Coxeter element and particle masses

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To Joseph Bernstein on his 70th birthday

Abstract. Let $\mathfrak{g}$ be a simple Lie algebra of rank r over $\mathbb{C}$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We construct a family of r commuting Hermitian operators acting on $\mathfrak{h}$ whose eigenvalues are equal to the coordinates of the eigenvectors of the Cartan matrix of $\mathfrak{g}$.

(2010) Mathematics Subject classification. 17B20, 17B22, 37K10

Introduction

This note arose from our attempt to understand a theorem discovered by the physicists, [BCDS], [Fr] (a), [FLO], to the effect that the masses of particles (the first r excitations) in affine Toda field theories are equal to the coordinates of the Perron – Frobenius eigenvector of the Cartan matrix A. In the text below we review the proof of this elegant result (together with a little generalization), and write down differential equations, similar to the Toda field equations, giving rise to particles whose masses are absolute values of the coordinates of all other eigenvectors of A. One observes some interesting regularities in their shape related to the geometry of the action of the Coxeter element on the Cartan algebra.

Let $A = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^r$ be the Cartan matrix of the root system $R \subset \mathfrak{h}^*$ corresponding to a simple finite dimensional complex Lie algebra $\mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$. Let $k_1 < k_2 < \ldots < k_r$ be the exponents of R . Here and below we suppose for simplicity that R is not of type D_{2n}, to

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avoid the case when one of the exponents has multiplicity 2. The eigenvalues of A are

$$\lambda_i = 2(1 - \cos(k_i\theta)),\ \theta = \pi/h,$$

where h is the Coxeter number of R. Let $* :\ \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be a Cartan antilinear involution and $H(x,y) = (x, y^*)$ the corresponding Hermitian form, (x,y) being the Killing form.

The principal result of this paper (see Theorem 5.2) is a construction of certain family of mutually commuting hermitian operators $M^{(i)} :\ \mathfrak{h} \longrightarrow \mathfrak{h},\ 1 \leq i \leq r$, such that for each i the eigenvalues $\mu_1^{(i)}, \dots, \mu_r^{(i)}$ of $M^{(i)}$ (in the appropriate order) form an eigenvector of A with eigenvalue λ_i.

Actually, the definition of these operators is quite simple (we give a sketch here in the introduction, the details the reader will find in the main body of the paper). One starts with a *cyclic element* in the sense of Kostant, [K],

$$e = \sum_{i=0}^{r} c_i e_i,\ c_i \neq 0,$$

where $e_i \in \mathfrak{g}_{\alpha_i}$, $\alpha_0 := -\theta$, θ being the highest root. Its centralizer $\mathfrak{h}' := Z(e)$ is a Cartan subalgebra which is, as Kostant puts it, *in apposition* to $\mathfrak{h}$. Let $\mathfrak{g} = \oplus_{i=0}^{h-1}\mathfrak{g}_i$ be the principal gradation, (cf. 1.2). The spaces $\mathfrak{h}'^{(i)} := \mathfrak{h}' \cap \mathfrak{g}_{k_i}$, $1 \leq i \leq r$, are one-dimensional.

Let $e^{(i)} \in \mathfrak{h}'^{(i)}$ be a nonzero vector, for example $e^{(1)} = e$. The operators $\mathrm{ad}_{e^{(i)}}\,\mathrm{ad}_{e^{(i)*}}$ preserve $\mathfrak{h}$; let $\tilde{M}^{(i)}$ denote its restriction to $\mathfrak{h}$. By definition $M^{(i)}$ is a suitable square root of $\tilde{M}^{(i)}$.

The relation to eigenvectors of A is based on a wellknown relation between A and the Coxeter transformation c, due to Coxeter, cf. [Co], (1.5), (1.7), see §3 below for the details[3].

The Coxeter element plays a ubiquitous role throughout various domains of Representation theory, cf. [BGP].

