gen},\vee}$$

which is derived from the usual composition product of symmetric sequences of vector spaces $\circ\colon k[\mathcal{O}_\Sigma] \times k[\mathcal{O}_\Sigma] \longrightarrow k[\mathcal{O}_\Sigma]$. We will refer to a (co)algebra with respect to this monoidal structure as a *derived (co)operad.* Given a suitable action of $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ on Mod_k, we can then define algebras over such derived operads.

Definition 2.7. Let $\odot\colon k[\mathcal{O}_\Sigma] \times \mathrm{Vect}_k^\omega \longrightarrow \mathrm{Vect}_k^\omega$ be an action of the monoidal category $\big(k[\mathcal{O}_\Sigma], \circ\big)$. We will say that $\odot$ is *of composition type* if the following holds.

(1) It is additive in the first variable.
(2) Each functor $k[\Sigma_r/H]\odot -\colon \mathrm{Vect}_k^\omega \longrightarrow \mathrm{Vect}_k^\omega$ is of degree $\leq r$.

By [4, Theorem 2.52], these conditions imply that the action extends to an action

$$\odot\colon \mathrm{sSeq}_k^{\mathrm{gen},\vee}\times \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$$

preserving sifted colimits and totalisations of cosimplicial diagrams in $k[\mathcal{O}_\Sigma]\times \mathrm{Vect}_k^\omega$.

Example 2.8. We have the following examples of actions of composition type:

(1) The composition product and restricted composition product [7]

$$X\circ V=\bigoplus_{r\geq 1}\big[X(r)\otimes V^{\otimes r}\big]_{\Sigma_r} \qquad X\mathbin{\bar\circ} V=\bigoplus_{r\geq 1}\big[X(r)\otimes V^{\otimes r}\big]^{\Sigma_r}.$$

(2) The image of the norm map $\mathrm{Nm}\colon X\circ V\twoheadrightarrow X\circ^{(1)}V\hookrightarrow X\mathbin{\bar\circ} V$ [6].

(3) For every action of composition type $\odot$, one obtains another action $X\mathbin{\overline{\odot}}V=(X^\vee\odot V^\vee)^\vee$ by conjugating with linear duality. This relates $\circ$ with $\bar\circ$ and $\circ^{(1)}$ with itself.

Example 2.9. The symmetric sequence Com from Example 2.5 has the natural structure of a derived operad (in arity ≥ 1). Its ∞-categories of algebras with respect to $\circ$, $\circ^{(1)}$ and $\bar\circ$ are precisely the ∞-category of derived (augmented, truncated, divided power) algebras over k from Example 2.2; here we use the equivalence between augmented algebras and non-unital algebras by taking augmentation ideals.

Finally, let us recall the adjoint pair $(-)^\vee\colon \mathrm{sSeq}_k^{\mathrm{gen},\vee}\leftrightarrows \big(\mathrm{sSeq}_k^{\mathrm{gen},\vee}\big)^{\mathrm{op}}:(-)^\vee$ sending each object to its *k-linear dual.* This functor is derived from the equivalence $k[\mathcal{O}_\Sigma]\longrightarrow k[\mathcal{O}_\Sigma]^{\mathrm{op}}$ taking k-linear duals and (hence) preserves colimits and totalisations of cosimplicial objects in $k[\mathcal{O}_\Sigma]$.

Definition 2.10. A connective derived symmetric sequence $X\in \mathrm{sSeq}_k^{\mathrm{gen,cn}}$ is said to be *(almost) perfect* if it defines an (almost) compact object in the ∞-category $\mathrm{sSeq}_k^{\mathrm{gen,cn}}$. Equivalently, X can be written as the geometric realisation of a simplicial object in $k[\mathcal{O}_\Sigma]$, cf. [11, Lemma C.6.6.3].

Lemma 2.11 ([4, Proposition 3.75])**.** *Let $\odot$ be an action of composition type. Then:*

(1) The restriction of $(-)^\vee\colon \mathrm{sSeq}_k^{\mathrm{gen},\vee}\longrightarrow \big(\mathrm{sSeq}_k^{\mathrm{gen},\vee}\big)^{\mathrm{op}}$ to the almost perfect connective derived symmetric sequences is fully faithful and strong monoidal with respect to $\circ$.

(2) The equivalence $(-)^\vee\colon \mathrm{Mod}_k^{\mathrm{ft,cn}}\xrightarrow{\sim}\mathrm{Mod}_k^{\mathrm{ft,ccn}}$ is compatible with the action of almost perfect connective derived symmetric sequences, in the following sense: if $X\in\mathrm{sSeq}_k^{\mathrm{gen,cn}}$ is almost perfect and $M\in \mathrm{Mod}_k^{\mathrm{ft,cn}}$, there is a natural equivalence $(X\odot M)^\vee\simeq X^\vee\mathbin{\overline{\odot}}M^\vee$.

Remark 2.12. Let $\mathrm{Mod}_k^{\mathrm{gr}}$ and $\mathrm{Mod}_k^{\mathrm{filt}}=\mathrm{Fun}((\mathbb{Z},\geq),\mathrm{Mod}_k)$ denote the ∞-categories of graded and (decreasingly) filtered k-modules. The derived functor formalism from Definition 2.1 carries over verbatim to these ∞-categories, with the role of Vect_k^ω being played by:

(1) the full subcategory $\mathrm{Vect}_k^{\mathrm{gr},\omega}\subseteq \mathrm{Mod}_k^{\mathrm{gr}}$ of graded vector spaces of finite total dimension.
(2) the full subcategory $\mathrm{Vect}_k^{\mathrm{filt},\omega}\subseteq\mathrm{Mod}_k^{\mathrm{filt}}$ of diagrams F^*V such that $F^iV=0$ for $i\gg 0$, $F^iV=F^{i-1}V$ for $i\ll 0$ and each $F^iV\hookrightarrow F^{i-1}V$ is injective with finite dimensional cokernel.

Using this, any action of composition type extends to an action of $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ on $\mathrm{Mod}_k^{\mathrm{gr}}$ and $\mathrm{Mod}_k^{\mathrm{filt}}$: on the above two subcategories, we declare that acting by $k[\Sigma_r/H]\in k[\mathcal{O}_\Sigma]$ multiplies the (filtration) weight by r and then take derived functors. By construction, these actions of $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$ are preserved by the functors

$$\mathrm{colim}\colon \mathrm{Mod}_k^{\mathrm{filt}}\longrightarrow\mathrm{Mod}_k \qquad \mathrm{gr}\colon\mathrm{Mod}_k^{\mathrm{filt}}\longrightarrow\mathrm{Mod}_k^{\mathrm{gr}}$$

sending a filtered complex F^*V to its colimit and associated graded. Likewise, taking k-linear duals respects the action of almost perfect connective derived symmetric sequences on almost perfect objects in $\mathrm{Mod}_k^{\mathrm{filt}}$ and $\mathrm{Mod}_k^{\mathrm{gr}}$, as in Lemma 2.11. We refer to [8, 12] for a detailed description.

3. Koszul duality

In this section, we will discuss how for a suitable augmented derived operad $\mathcal{P}$, the ∞-category of formal moduli problems on Artin $\mathcal{P}$-algebras is equivalent to the ∞-category of algebras over its Koszul dual (Corollary 3.7); see [5, Section 5.3] for a similar discussion with slightly different hypotheses.

Given an augmented derived operad $\mathcal{P}$, its bar construction $\mathrm{Bar}(\mathcal{P}) = \mathbf{1} \circ_{\mathcal{P}} \mathbf{1}$ has the structure of a co-augmented derived cooperad [10, Section 5.2.2]. One can identify $\mathrm{Bar}(\mathcal{P})$ with the coendomorphism coalgebra of the right $\mathcal{P}$-module $\mathbf{1}$ in $\mathrm{sSeq}_k^{\mathrm{gen},\vee}$. In particular, the unit symmetric sequence $\mathbf{1}$ has a commuting right $\mathcal{P}$-module structure and a left $\mathrm{Bar}(\mathcal{P})$-comodule structure. For any action of composition type, this induces an adjoint pair between algebras and coalgebras

$$\mathrm{Bar}_{\mathcal{P}} \colon \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_k, \odot) \rightleftarrows \mathrm{Coalg}_{\mathrm{Bar}(\mathcal{P})}(\mathrm{Mod}_k, \odot) : \mathrm{Cobar}_{\mathcal{P}}$$

where the left adjoint sends A to $\mathbf{1} \odot_{\mathcal{P}} A = \left|\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{P}, A)\right|$ [4, Section 3.4].

Definition 3.1. Let us say that an augmented derived operad $\mathcal{P}$ is *almost of finite type* if it is connective, $\mathcal{P}(1) = \mathbf{1}$ and $\mathrm{Bar}(\mathcal{P})$ is an almost perfect derived symmetric sequence. It is *of finite type* if in addition, each $\mathrm{Bar}(\mathcal{P})(r)$ is perfect.

Lemma 3.2. *Let $\mathcal{P}$ be almost of finite type. Then then the right $\mathcal{P}$-module $\mathbf{1}$ admits a natural exhaustive filtration $F_*\mathbf{1}$ with associated graded $\mathrm{Gr}_r\mathbf{1} \simeq \mathrm{Bar}(\mathcal{P})(r) \circ \mathcal{P}$. In particular, $\mathbf{1}$ is an almost perfect right $\mathcal{P}$-module.*

Proof. Set $F_0\mathbf{1} = 0$ and proceed by induction as follows. For each $r \geq 1$, let X_r be the arity r part of the fibre $\mathrm{fib}(F_{r-1}\mathbf{1} \to \mathbf{1})(r)$ and define $F_r\mathbf{1}$ to be the cofibre of the map $X_r \circ \mathcal{P} \to F_{r-1}\mathbf{1}$. Each $F_r\mathbf{1} \to \mathbf{1}$ is then an equivalence in arities $\leq r$, so the filtration is exhaustive. Upon applying $- \circ_{\mathcal{P}} \mathbf{1}$, the filtration splits and one finds that $X_r \simeq \mathrm{Bar}(\mathcal{P})(r)$, as desired. In particular, we see that the associated graded consists of almost perfect right $\mathcal{P}$-modules whose connectivity tends to ∞. This implies that the colimit $\mathbf{1}$ is almost perfect as well. □

Lemma 3.3. *Let $\mathcal{P}$ be of finite type and let $A^\bullet \colon \mathbf{\Delta} \longrightarrow \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_k^{\mathrm{gr}}, \odot)$ be a cosimplicial diagram of graded $\mathcal{P}$-algebras (Remark 2.12). Suppose that each A^n is a graded vector space concentrated in strictly positive weights. Then the natural map $\mathrm{Bar}_{\mathcal{P}}(\mathrm{Tot}(A^\bullet)) \longrightarrow \mathrm{Tot}(\mathrm{Bar}_{\mathcal{P}}(A^\bullet))$ is an equivalence.*

Proof. For any graded $\mathcal{P}$-algebra A in strictly positive weights, the map $F_r\mathbf{1} \odot_{\mathcal{P}} B \to \mathrm{Bar}_{\mathcal{P}}(A)$ is an equivalence in weights $\leq r$. It thus suffices to verify that $F_r\mathbf{1} \odot_{\mathcal{P}} - \colon \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_k^{\mathrm{gr}}, \odot) \longrightarrow \mathrm{Mod}_k^{\mathrm{gr}}$ preserves the totalisation of $A^\bullet$ for each r. Note that functors with this property are closed under finite colimits and retracts and that the forgetful functor $\mathcal{P} \odot_{\mathcal{P}} -$ preserves such totalisations. The result then follows from $F_r\mathbf{1}$ being a perfect right $\mathcal{P}$-module, as it admits a finite filtration whose associated graded $\mathrm{Bar}(\mathcal{P})(i) \circ \mathcal{P}$ is perfect for each i. □

Definition 3.4. Let $\mathcal{P}$ be an augmented derived operad that is almost of finite type. For any action of composition type on Mod_k, let us write $\mathrm{Art}_{\mathcal{P},\odot} \subseteq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_k, \odot)$ for the smallest full subcategory of $\mathcal{P}$-algebras such that:

(1) it contains the zero algebra 0.

(2) for each $A \in \mathrm{Art}_{\mathcal{P},\odot}$, $V \in \mathrm{Vect}_k^{\omega}$ and $n \geq 1$, given a pullback square of $\mathcal{P}$-algebras

$$\begin{array}{ccc} B & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{triv}(\Sigma^n V) \end{array} \tag{3.5}$$

where $\mathrm{triv}(\Sigma^n V)$ is the trivial $\mathcal{P}$-algebra (i.e. acted upon via $\mathcal{P} \to \mathbf{1}$), we have that $B \in \mathrm{Art}_{\mathcal{P},\odot}$.

The k-module underlying an Artin $\mathcal{P}$-algebra is contained in $\mathrm{Mod}_k^{\mathrm{ft,cn}}$. Lemma 3.2 therefore implies that $\mathrm{Bar}_{\mathcal{P}}(A)$ is concentrated in $\mathrm{Mod}_k^{\mathrm{ft,cn}}$ as well. Let us now pass to the k-linear dual picture and consider the Koszul dual derived operad $\mathrm{KD}(\mathcal{P}) = \mathrm{Bar}(\mathcal{P})^{\vee}$. By Lemma 2.11, linear duality yields a (contravariant) equivalence between $\mathrm{Bar}(\mathcal{P})$-coalgebras in $\mathrm{Mod}_k^{\mathrm{ft,cn}}$ and $\mathrm{KD}(\mathcal{P})$-algebras in $\mathrm{Mod}_k^{\mathrm{ft,ccn}}$ with respect to the $\overline{\odot}$-action. In particular, we obtain a functor

$$\mathrm{KD}_{\mathcal{P}}\colon \mathrm{Art}_{\mathcal{P},\odot} \xrightarrow{\mathrm{Bar}_{\mathcal{P}}} \mathrm{CoAlg}_{\mathrm{Bar}(\mathcal{P})}(\mathrm{Mod}_k^{\mathrm{ft,cn}}, \odot) \xhookrightarrow{(-)^{\vee}} \mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{Mod}_k, \overline{\odot}\,\big)^{\mathrm{op}}.$$

The second functor is fully faithful, with essential image given by the $\mathrm{KD}(\mathcal{P})$-algebras in $\mathrm{Mod}_k^{\mathrm{ft,ccn}}$.

Theorem 3.6. *Let $\mathcal{P}$ be an augmented derived operad that is almost of finite type. Then $\mathrm{KD}_{\mathcal{P}}\colon \mathrm{Art}_{\mathcal{P},\odot} \longrightarrow \mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}(\mathrm{Mod}_k, \overline{\odot})^{\mathrm{op}}$ is fully faithful and preserves all pullbacks of the form (3.5).*

Proof. This type of result is well known, see e.g. [5, Section 5.3]. To see that $\mathrm{KD}_{\mathcal{P}}$ preserves pullbacks of the form (3.5), let us say that an Artin $\mathcal{P}$-algebra is good if the natural map of $\mathrm{KD}(\mathcal{P})$-algebras

$$\mathrm{KD}_{\mathcal{P}}(A) \amalg \mathrm{KD}_{\mathcal{P}}\big(\mathrm{triv}(\Sigma^m W)\big) \longrightarrow \mathrm{KD}_{\mathcal{P}}(A \times \mathrm{triv}(\Sigma^m W))$$

is an equivalence for all $m \geq 0$ and $W \in \mathrm{Vect}_k^{\omega}$. Note that $\mathrm{KD}_{\mathcal{P}}(\mathrm{triv}(M)) \simeq \mathrm{KD}_{\mathcal{P}} \,\overline{\odot}\, M^{\vee}$ is the free $\mathrm{KD}(\mathcal{P})$-algebra on $M^{\vee}$ for any $M \in \mathrm{Mod}_k^{\mathrm{ft,cn}}$. This implies that $\mathrm{triv}(M)$ is good.

We now claim the following: given a pullback square (3.5) such that A is good, then (a) its image under $\mathrm{KD}_{\mathcal{P}}$ is a pushout square and (b) B is good. Assuming this, it follows by induction that $\mathrm{KD}_{\mathcal{P}}$ preserves all pullbacks of the form (3.5).