An eigenvector $p \in \mathfrak{h}^*$ with the lowest eigenvalue λ_1, a *Perron – Frobenius vector*, plays a distinguished role. The assertion that the eigenvalues of $M^{(1)}$ coincide with its coordinates has been proven in [Fr] (a), [FLO]; a generalization to $i > 1$ is straightforward. The coordinates of p have a remarkable physical meaning, cf. [Cor]. There exist some mysterious

[3]Note that A is (close to) a symmetric matrix, whereas c is an orthogonal matrix, the passage from one to another is somewhat similar to the classical Cayley transform.

formulas expressing them as certain products of values of Gamma function, cf. [CAS].

In the *Goddard – Nuyts – Olive dual* picture the numbers $\mu_j^{(i)}$ appear as the charges of static soliton solutions of Toda field equations with the purely imaginary coupling constant, corresponding to the Langlands dual Lie algebra $\mathfrak{g}^\vee$, cf. [Fr] (b).

In the last §7 we describe some factorization patterns in the shape of the Cartan eigenvectors. Namely, among them there are exactly $\phi(h)$ vectors *of PF type* whose coordinates are, up to signs, permutations of the coordinates of the PF eigenvector. The nonzero components of the other eigenvectors consist of several *clusters*, each cluster corresponding to a PF eigenvector of a root subsystem $R' \subset R$ with the Coxeter number h' dividing h.

We are grateful to M. Finkelberg for stimulating discussions, and to P. Dorey and V. Kac for a useful correspondence. We understand that P. Dorey has obtained some results close to our Theorem 5.2. Our special thanks go to the referee for the useful criticism which allowed us to improve the exposition.

1 Principal element and Cartan subalgebras in apposition

1.1. Setup. Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra; $(,)$ will denote the Killing form on $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; let $R \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\{\alpha_1, \ldots, \alpha_r\} \subset R$ a base of simple roots,

$$\mathfrak{g} = (\oplus_{\alpha<0}\mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus (\oplus_{\alpha>0}\mathfrak{g}_\alpha),$$

the root decomposition. For $\alpha = \sum_{i=1}^r m_i\alpha_i$ we set

$$\operatorname{ht}\alpha = \sum_{i=1}^r m_i.$$

Let

$$\theta = \sum_{i=1}^r n_i\alpha_i,$$

be the longest root; we set

$$\alpha_0 := -\theta, n_0 := 1.$$

The number

$$h = \sum_{i=0}^{r} n_i = 1 + \operatorname{ht}\theta,$$

is the Coxeter number of $\mathfrak{g}$; set $\zeta = \exp(2\pi i/h)$.

For each $\alpha \in R$ choose a base vector $E_\alpha \in \mathfrak{g}_\alpha$.

The Killing form $(,)$ induces a W-invariant scalar product on $\mathfrak{h}$. Using it we identify $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$, and each root may be considered as an element of $\mathfrak{h}$. Thus,

$$[h, x] = (h, \alpha)x, \ \alpha \in R, \ x \in \mathfrak{g}_\alpha. \tag{1.1.1}$$

For $x \in \mathfrak{g}$, $Z(x) = \mathfrak{g}^x$ will denote the centralizer of x.

Let G denote the adjoint group of $\mathfrak{g}$, e.g., the (Zarisky closure of the) subgroup $G \subset GL(\mathfrak{g})$ generated by the elements e^{ad_x}, $x \in \mathfrak{g}$, and

$$\exp : \ \mathfrak{g} \longrightarrow G$$

the exponential map. For $g \in G$ and $x \in \mathfrak{g}$, the result of the action of g on x will be denoted $\operatorname{Ad}_g(x)$. If $\mathfrak{g}$ is realized as a Lie subalgebra of a matrix algebra then

$$\operatorname{Ad}_{\exp(y)}(x) = e^y x e^{-y}. \tag{1.1.2}$$

1.2. Principal element and principal gradation. Let $\rho^\vee \in \mathfrak{h}$ be defined by

$$\langle \alpha_i, \rho^\vee \rangle = 1, \ i = 1, \dots, r.$$

Another definition of $\rho^\vee$:

$$\rho^\vee = \frac{1}{2} \sum_{\alpha^\vee \in R^\vee_{>0}} \alpha^\vee$$

where $R^\vee \subset \mathfrak{h}$ is the dual root system. It follows that for all $\alpha \in R$