To prove claim (a) and (b), we will endow the objects involved with a decreasing filtration, as follows. We put A and $\mathrm{triv}(\Sigma^m W)$ in filtration weight 0 (i.e. $F^1 = 0$), while $\mathrm{triv}(\Sigma^n V)$ is put in filtration weight 1 ($F^2 = 0$). Taking the fibre product then yields a filtration on $B \times \mathrm{triv}(\Sigma^m W)$ with $F^1 = \Sigma^{n-1} V$ and $F^2 = 0$.

Let R be any of the $\mathcal{P}$-algebras from the previous paragraph and let F^*R be its (finite) filtration. By Remark 2.12, the functors $\mathrm{Bar}_{\mathcal{P}}$ and $\mathrm{KD}_{\mathcal{P}}$ have natural analogues in the filtered and graded setting. Since F^*R defines an almost perfect connective object in $\mathrm{Mod}_k^{\mathrm{filt}}$, Lemma 3.2 implies that the same holds for $\mathrm{Bar}_{\mathcal{P}}(F^*R)$. Consequently, $\mathrm{Bar}_{\mathcal{P}}(F^*R)$ arises as the geometric realisation of a simplicial object in $\mathrm{Vect}_k^{\mathrm{filt},\omega}$ [11, Lemma C.6.6.3]. Taking linear duals, it follows that $\mathrm{KD}_{\mathcal{P}}(F^*R)$ is the totalisation of a cosimplicial diagram in $\mathrm{Vect}_k^{\mathrm{filt},\omega}$. This implies that $\mathrm{KD}_{\mathcal{P}}(F^*R)$ gives an exhaustive filtration on $\mathrm{KD}_{\mathcal{P}}(R)$, whose associated graded can be identified with $\mathrm{KD}_{\mathcal{P}}(\mathrm{gr}(F^*R))$.

We thus obtain natural increasing filtrations on the Koszul duals of all $\mathcal{P}$-algebras involved in (3.5). It now suffices to verify claim (a) and (b) at the level of the associated graded, i.e. we need to check that the following maps are equivalences:

$$\mathrm{KD}_{\mathcal{P}}(A) \amalg_{\mathrm{KD}_{\mathcal{P}}(\mathrm{triv}(\Sigma^n V))} 0 \longrightarrow \mathrm{KD}_{\mathcal{P}}(\mathrm{gr}(B))$$

$$\mathrm{KD}_{\mathcal{P}}\big(\mathrm{gr}(B)\big) \amalg \mathrm{Bar}_{\mathcal{P}}\big(\mathrm{triv}(\Sigma^m W)\big) \longrightarrow \mathrm{KD}_{\mathcal{P}}\big(\mathrm{gr}(B) \times \mathrm{triv}(\Sigma^m W)\big)$$

Here A and $\mathrm{triv}(\Sigma^m W)$ are concentrated in weight 0 and $\mathrm{triv}(\Sigma^n V)$ is in weight 1. The above maps are then equivalences because $\mathrm{gr}(B) \simeq A \times \mathrm{triv}(\Sigma^{n-1} V)$ and A was good by assumption.

Finally, we need to verify that $\mathrm{Map}_{\mathcal{P}}(A, B) \to \mathrm{Map}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{KD}_{\mathcal{P}}(B), \mathrm{KD}_{\mathcal{P}}(A)\big)$ is an equivalence for all $A, B \in \mathrm{Art}_{\mathcal{P},\odot}$. Since $\mathrm{KD}_{\mathcal{P}}$ sends pullbacks to pushouts of $\mathrm{KD}(\mathcal{P})$-algebras, we can reduce to the case where $B = \mathrm{triv}(\Sigma^n V)$ for some vector space V. In this case, $\mathrm{KD}_{\mathcal{P}}(B) = \mathrm{KD}(\mathcal{P}) \,\overline{\odot}\, \Sigma^{-n} V^\vee$ is free and the map can be identified with the composite

$$\mathrm{Map}_{\mathcal{P}}(A, \mathrm{triv}(\Sigma^n V)) \longrightarrow \mathrm{Map}_k\big(\mathrm{Bar}_{\mathcal{P}}(A), \Sigma^n V\big) \longrightarrow \mathrm{Map}_k\big(\Sigma^{-n} V^\vee, \mathrm{KD}_{\mathcal{P}}(A)\big).$$

This is an equivalence by definition of the bar construction and $\Sigma^n V$ being dualisable. □

Corollary 3.7. *Let $\mathcal{P}$ be almost of finite type and let $\odot$ be an action of composition type. Then there is an equivalence between the category $\mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{Mod}_k, \overline{\odot}\big)$ and the category of* formal moduli problems *on $\mathrm{Art}_{\mathcal{P},\odot}$, that is, functors $\mathrm{Art}_{\mathcal{P},\odot} \longrightarrow \mathcal{S}$ preserving the terminal object and all pullback squares* (3.5)*.*

Proof. Extending $\mathrm{KD}_{\mathcal{P}} \colon \mathrm{Art}_{\mathcal{P},\odot} \hookrightarrow \mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{Mod}_k, \overline{\odot}\big)^{\mathrm{op}}$ freely by colimits and taking opposites, we obtain an adjoint pair $\mathfrak{D}^* \colon \mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{Mod}_k, \overline{\odot}\big) \leftrightarrows \mathcal{P}(\mathrm{Art}_{\mathcal{P},\odot})^{\mathrm{op}} \colon \mathfrak{D}_*$. By Theorem 3.6 and the fact that $\mathrm{oblv} \colon \mathrm{Alg}_{\mathrm{KD}(\mathcal{P})}\big(\mathrm{Mod}_k, \overline{\odot}\big) \longrightarrow \mathrm{Mod}_k$ preserves sifted colimits, $(\mathfrak{D}^*, \mathfrak{D}_*)$ defines a deformation theory in the sense of [11, Definition 12.3.3.2]. The result then follows from [11, Theorem 12.3.3.5]. □

4. Derived (restricted) Lie algebras

We will now spell out the results of the previous section in the particular case of the derived (non-unital) commutative operad, whose Koszul dual will be denoted

$$\mathrm{Lie}^{\pi}_{\Delta} := \mathrm{KD}(\mathrm{Com}).$$

The three actions of the derived commutative operad on Mod_k (Example 2.9) give rise to three categories of formal moduli problems

$$\mathrm{FMP}_k \xrightarrow{\mathrm{oblv}} \mathrm{FMP}^{\mathrm{tr}}_k \xrightarrow{\mathrm{oblv}} \mathrm{FMP}^{\mathrm{pd}}_k$$

defined on Artin augmented k-algebras, truncated k-algebras and divided power algebras, respectively. In the last case, note that if an (augmented) divided power algebra A is Artin in the sense of Definition 3.4, then $\pi_0(\mathfrak{m}_A)$ is also nilpotent with respect to its divided power structure. We will apply Corollary 3.7 in this setting:

Lemma 4.1. *Let k be a field of characteristic $p > 0$. Then the derived k-linear operad* Com *is of finite type.*

Proof. We need to verify that each $\mathrm{Bar}(\mathrm{Com})(r)$ is perfect and that its connectivity tends to ∞ if $r \to \infty$. To see this, recall that $\mathrm{Bar}(\mathrm{Com})(r) \simeq k[T_r]$ arises as the k-linearisation of the reduced-unreduced suspension of the r-th partition complex. Since T_r has finitely many non-degenerate simplices, each $k[T_r]$ is a perfect derived symmetric sequence.

For the connectivity statement, let $T_{r,\mathcal{F}} \to T_r$ be the counit map for restriction and left Kan extension along the inclusion $\mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}_{\Sigma_r}$, where $\mathcal{F}$ is the family of p-subgroups. Then the map $k[T_{r,\mathcal{F}}] \longrightarrow k[T_r]$ is an equivalence [3, Proposition 4.6]. It hence suffices to verify that T_r becomes increasingly connective as a genuine $\mathcal{F}$-equivariant space, which follows from [2, Corollary 6.8]. □

Corollary 4.2. *There are equivalences of ∞-categories*

$$\begin{array}{ccccc} \mathrm{FMP}_k & \xrightarrow{\mathrm{oblv}} & \mathrm{FMP}_k^{\mathrm{tr}} & \xrightarrow{\mathrm{oblv}} & \mathrm{FMP}_k^{\mathrm{pd}} \\ {\scriptstyle T}\downarrow\wr & & {\scriptstyle T}\downarrow\wr & & \wr\downarrow{\scriptstyle T} \\ \mathrm{Alg}_{\mathrm{Lie}^\pi_\Delta}(\mathrm{Mod}_k, \overline{\circ}) & \xrightarrow{\mathrm{oblv}} & \mathrm{Alg}_{\mathrm{Lie}^\pi_\Delta}(\mathrm{Mod}_k, \circ^{(1)}) & \xrightarrow{\mathrm{oblv}} & \mathrm{Alg}_{\mathrm{Lie}^\pi_\Delta}(\mathrm{Mod}_k, \circ). \end{array}$$

The ∞-category of Lie^π_Δ-algebras with respect to the action $\overline{\circ}$ is precisely the ∞-category of *partition Lie algebras* from [5]. Our main goal is to identify the other two types of Lie^π_Δ-algebras with derived (restricted) Lie algebras (as defined in Example 2.3):

Theorem 4.3. *There are equivalences of ∞-categories, acting as the desuspension on the underlying complex*

$$\mathrm{Alg}_{\mathrm{Lie}^\pi_\Delta}(\mathrm{Mod}_k, \circ) \xrightarrow{\Sigma^{-1}} \mathrm{DLie} \qquad\qquad \mathrm{Alg}_{\mathrm{Lie}^\pi_\Delta}(\mathrm{Mod}_k, \circ^{(1)}) \xrightarrow{\Sigma^{-1}} \mathrm{DLie}^{\mathrm{res}}.$$

The main idea will be to show that the monads $\mathrm{Lie}^\pi_\Delta \circ (-)$ and $\mathrm{Lie}^\pi_\Delta \circ^{(1)} (-)$ send the suspension of a vector space V to the suspension of the free (restricted) Lie algebra on V. To do this, let us start by recalling some facts about the classical (restricted) Lie monad. Let us write $\overline{T}^\bullet \subseteq T^\bullet$ for the usual (non-unital, resp. unital) associative monad on k-vector spaces. For any vector space V, there are inclusions

$$\mathrm{Lie}(V) \hookrightarrow \mathrm{Lie}^{\mathrm{res}}(V) \hookrightarrow T^\bullet(V)$$

exhibiting the free (restricted) Lie algebra as the smallest subspace of the tensor algebra that contains V and is closed under the commutator bracket (and p-th powers). Using this, we can identify the (restricted) Lie monad as a sub-monad of the associative monad:

Lemma 4.4. *Let $\mathcal{L} \hookrightarrow T^\bullet$ be a sub-monad on Vect_k that preserves filtered colimits. For any $V \in \mathrm{Vect}_k^\omega$ and $r \geq 0$, let us write $\mathcal{L}(V)_{(r)} = \mathcal{L}(V) \times_{T^\bullet(V)} V^{\otimes r}$. If*

$$\dim(\mathcal{L}(V)_{(r)}) = \dim(\mathrm{Lie}(V)_{(r)}) \qquad \textit{resp.} \qquad \dim(\mathcal{L}(V)_{(r)}) = \dim(\mathrm{Lie}^{\mathrm{res}}(V)_{(r)}),$$

for each $r \geq 1$, then $\mathcal{L} = \mathrm{Lie}$ or $\mathcal{L} = \mathrm{Lie}^{\mathrm{res}}$, respectively.

Proof. It suffices to verify that $\mathcal{L}(V) \subseteq T^\bullet(V)$ is closed under the commutator brackets (and p-th powers); this will yield an inclusion $\mathrm{Lie}^{(\mathrm{res})} \subseteq \mathcal{L}$ which is an isomorphism for dimension reasons. Looking at $\mathcal{L}(k^{\oplus 2})$ and using naturality with respect to the projection and sum maps $k^{\oplus 2} \longrightarrow k$, one sees that $\mathcal{L}(k^{\oplus 2})$ contains the element $x \otimes y - y \otimes x$. By naturality and the monad structure on $\mathcal{L}$, this implies that $\mathcal{L}(V)$ is closed under the commutator bracket. Similarly, $\mathcal{L}_{(p^n)}(k)$ being one-dimensional implies that $\mathcal{L}(V)$ is closed under p-th powers. □

Remark 4.5. If V is of dimension n, the dimensions appearing above have well-known descriptions in terms of words on n letters: the dimension $\dim(\mathrm{Lie}(V)_{(r)})$ is the number of Lyndon words of length r and $\dim(\mathrm{Lie}^{\mathrm{res}}(V)_{(r)})$ is the number of words of length r of the form w^{p^k}, where w is a Lyndon word [15].

To apply Lemma 4.4, let us consider the map $\mathrm{Lie}^\pi_\Delta = \mathrm{KD}(\mathrm{Com}) \longrightarrow \mathrm{KD}(\mathrm{Ass})$ that is Koszul dual to the usual map $\mathrm{Ass} \longrightarrow \mathrm{Com}$ from the derived (non-unital) associative operad to the derived commutative operad. Since the associative operad is Σ-free, it has the same algebras with respect to $\circ, \overline{\circ}$ and $\circ^{(1)}$, and its Koszul dual is the usual shifted associative operad [9]. In other words, $\mathrm{KD}(\mathrm{Ass})$ restricts to the usual (non-unital) associative monad

$$\overline{T}^\bullet\colon \mathrm{Vect}_k \xrightarrow{\Sigma} \Sigma\mathrm{Vect}_k \xrightarrow{\mathrm{KD}(\mathrm{Ass})} \Sigma\mathrm{Vect}_k \xrightarrow{\Sigma^{-1}} \mathrm{Vect}_k.$$

For every vector space V, we thus obtain a natural map of k-modules $\Sigma^{-1}\mathrm{Lie}^\pi_\Delta \circ \Sigma V \longrightarrow \overline{T}^\bullet V$, whose pullback along $V^{\otimes r} \to \overline{T}^\bullet V$ coincides with $\Sigma^{-1}\mathrm{Lie}^\pi_\Delta(r) \circ \Sigma V$.

Proposition 4.6. *Let $V \in \mathrm{Vect}_k^{\omega}$. For each $r \geq 1$, the maps*

$$\mathrm{Lie}^{\pi}_{\Delta}(r) \circ \Sigma V \longrightarrow \Sigma(V^{\otimes r}) \qquad\qquad \mathrm{Lie}^{\pi}_{\Delta}(r) \circ^{(1)} \Sigma V \longrightarrow \Sigma(V^{\otimes r}) \tag{4.7}$$

are inclusions of linear subspaces, whose dimensions coincide with $\dim(\mathrm{Lie}(V)_{(r)})$ *and* $\dim(\mathrm{Lie}^{\mathrm{res}}(V)_{(r)})$.

Proof. We will just treat the case of $\mathrm{Lie}^{\pi}_{\Delta}(r) \circ \Sigma V$; the case of $\mathrm{Lie}^{\pi}_{\Delta}(r) \circ^{(1)} \Sigma V$ is similar and we will indicate the necessary changes in Remark 4.14. Since each $\mathrm{Lie}^{\pi}_{\Delta}(r)$ is a perfect derived symmetric sequence (Lemma 4.1), the domain of (4.7) is an almost perfect k-module. It therefore suffices to show that the k-linear dual map

$$\Sigma^{-1}V^{\otimes r} \simeq \mathrm{Bar}(\mathrm{Ass})(r) \,\overline{\circ}\, \Sigma^{-1}V \longrightarrow \mathrm{Bar}(\mathrm{Com})(r) \,\overline{\circ}\, \Sigma^{-1}V \tag{4.8}$$

is a surjective map between desuspended vector spaces, with codomain of the correct dimension.