$$\langle \alpha, \rho^\vee \rangle = \operatorname{ht}\alpha.$$

We set

$$P = \exp(2\pi i \rho^\vee/h) \in G.$$

For all $\alpha \in R$,

$$\operatorname{Ad}_P(E_\alpha) = \zeta^{\operatorname{ht}\alpha} E_\alpha.$$

Thus, $\operatorname{Ad} P$ defines a $\mathbb{Z}/h\mathbb{Z}$-grading on $\mathfrak{g}$,

$$\mathfrak{g} = \oplus_{k=0}^{h-1} \mathfrak{g}_k, \ \mathfrak{g}_k = \{x \in \mathfrak{g} | \ \operatorname{Ad}_P(x) = \zeta^k x\}.$$

We have $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g}_1$ admits as a base the set

$$E_{\alpha_0}, E_{\alpha_1}, \ldots, E_{\alpha_r}.$$

(Note that $\operatorname{ht} \alpha_0 = -\operatorname{ht} \theta = 1 - h$, so that $\operatorname{Ad}_P(E_{\alpha_0}) = \zeta E_{\alpha_0}$.)

1.3. The Cartan subalgebra $\mathfrak{h}'$. Fix complex numbers m_i, m_i' such that $m_i m_i' = n_i$, $i = 0, \ldots, r$, $m_0 = m_0' = 1$ and define elements

$$E = \sum_{i=0}^{r} m_i E_{\alpha_i}, \tilde{E} = \sum_{i=0}^{r} m_i' E_{-\alpha_i}.$$

We have $E \in \mathfrak{g}_1$, $\tilde{E} \in \mathfrak{g}_{h-1}$.

1.3.1. Lemma. $[E, \tilde{E}] = 0$.

Kostant calls $E, \tilde{E}$ *cyclic elements*; these are $z_0, \tilde{z}_0$ in the notation of [K], Thm. 6.7.

We define, with Kostant, [K], the subspace

$$\mathfrak{h}' := Z(E) = Z(\tilde{E}) \subset \mathfrak{g}.$$

It is proven in [K], Thm. 6.7, that $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$, called *the Cartan subalgebra in apposition to* $\mathfrak{h}$ *with respect to the principal element* P.

The subspace $\mathfrak{h}' \cap \mathfrak{g}_i$ is nonzero iff $i \in \{k_1, k_2, \ldots, k_r\}$ where $1 = k_1 < k_2 < \ldots < k_r = h - 1$ are the *exponents* of $\mathfrak{g}$. We have $k_i + k_{r+1-i} = h$.

Set

$$\mathfrak{h}'^{(i)} := \mathfrak{h}' \cap \mathfrak{g}_{k_i}, \ 1 \leq i \leq r;$$

these are the subspaces of dimension 1.

We denote by

$$T' = \exp(\mathfrak{h}') \subset G,$$

the maximal torus corresponding to $\mathfrak{h}'$.

If $x \in \mathfrak{h}'$, that is, $[x, E] = 0$, then

$$0 = [\operatorname{Ad}_P(x), \operatorname{Ad}_P(E)] = \zeta[\operatorname{Ad}_P(x), E],$$

whence $\mathrm{Ad}_P(\mathfrak{h}') \subset \mathfrak{h}'$, e.g., $P \in N_{T'}$.

Let $R' \subset \mathfrak{h}'^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}'$, and denote by $\mathfrak{h}'^*_{\mathbb{R}} \subset \mathfrak{h}'^*$ the real linear subspace generated by R'.

Recall that the set of (unordered) bases of simple roots in R' is in bijection with the set of chambers, the connected components of

$$\mathfrak{h}'^*_{\mathbb{R}} \setminus \cup_{\alpha' \in R'} \alpha'^{\perp}.$$

The set of bases is a W'-torsor where W' is the Weyl group of R'.