To identify this map, let us work throughout in the positively graded setting from Remark 2.12. For psychological reasons, we will furthermore work with the augmented versions of associative and divided power algebras, rather than the non-unital versions that fit into the operadic formalism. Throughout, we consider V as a graded vector space concentrated in weight 1 and write $\mathrm{triv}(\Sigma^{-1}V) = k \oplus \Sigma^{-1}V$ for the trivial graded algebra, with k in weight 0 and $\Sigma^{-1}V$ in weight 1. The map (4.8) is then precisely the weight r part of the natural comparison map $\varphi\colon \mathrm{Bar}_{\mathrm{Ass}}(\mathrm{triv}(\Sigma^{-1}V)) \longrightarrow \mathrm{Bar}_{\mathrm{Com}}(\mathrm{triv}(\Sigma^{-1}V))$ between the bar constructions of the trivial algebra, viewed as an associative and as a divided power algebra, respectively.

We will use explicit resolutions to show that φ is a surjective map of desuspended graded vector spaces. To this end, let us consider the classical (graded) *shuffle Hopf algebra*

$$H(V) = \bigoplus_{r \geq 0} V^{\otimes r}.$$

This is a commutative Hopf algebra with divided powers [1], whose coproduct is given by de-concatenation and whose product is the shuffle product (together with its natural system of divided powers). The graded dual of $H(V)$ is the tensor algebra $T^{\bullet}(V^{\vee})$, so one obtains a natural surjection of graded vector spaces (finite dimensional in each weight)

$$H(V) = T^{\bullet}(V^{\vee})^{\vee} \twoheadrightarrow \mathrm{Lie}(V^{\vee})^{\vee} =: \mathrm{coLie}(V). \tag{4.9}$$

We now recall that $H(V)$ is free as a divided power algebra and that (4.9) is the natural projection onto its indecomposables. The second assertion follows from [1, Proposition 8, 9 and 21], and [14, Theorem 1] then shows that any section of (4.9) induces an isomorphism $\Gamma(\mathrm{coLie}(V)) \cong H(V)$ of divided power algebras. Alternatively, one can prove this (over $\mathbb{Z}$) using the Lyndon basis [13, Theorem 5.3].

Let us now consider the cosimplicial cobar construction of the coalgebra $H(V)$. This comes with a natural map of diagrams of graded vector spaces

$$\begin{array}{ccccccc} k & \rightrightarrows & H(V) & \rightrightarrows & H(V) \otimes H(V) & \rightrightarrows & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ k & \rightrightarrows & \mathrm{triv}(V) & \rightrightarrows & \mathrm{triv}(V \oplus V) & \rightrightarrows & \cdots \end{array} \tag{4.10}$$

sending all higher tensor powers of V to 0. Note that (4.10) is an isomorphism in weights 0 and 1 and that the cobar construction of $H(V)$ is acyclic in weights ≥ 2 because $H(V)$ is a *cofree* coalgebra. Consequently, (4.10) induces an equivalence on totalisations. Using that $H(V)$ is a divided power Hopf algebra, (4.10) is a diagram of divided power algebras, where the bottom row consists of trivial divided power algebras [1, Lemma 22]. We conclude that the top row of (4.10) provides a cosimplicial resolution of $\mathrm{triv}(\Sigma^{-1}V)$ by free divided power algebras.

Next, let us regard the diagram (4.10) not as a diagram of divided power algebras, but of augmented associative algebras. Of course, in this case the top row no longer provides a resolution of $\mathrm{triv}(\Sigma^{-1}V)$ by free algebras. Instead, let us observe that as a diagram of associative algebras, (4.10) can be extended naturally by

$$\begin{array}{ccccccc} k & \rightrightarrows & T^\bullet(\overline{H}(V)) & \rightrightarrows & T^\bullet\big(\overline{H}(V)\oplus\overline{H}(V)\big) & \rightrightarrows & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ k & \rightrightarrows & H(V) & \rightrightarrows & H(V)\otimes H(V) & \rightrightarrows & \cdots \end{array} \tag{4.11}$$

Here we write $\overline{H}(V)$ for the augmentation ideal of the Hopf algebra $H(V)$ and the vertical maps are given on generators by sending the i-th summand $\overline{H}(V)$ to the augmentation ideal of the i-th factor $H(V)$. In the top row, each $\alpha\colon [m]\to[n]$ in $\mathbf{\Delta}$ induces a map of tensor algebras

$$\alpha_*\colon T^\bullet\big(\overline{H}(V)^{\oplus m}\big) \longrightarrow T^\bullet\big(\overline{H}(V)^{\oplus n}\big)$$

as follows. Let us write $(v_1\dots v_k)^{(i)}$ for an element from the i-th summand $\overline{H}(V)\subseteq\overline{H}(V)^{\oplus m}$. Then $\alpha_*(v_1\dots v_k)^{(i)}$ is given by

$$\sum_{0\leq j_1\leq\dots\leq j_t\leq k} \big(v_1\dots v_{j_1}\big)^{(\alpha(i))}\otimes\big(v_{j_1+1}\dots v_{j_2}\big)^{(\alpha(i)+1)}\otimes\dots\otimes\big(v_{j_t+1}\dots v_k\big)^{(\alpha(i+1)-1)}$$

where $t=\alpha(i+1)-\alpha(i)$ and $\otimes$ denotes the product in the tensor algebra. The composite map from the top row of (4.11) to the bottom row of (4.10) induces an isomorphism in weights ≤ 1, and the top row of (4.11) is acyclic in weights ≥ 2 because it computes the cobar construction of the cofree coalgebra $H(V)$ as well. Consequently, the top row of (4.11) provides a cosimplicial resolution of $\mathrm{triv}(\Sigma^{-1}V)$ by free graded (augmented) associative algebras.

We will use these resolutions to compute the associative and commutative bar construction of the trivial graded algebra $\mathrm{triv}(\Sigma^{-1}V)$. More precisely, taking bar constructions we obtain the following diagram of k-modules

$$\begin{array}{ccccccccc} \mathrm{Bar}_{\mathrm{Ass}}(\mathrm{triv}(\Sigma^{-1}V)) & \longrightarrow & 0 & \rightrightarrows & \overline{H}(V) & \rightrightarrows & \overline{H}(V)^{\oplus 2} & \rightrightarrows & \cdots \\ \varphi\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Bar}_{\mathrm{Com}}(\mathrm{triv}(\Sigma^{-1}V)) & \longrightarrow & 0 & \rightrightarrows & \mathrm{coLie}(V) & \rightrightarrows & \mathrm{coLie}(V)^{\oplus 2} & \rightrightarrows & \cdots. \end{array} \tag{4.12}$$

Here the vertical maps arise from (4.11) and are given by the projection $\overline{H}(V)\to\mathrm{coLie}(V)$. The rows of (4.12) are limit diagrams in Mod_k by Lemma 3.3 and Lemma 4.1 (and its easier analogue for the associative operad). The map φ can thus be identified with the desuspension of the map (4.9) and the result follows. □

Remark 4.13 (Cobar constructions)**.** Let us briefly comment on the claims about the acyclicity of the cobar construction of the (cofree) graded coalgebra $H(V)$ appearing in the proof of Proposition 4.6. Recall that for any graded coaugmented coalgebra C, corestriction along the coaugmentation $k\longrightarrow C$ has a right adjoint functor

$$k\otimes^C(-)\colon \mathrm{Comod}_C(\mathrm{Mod}_k^{\mathrm{gr}})\longrightarrow\mathrm{Mod}_k^{\mathrm{gr}}.$$

By definition, this sends each cofree comodule $C\otimes W$ to W. The cobar construction of C is the value of this functor on the trivial comodule k, and can be computed in various ways:

(1) The trivial comodule arises as the totalisation of the usual cobar resolution $B^\bullet(C,C,k)$, because its image in Mod_k admits extra codegeneracies. Since $k\otimes^C(-)$ is a right adjoint, we have $k\otimes^C k\simeq\mathrm{Tot}\big(B^\bullet(k,C,k)\big)$. When $C=H(V)$, this is precisely the bottom row of (4.11).

(2) The trivial comodule arises as the totalisation of a cosimplicial diagram of cofree C-comodules of the form

$$C \rightrightarrows C \otimes T^\bullet(\overline{C}) \rightrightarrows C \otimes T^\bullet(\overline{C}^{\oplus 2}) \rightrightarrows \dots$$

Let us denote an element in cosimplicial degree n by $c^{(0)} \otimes c_1^{(i(1))} \dots c_t^{(i(t))}$, where $c^{(0)}$ is contained in the tensor factor of $C = k \cdot 1 \oplus \overline{C}$ and each $c_s^{(i(s))}$ denotes an element in the $i(s)$-th copy of the augmentation coideal $\overline{C}$, for $1 \leq i(s) \leq n$. The coface and codegeneracy maps act on each of the factors of such an element as follows (and are extended multiplicatively):

(a) σ_i: we send each $c^{(j)}$ to itself if $j \leq i$, to zero if $j = i+1$ and to $c^{(j-1)}$ if $j > i+1$.

(b) δ_i for $i > 0$: for $c \in \overline{C}$, let us denote its coproduct by

$$\Delta(c) = 1 \otimes c + c \otimes 1 + \sum_\alpha a_\alpha \otimes b_\alpha \qquad \text{where } a_\alpha, b_\alpha \in \overline{C}.$$

Then we send $c^{(j)}$ to itself if $j < i$, to $c^{(j+1)}$ if $j > i$ and to $c^{(i+1)} + c^{(i)} + \sum a_\alpha^{(i)} \otimes b_\alpha^{(i+1)}$ if $j = i$.

(c) δ_0: we send each factor $c^{(i)}$ with $i > 0$ to $c^{(i+1)}$ and $c^{(0)}$ with $c \in \overline{C}$ to $1^{(0)} \otimes c^{(1)} + c^{(0)} + \sum_\alpha a_\alpha^{(0)} \otimes b_\alpha^{(1)}$. Finally, we send $1^{(0)}$ to $1^{(0)}$.

(d) At the level of k-vector spaces, this admits an extra codegeneracy σ_{-1}: this sends

$$1^{(0)} \otimes c^{(1)} c_2^{(i(2))} \dots c_t^{(i(t))} \longmapsto c^{(0)} \otimes c_2^{(i(2)-1)} \dots c_t^{(i(t)-1)}$$
$$1^{(0)} \otimes c_1^{(i(1))} \dots c_t^{(i(t))} \longmapsto 1^{(0)} \otimes c_1^{(i(1)-1)} \dots c_t^{(i(t))}$$

in the case where all $i(s) \geq 2$. All other elements (for example, starting with $c^{(0)}$ where $c \in \overline{C}$ or containing multiple factors $c_i^{(1)}$) are sent to 0.

The functor $k \otimes^C (-)$ therefore sends this to a cosimplicial diagram of tensor algebras $T^\bullet(\overline{C}^{\oplus \bullet})$ whose totalisation is $k \otimes^C k$. When $C = H(V)$, this is precisely the top row of (4.11).

(3) Finally, suppose that $C = H(V) = \bigoplus V^{\otimes n}$ is the *cofree* graded coalgebra on V. Then the trivial module arises as the fibre of the map of cofree $H(V)$-comodules $H(V) \to H(V) \otimes V$ sending $v_1 \dots v_n$ to $(v_1 \dots v_{n-1}) \otimes v_n$. Using this, one finds that $k \otimes^{H(V)} k \simeq k \oplus \Sigma^{-1} V$. In particular, it is acyclic in weights ≥ 2.

Remark 4.14. For the case of restricted Lie algebras, we apply the same argument, but viewing $H(V)$ as a truncated commutative algebra instead (i.e. we forget the divided power operations). Note that the free divided power algebra $H(V)$ (generated by x_i) is also free as a truncated polynomial ring (generated by $\gamma_{p^n}(x_i)$). The comparison map between the bar constructions of $\mathrm{triv}(\Sigma^{-1}V)$ as an associative and a truncated commutative algebra can then be identified with the desuspension of a surjective map of graded vector spaces $\overline{T}^\bullet(V) \longrightarrow \mathrm{coLie}^{\mathrm{res}}(V)$, whose codomain has the correct dimension in each weight by Remark 4.5.

Proof of Theorem 4.3. Let us consider the monad $\mathcal{L}\colon \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$ given by either $\mathcal{L}(V) = \Sigma^{-1}\mathrm{Lie}^\pi_\Delta \circ \Sigma V$ or $\mathcal{L}(V) = \Sigma^{-1}\mathrm{Lie}^\pi_\Delta \circ^{(1)} \Sigma V$. We need to show that this coincides with the derived (restricted) Lie algebra monad. Lemma 4.4 and Proposition 4.6 imply that $\mathcal{L}$ restricts to the usual (restricted) Lie algebra monad on Vect_k. It then suffices to observe that $\mathcal{L}$ preserves sifted colimits and totalisations of m-coskeletal diagrams of vector spaces (see Definition 2.1). The first assertion follows from $\mathrm{Lie}^\pi_\Delta \circ (-)$ preserving sifted colimits, and the second from $\mathcal{L}$ being a direct sum of the r-excisive functors $\Sigma^{-1}\mathrm{Lie}^\pi_\Delta(r) \circ \Sigma(-)$ [5, Proposition 3.37]. □

5. Frobenius neighbourhoods

From a geometric point of view, one can think of a formal moduli problem defined on Artin augmented k-algebras as a pointed stack (X, x) over k describing a formal neighbourhood of its basepoint. The restriction of X to Artin truncated k-algebras can then be viewed as encoding the Frobenius neighbourhood of the basepoint, i.e. the fibre of the relative Frobenius $\phi\colon X \to X^{(1)}$ over x. The aim of this section is to flesh out this idea in the case where k is a perfect field. Let us start with a categorical description of the process of trivialising the relative Frobenius.

Definition 5.1. Let us refer to an action of the monoidal category $(\mathbb{Z}, \leq, \otimes = +)$ on an ∞-category $\mathcal{C}$ as a $\mathbb{Z}$-*twisting*. We will write $(-)^{(1)}\colon \mathcal{C} \longrightarrow \mathcal{C}$ for the automorphism associated to $1 \in \mathbb{Z}$, with inverse $(-)^{(-1)}$, and $\phi\colon \mathrm{id}_{\mathcal{C}} \to (-)^{(1)}$ for the natural transformation associated to $0 \leq 1$. A $\mathbb{Z}^{\mathrm{op}}$-twisting is defined similarly.

Lemma 5.2. *Let $\mathcal{C}$ be a pointed $\mathbb{Z}$-twisted ∞-category with finite limits (colimits). Then the endofunctor*

$$\mathrm{fib}_\phi = \mathrm{fib}(\phi\colon \mathrm{id}_{\mathcal{C}} \to (-)^{(1)}) \qquad \textit{resp.} \qquad \mathrm{cofib}_\phi = \mathrm{cofib}(\phi\colon (-)^{(-1)} \to \mathrm{id}_{\mathcal{C}})$$

has the natural structure of a comonad (monad).

If $\mathcal{C}$ has finite limits and colimits, then the comonad fib_ϕ is right adjoint to the monad cofib_ϕ.

Proof. We will only check the case of cofib_ϕ, the other case follows by taking opposites. The action of $(\mathbb{Z}, \leq)$ on $\mathcal{C}$ extends to an action of $\mathcal{P}(\mathbb{Z}, \leq)^\omega_*$ on $\mathcal{C}$ that preserves finite colimits in each variable; here $\mathcal{P}(\mathbb{Z}, \leq)^\omega_*$ is the category of compact objects in pointed presheaves, with the Day convolution product. In $\mathcal{P}(\mathbb{Z}, \leq)^\omega_*$, the cofibre of the map of (pointed) representable presheaves $(h_{-1})_+ \to (h_0)_+$ is the presheaf A whose value is S^0 at 0 and $*$ for all $i \in \mathbb{Z} \setminus \{0\}$. One readily verifies that A has a unique unital algebra structure. □

Definition 5.3. Let $\mathcal{C}$ be a pointed ∞-category with finite limits and colimits equipped with a $\mathbb{Z}$-twisting. We will write $\mathcal{C}^{\phi=0}$ for the ∞-category of algebras over the monad cofib_ϕ. The forgetful functor $\mathrm{oblv}\colon \mathcal{C}^{\phi=0} \longrightarrow \mathcal{C}$ admits a left and right adjoint denoted (abusively) cofib_ϕ and fib_ϕ; the compositions of these functors with oblv are indeed the functors from Lemma 5.2.