A *Coxeter element* in W' is an element of the form

$$c = s_{\alpha'_1} \dots s_{\alpha'_r}$$

where $\{\alpha'_1, \dots, \alpha'_r\} \subset R'$ is some base of simple roots. All Coxeter elements are conjugate, cf. [B], Ch. V, §6, Prop. 1.

1.4. Theorem. (Kostant) The image of P in $N_{T'}/T' = W'$ is a Coxeter element.

Proof. *See [K], Corollary 8.6.* □

As Kostant shows, one can go backwards: starting from a Cartan subalgebra $\mathfrak{h}'$ and from an arbitrary Coxeter element $c \in W(\mathfrak{h}')$, one can reconstruct $\mathfrak{h}$. We shall use this in §5 below.

1.5. Thus we have

$$\mathfrak{h} = \mathfrak{g}^P := \{x \in \mathfrak{g} |\ \mathrm{Ad}_P(x) = x\},$$

and

$$\mathfrak{h}' = \mathfrak{g}^E := \{x \in \mathfrak{g} |\ \mathrm{Ad}_E(x) = [E, x] = 0\}.$$

So,

$$E \in \mathfrak{h}' \cap \mathfrak{g}_1,\ \tilde{E} \in \mathfrak{h}' \cap \mathfrak{g}_{h-1}.$$

It follows that E (resp. $\tilde{E}$) is an eigenvector of c with eigenvalue ζ (resp. ζ^{-1}).

2 Diagonalization of some operators

2.1. Consider the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}'$:

$$\mathfrak{g} = \mathfrak{h}' \oplus_{\alpha' \in R'} \mathfrak{g}_{\alpha'},\ R' \subset \mathfrak{h}'^*.$$

Recall that Ad_P leaves $\mathfrak{h}'$ stable and induces the action of a Coxeter element $c \in W'$ on $\mathfrak{h}'$,

$$W' = N_{T'}/T' \tilde{=} N_{\mathfrak{h}'}/\mathfrak{h}'.$$

Recall the the order of c in W' is equal to h.

2.2. Proposition. It is possible to choose nonzero root vectors $e_{\alpha'} \in \mathfrak{g}_{\alpha'}$ in such a way that

$$\mathrm{Ad}_P(e_{\alpha'}) = e_{c(\alpha')}$$

for all $\alpha' \in R'$.

Proof. *See [K], Theorem 8.4.* □

2.3. According to [K], the action of c on R' has r orbits $\Omega_i,\ i = 1, \ldots, r$, each of them containing h elements:

$$R' = \coprod_{i=1}^{r} \Omega_i;$$

here the prime reminds us that the orbits lie in $R' \subset \mathfrak{h}'^*$.

By the way, it follows that $|R'| = \dim \mathfrak{g} - r = hr$, whence

$$\dim \mathfrak{g} = h(r+1).$$

For example, for $\mathfrak{g} = \mathfrak{sl}(n)$, $r = n - 1$, $h = n$ and $\dim \mathfrak{g} = n^2 - 1$.

For every $1 \leq i \leq r$, define with Kostant an element

$$a_i = \sum_{\alpha' \in \Omega_i} e_{\alpha'}.$$

It follows from Prop. 2.2 that $\mathrm{Ad}_P(a_i) = a_i$, e.g., all $a_i \in \mathfrak{h} = \mathfrak{g}^P$.

According to [K], Theorem 8.4, $a_1, \ldots, a_r$ forms a base of $\mathfrak{h}$.

Let us pick an element $\gamma_i \in \Omega_i$, so that $\Omega_i = \{c^k(\gamma_i) | k = 0, \ldots, h-1\}$.

For any $x \in \mathfrak{g}_m \cap \mathfrak{h}'$, $m \in \mathbb{Z}/h\mathbb{Z}$ we have (cf. [Fr] (a))

$$[x, e_{c^k(\gamma_i)}] = \zeta^{-km} \langle \gamma_i, x \rangle e_{c^k(\gamma_i)}. \tag{2.3.1}$$

Indeed,

$$\begin{aligned} [x, e_{c^k(\gamma_i)}] &= [x, \mathrm{Ad}_P^k(e_{\gamma_i})] = \mathrm{Ad}_P^k[\mathrm{Ad}_P^{-k}(x), e_{\gamma_i}] \\ &= \zeta^{-km} \mathrm{Ad}_P^k[x, e_{\gamma_i}] = \zeta^{-km} \mathrm{Ad}_P^k(\langle \gamma_i, x \rangle e_{\gamma_i}) \\ &= \zeta^{-km} \langle \gamma_i, x \rangle e_{c^k(\gamma_i)}. \end{aligned}$$

It follows that for any $y \in \mathfrak{g}_{-m} \cap \mathfrak{h}'$,

$$[y, [x, e_{c^k(\gamma_i)}]] = \langle \gamma_i, y \rangle \langle \gamma_i, x \rangle e_{c^k(\gamma_i)}.$$

Summing up by k, we get the following theorem.

2.4. Theorem. For all $m \in \mathbb{Z}/h\mathbb{Z}$, $x \in \mathfrak{g}_m \cap \mathfrak{h}'$, $y \in \mathfrak{g}_{-m} \cap \mathfrak{h}'$, $1 \leq i \leq r$,

$$[y, [x, a_i]] = \langle \gamma_i, y \rangle \langle \gamma_i, x \rangle a_i.$$

In other words, $\{a_1, \ldots, a_r\}$ is a base of $\mathfrak{h}$ which diagonalizes the operator $\mathrm{ad}_y \, \mathrm{ad}_x$.

3 Coxeter element, Cartan matrix, and their eigenvectors

3.1. Let $R \subset V$ be a reduced irreducible root system in a real vector space V of dimension r (in particular R generates V), W the Weyl group of R, $(,)$ a W-invariant scalar product on V. We identify V with V^* using $(,)$ so that $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, cf. [B], Chapitre V, §1, Lemme 2.

Let $\{\alpha_1, \ldots, \alpha_r\} \subset R$ be a base of simple roots.

Choose a black/white *colouring* of the set I of vertices of the Dynkin graph of R (which is a tree) such that neighbouring vertices have different colours. Identify I with $\{1, \ldots, r\}$ in such a way that the vertices $\{1, \ldots, p\}$ are black, and the vertices $\{p+1, \ldots, r\}$ are white.

We denote $s_i := s_{\alpha_i}$. Consider a Coxeter element

$$c = c_b c_w, \ c_b = \prod_{i=1}^{p} s_i, \ c_w = \prod_{i=p+1}^{r} s_i,$$

the order inside the products defining c_b and c_w does not matter since reflections s_i, s_j commute once i and j have the same colour. Evidently

$$c_w^2 = c_b^2 = 1,$$

whence

$$(c_b + c_w)^2 = c + c^{-1} + 2.$$

Let $A = (n_{ij}) = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^r$ be the Cartan matrix of R. We denote by $\hat{A} : V \longrightarrow V$ an operator defined by

$$\hat{A}(\alpha_i) = \sum_{j=1}^{r} n_{ij} \alpha_j.$$

3.2. Lemma. We have

$$c_b + c_w = 2I - \hat{A}.$$

Proof. *The matrix A has a block form*

$$A = \begin{pmatrix} 2I_p & X \\ Y & 2I_{r-p} \end{pmatrix}.$$

On the other hand, calculating the action of the (commuting) simple reflections, one finds the matrices of the operators c_b and c_w in the base $\alpha_1, \ldots, \alpha_r$ to be

$$c_b = \begin{pmatrix} -I & -X \\ 0 & I \end{pmatrix}, c_w = \begin{pmatrix} I & 0 \\ -Y & -I \end{pmatrix},$$

whence

$$c_b + c_w = \begin{pmatrix} 0 & -X \\ -Y & 0 \end{pmatrix} = 2I - \hat{A}.$$

□

3.3. Lemma. All the eigenvalues of $\hat{A}$ have the form $2(1 - \cos k_i\theta_1)$, $i \in \{1, \ldots, r\}$ where k_i are the exponents of $\mathfrak{g}$ and $\theta_1 = \pi/h$.