Example 5.4. Let k be a perfect field. The category Vect_k^ω carries a $\mathbb{Z}^{\mathrm{op}}$-twisting where $V^{(n)} = (\phi^n)^*V$ and each $V^{(n+1)} \to V^{(n)}$ is zero. Taking derived functors, this extends to a $\mathbb{Z}$-twisting of Mod_k. The Schwede–Shipley theorem [10, Theorem 7.1.2.1] identifies $\mathrm{oblv}\colon \mathrm{Mod}_k^{\phi=0} \to \mathrm{Mod}_k$ with the forgetful functor $\mathrm{Mod}_{k[\eta]} \to \mathrm{Mod}_k$, where $k[\eta] = \mathrm{LSym}_k(\Sigma k^{(1)}) = \mathrm{triv}(\Sigma k^{(1)})$ denotes the free (equivalently, square zero) algebra generated by $\Sigma k^{(1)}$.

Example 5.5. The category Poly_k of finite type free augmented k-algebras carries a $\mathbb{Z}^{\mathrm{op}}$-twisting, where $A^{(n)} = (\phi^n)^*A$ and $\phi\colon A^{(n+1)} \to A^{(n)}$ is the relative Frobenius. Taking derived functors, this induces a $\mathbb{Z}^{\mathrm{op}}$-twisting on the ∞-category $\mathrm{DAlg}_k^{\mathrm{cn}}$ of connective derived augmented k-algebras. The composite

$$\mathrm{DAlg}_k^{\mathrm{cn},\phi=0} \to \mathrm{DAlg}_k^{\mathrm{cn}} \to \mathrm{Mod}_k^{\mathrm{cn}}$$

exhibits the domain as the category of algebras over a sifted colimit-preserving monad. This monad restricts to the $\mathrm{Sym}^{\mathrm{tr}}$-monad on Vect_k, as $\mathrm{Sym}^{\mathrm{tr}}(V) \cong k \otimes^\phi_{\mathrm{Sym}(V)} \mathrm{Sym}(V)$. We therefore obtain

$$\mathrm{DAlg}_k^{\mathrm{cn},\phi=0} \simeq \mathrm{DAlg}_k^{\mathrm{tr,cn}}. \tag{5.6}$$

The $\mathbb{Z}^{\mathrm{op}}$-twisting of $\mathrm{DAlg}_k^{\mathrm{cn}}$ from Example 5.5 restricts to a $\mathbb{Z}^{\mathrm{op}}$-twisting of the full subcategory Art_k. Indeed, each equivalence $(-)^{(n)}$ preserves pullbacks of the form (3.5): the Frobenius twist of a square zero extension $I \to B \to A$ is a square zero extension of $A^{(n)}$ by $(\phi^n)^*I$. This extends uniquely to a $\mathbb{Z}$-twisting of the free sifted cocompletion $\mathcal{P}_\Sigma(\mathrm{Art}_k^{\mathrm{op}})$, as well as its left Bousfield localisation FMP_k at the maps arising from (3.5). Using this, we obtain a diagram

$$\mathcal{O}\colon \mathrm{FMP}_k \rightleftarrows \mathrm{DAlg}_k^{\mathrm{cn,op}} : \mathrm{Spf}$$

of $\mathbb{Z}$-twisted ∞-categories. Here $\mathcal{O}$ is the unique sifted colimit preserving extension of the inclusion $\mathrm{Art}_k^{\mathrm{op}} \hookrightarrow \mathrm{DAlg}_k^{\mathrm{cn,op}}$; it is compatible with the $\mathbb{Z}$-twistings since the $\mathbb{Z}$-twisting on FMP_k is extended from the one on $\mathrm{Art}_k^{\mathrm{op}}$ by sifted colimits. Its right adjoint Spf (automatically compatible with $\mathbb{Z}$-twistings) can be identified with the restricted Yoneda embedding.

Passing to categories of objects with a trivialisation of ϕ and using the equivalence (5.6), we then obtain a fully faithful functor

$$\mathrm{Spf}\colon \mathrm{Art}_k^{\mathrm{tr,op}} \subseteq \mathrm{DAlg}_k^{\mathrm{cn,op}} \longrightarrow \mathrm{FMP}_k^{\phi=0}$$

sending each Artin truncated k-algebra A to the (corepresentable) formal moduli problem $\mathrm{Spf}(A)$, with trivialisation of the Frobenius determined by the truncated algebra structure on A. This sends each pullback square (3.5) to a pushout square in $\mathrm{FMP}_k^{\phi=0}$, since $\mathrm{oblv}\colon \mathrm{FMP}_k^{\phi=0} \longrightarrow \mathrm{FMP}_k$ detects limits and colimits. Consequently, we obtain a commuting diagram of right adjoint functors

$$\mathrm{FMP}_k \underset{\mathrm{fib}_\phi}{\overset{\mathrm{oblv}}{\rightleftarrows}} \mathrm{FMP}_k^{\phi=0} \xrightarrow{\ \Psi\ } \mathrm{FMP}_k^{\mathrm{tr}}.$$

were Ψ sends each $X \in \mathrm{FMP}_k^{\phi=0}$ to the formal moduli problem given by

$$\Psi(X)(A) = \mathrm{Map}_{\mathrm{FMP}_k^{\phi=0}}(\mathrm{Spf}(A), X).$$

By construction, the top horizontal composite $\Psi \circ \mathrm{fib}_\phi$ is naturally equivalent to the forgetful functor appearing in Corollary 4.2.

Theorem 5.7. *The functor Ψ is fully faithful. Furthermore, its essential image contains all truncated formal moduli problems whose tangent complex is eventually coconnective.*

In particular, this has the following consequence. If X is a formal moduli problem classified by a partition Lie algebra $\mathfrak{g}$, then the Frobenius neighbourhood $\mathrm{fib}(\phi\colon X \to X^{(1)})$, *together with its trivialisation of ϕ*, is classified by the shifted derived restricted Lie algebra underlying $\mathfrak{g}$. Without the data of a trivialisation of ϕ, the formal moduli problem $\mathrm{fib}(\phi\colon X \to X^{(1)})$ is classified by a partition Lie algebra of the form $\mathfrak{g} \oplus \Sigma^{-1}\mathfrak{g}^{(1)}$.

Construction 5.8. Consider the functor $\mathrm{triv}\colon \mathrm{Mod}_k^{\mathrm{cn}} \longrightarrow \mathrm{DAlg}_k^{\mathrm{cn}}$ taking trivial augmented algebras. This intertwines the $\mathbb{Z}^{\mathrm{op}}$-twistings from Example 5.4 and Example 5.5. We therefore obtain a functor

$$\mathrm{triv}_{k[\eta]}^{\mathrm{tr}}\colon \mathrm{Mod}_{k[\eta]}^{\mathrm{cn}} \simeq \mathrm{Mod}_k^{\mathrm{cn},\phi=0} \longrightarrow \mathrm{DAlg}_k^{\mathrm{cn},\phi=0} \simeq \mathrm{DAlg}_k^{\mathrm{tr,cn}}$$

which commutes both with forgetting the ϕ-trivialisation and with the functors fib_ϕ. The composite

$$\mathrm{triv}^{\mathrm{tr}}\colon \mathrm{Mod}_k^{\mathrm{cn}} \xrightarrow{\ \mathrm{triv}\ } \mathrm{Mod}_{k[\eta]}^{\mathrm{cn}} \xrightarrow{\ \mathrm{triv}_{k[\eta]}^{\mathrm{tr}}\ } \mathrm{DAlg}_k^{\mathrm{tr,cn}}$$

sends a connective k-module to the trivial truncated algebra. Indeed, this holds because $\mathrm{triv}^{\mathrm{tr}}$ preserves sifted colimits and $\mathrm{triv}^{\mathrm{tr}}(N)$ is a trivial truncated algebra when N is a vector space.

Now let M be a 1-connective perfect k-module and consider the connective $k[\eta]$-module $\mathrm{fib}_\phi(M) \simeq \mathrm{Hom}_k\big(k[\eta], M\big) \simeq M \oplus \Sigma^{-1}\phi_* M$. Using that $k[\eta]$ is the cofibre of a map $\Sigma^{-1}k \to \Sigma\phi^* k$ of trivial $k[\eta]$-modules, one obtains a natural fibre sequence of truncated algebras

$$\mathrm{fib}_\phi\big(\mathrm{triv}(M)\big) \simeq \mathrm{triv}^{\mathrm{tr}}_{k[\eta]}\big(\mathrm{fib}_\phi(M)\big) \longrightarrow \mathrm{triv}^{\mathrm{tr}}(\Sigma^{-1}\phi_* M) \longrightarrow \mathrm{triv}^{\mathrm{tr}}(\Sigma M). \tag{5.9}$$

Consequently, $\mathrm{fib}_\phi\big(\mathrm{triv}(M)\big)$ is contained in $\mathrm{Art}^{\mathrm{tr}}_k$.

Lemma 5.10. *Let $\Phi\colon \mathrm{FMP}^{\mathrm{tr}}_k \longrightarrow \mathrm{FMP}^{\phi=0}_k$ denote the left adjoint of Ψ. For any $Y \in \mathrm{FMP}^{\mathrm{tr}}_k$ and any 1-connective perfect k-module M, there is a natural equivalence*

$$\mathrm{Map}_k\big(M^\vee, T_{\Phi(Y)}\big) \simeq Y\big(\mathrm{fib}_\phi\big(\mathrm{triv}(M)\big)\big)$$

where $T_{\Phi(Y)}$ denotes the tangent complex of the formal moduli problem underlying $\Phi(Y) \in \mathrm{FMP}^{\phi=0}_k$.

Proof. For each $Y \in \mathrm{FMP}^{\mathrm{tr}}_k$, consider the reduced excisive functor

$$F_Y\colon \mathrm{Perf}^{1-\mathrm{cn}}_k \longrightarrow \mathcal{S}; \quad F_Y(M) = Y\big(\mathrm{fib}_\phi(\mathrm{triv}(M))\big).$$

We have to show that under the equivalence $\mathrm{Mod}_k \simeq \mathrm{Exc}_*(\mathrm{Perf}^{1-\mathrm{cn}}_k, \mathcal{S})$ that identifies a k-module N with the reduced excisive functor $M \mapsto \mathrm{Map}_k(M^\vee, N)$, this corresponds to $T_{\Phi(Y)}$.

If $Y = \mathrm{Spf}(A)$ is corepresentable by $A \in \mathrm{Art}^{\mathrm{tr}}_k$, this follows from the natural equivalences

$$\mathrm{Map}_k\big(M^\vee, T_{\Phi(\mathrm{Spf}(A))}\big) \simeq \mathrm{Map}_{\mathrm{DAlg}^{\mathrm{cn}}_k}\big(\mathrm{oblv}(A), \mathrm{triv}(M)\big) \simeq \mathrm{Map}_{\mathrm{DAlg}^{\mathrm{tr,cn}}_k}\big(A, \mathrm{fib}_\phi\big(\mathrm{triv}(M)\big)\big).$$

For a general Y, let $\int Y$ denote the full subcategory of $\big(\mathrm{FMP}^{\mathrm{tr}}_k\big)_{/Y}$ on the maps $\mathrm{Spf}(A) \to Y$, where A is an Artin truncated algebra. This ∞-category is sifted, because it has finite coproducts, and Y is the pointwise colimit of the canonical diagram $\int Y \longrightarrow \mathrm{FMP}^{\mathrm{tr}}_k$ sending $\mathrm{Spf}(A) \to Y$ to $\mathrm{Spf}(A)$.

We can thus write $F_Y \simeq \mathrm{colim}_{\int Y} F_{\mathrm{Spf}(A)}$ as a sifted colimit in $\mathrm{Exc}_*(\mathrm{Perf}^{1-\mathrm{cn}}_k, \mathcal{S})$. Consequently, the module classifying F_Y is the colimit of the modules classifying the $F_{\mathrm{Spf}(A)}$. The result then follows from the fact that $T_{\Phi(Y)} \simeq \mathrm{colim}_{\int Y} T_{\Phi(\mathrm{Spf}(A))}$, since Φ and taking the tangent complex of a formal moduli problem are both functors that preserve sifted colimits. □

Lemma 5.11. *For each $Y \in \mathrm{FMP}^{\mathrm{tr}}_k$, there is a natural fibre sequence $T_Y \to T_{\Phi(Y)} \to \Sigma^{-1}\phi^* T_Y$ in Mod_k.*

Proof. For each 1-connective perfect k-module M, the fibre sequence (5.9) gives rise to a fibre sequence of spaces $Y(\mathrm{triv}^{\mathrm{tr}}(M)) \to Y(\mathrm{fib}_\phi(\mathrm{triv}(M))) \to Y(\mathrm{triv}^{\mathrm{tr}}(\Sigma^{-1}\phi_* M))$. Varying M, the outer terms are classified by T_Y and $\Sigma^{-1}\phi^* T_Y$ and the middle term is classified by $T_{\Phi(Y)}$ by Lemma 5.10. □

Proof of Theorem 5.7. Let $X \in \mathrm{FMP}^{\phi=0}_k$ and consider the counit $\epsilon\colon \Phi\Psi(X) \to X$. It suffices to verify that ϵ induces an equivalence between the tangent complexes of the underlying formal moduli problems. To see this, let M be a 1-connective perfect k-module and consider the map $\epsilon_*\colon \mathrm{Map}_k(M^\vee, T_{\Phi\Psi(X)}) \longrightarrow \mathrm{Map}_k(M^\vee, T_X)$. Using Lemma 5.10, we can identify ϵ_* with the map

$$\mathrm{Map}_{\mathrm{FMP}^{\phi=0}_k}\Big(\mathrm{Spf}\big(\mathrm{fib}_\phi(\mathrm{triv}(M))\big), X\Big) \longrightarrow \mathrm{Map}_{\mathrm{FMP}_k}\Big(\mathrm{Spf}\big(\mathrm{triv}(M)\big), X\Big)$$

restricting along $\mathrm{Spf}\big(\mathrm{triv}(M)\big) \longrightarrow \mathrm{Spf}\big(\mathrm{fib}_\phi(\mathrm{triv}(M))\big)$. Since the ϕ-trivial formal moduli problem $\mathrm{Spf}\big(\mathrm{fib}_\phi(\mathrm{triv}(M))\big) \simeq \mathrm{cofib}_\phi\big(\mathrm{Spf}(\mathrm{triv}(M))\big)$ is the universal ϕ-trivial formal moduli problem receiving a map from $\mathrm{Spf}\big(\mathrm{triv}(M)\big)$, the above map is an equivalence. Consequently, $T_{\Phi\Psi(X)} \to T_X$ is an equivalence.

Next, let us fix $Y \in \mathrm{FMP}_k^{\mathrm{tr}}$ such that T_Y is n-coconnective. By the fibre sequence from Lemma 5.11, $T_{\Phi(Y)}$ is n-coconnective as well. Since $\mathrm{oblv}\colon \mathrm{FMP}_k^{\phi=0} \longrightarrow \mathrm{FMP}_k$ is comonadic, we can write $\Phi(Y)$ as a totalisation of a cosimplicial diagram of iterated fibres $\mathrm{fib}^n_\phi(\Phi(Y))$. Applying Ψ and using that $T_{\Psi(\mathrm{fib}_\phi(X))} \simeq T_X$ for any $X \in \mathrm{FMP}_k$, it follows that $T_{\Psi\Phi(Y)}$ is the totalisation of a cosimplicial diagram of n-coconnective modules, and hence n-coconnective itself. Using Lemma 5.11 once more, we obtain a fibre sequence $F \to G \to \Sigma^{-1}\phi^*F$ where $F = \mathrm{fib}(T\eta\colon T_Y \to T_{\Psi\Phi(Y)})$ and $G = \mathrm{fib}(T_{\Phi(Y)} \to T_{\Phi\Psi\Phi(Y)})$. The triangle identities show that $G \simeq 0$. Because F is n-coconnective and ϕ^* is an (exact) equivalence, it then follows by induction that $F \simeq 0$. We conclude that the unit $Y \to \Psi\Phi(Y)$ is an equivalence, as desired. □

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