Proof. *We shall use the identity*

$$(2I - \hat{A})^2 = c + c^{-1} + 2. \qquad (3.3.1)$$

The eigenvalues of c are $e^{2k_j\pi i/h}, 1 \leq j \leq r$, cf. [Co]. It follows from (3.3.1) that if $e^{2i\theta}$ is an eigenvalue of c, then $4\cos^2\theta$ is an eigenvalue of $(2I - \hat{A})^2$, so $2(1 \pm \cos\theta)$ is an eigenvalue of $\hat{A}$.

Note that

$$2(1 + \cos\theta) = 2(1 - \cos(\pi - \theta)),$$

and $k_{r-i} = h - k_i$, which implies the assertion of the lemma. □

3.4. For a vector

$$x = \sum_{i=1}^{r} x_i\alpha_i \in V,$$

we have

$$\hat{A}x = \sum_i x_i(\sum_j n_{ij})\alpha_j = \sum_j(\sum_i x_i n_{ij})\alpha_j.$$

Thus, $\hat{A}x = \lambda x$ is equivalent to

$$\sum_i x_i n_{ij} = \lambda x_j,\ j = 1, \ldots, r. \tag{3.4.1}$$

Define a *colour function* $\epsilon : \{1, \ldots, r\} \longrightarrow \{\pm 1\}$ by

$$\epsilon(i) = \begin{cases} 1 & \text{if } i \leq p \\ -1 & \text{if } i > p. \end{cases}$$

3.5. Duality Lemma. Let x satisfy (3.4.1) with $\lambda = 2(1 - \cos\theta)$, that is,

$$\sum_{i=1}^{r} x_i n_{ij} = 2(1 - \cos\theta)x_j,\ j = 1, \ldots, r. \tag{3.5.1}$$

Then

$$\sum_{i=1}^{r} \epsilon(i)x_i n_{ij} = 2(1 + \cos\theta)\epsilon(j)x_j,\ j = 1, \ldots, r. \tag{3.5.2}$$

Proof. *Recall that the matrix A has a block form $\begin{pmatrix} 2I_p & X \\ Y & 2I_{r-p} \end{pmatrix}$ and write $x_b = (x_1, .., x_p)$ and $x_w = (x_{p+1}, .., x_r)$. The identity $xA = \lambda x$ gives :*

$$\begin{cases} 2x_b + x_w Y = 2(1 - \cos\theta)x_b \\ x_b X + 2x_w = 2(1 - \cos\theta)x_w. \end{cases}$$

Then

$$\begin{cases} 2x_b - x_w Y = 2(1 + \cos\theta)x_b \\ x_b X - 2x_w = 2(1 + \cos\theta)(-x_w). \end{cases}$$

This means that $\tilde{x} = (\epsilon(1)x_1, ..., \epsilon(r)x_r)$ satisfies (3.4.1) with $\lambda = 2(1 + \cos\theta)$. □

3.1

Now set

$$x_b = \sum_{i=1}^{p} x_i \alpha_i,\ x_w = \sum_{i=p+1}^{r} x_i \alpha_i.$$

Lemma. (a) $c_w(x_w) = -x_w,\ c_b(x_b) = -x_b$. (b) $c_w(x_b) = x_w + 2\cos\theta x_b, c_b(x_w) = x_b + 2\cos\theta x_w$.

3.7. Corollary. Define $y = e^{-i\theta/2}x_w + e^{i\theta/2}x_b$. Then

$$c(y) = e^{2i\theta}y.$$

3.8. Lemma. For all $j = 1, \dots, r$,

$$(y, \alpha_j) = i\epsilon(j)e^{-i\epsilon(j)\theta/2}\sin\theta \cdot (\alpha_j, \alpha_j)x_j.$$

Proof. *Recall that* $(\alpha_k, \alpha_j) = \frac{1}{2}(\alpha_j, \alpha_j)n_{kj}, k, j = 1, \dots, r$. *We have*

$$(y, \alpha_j) = \frac{(\alpha_j, \alpha_j)}{2}\left(e^{i\theta/2}\sum_{k=1}^{p} x_k n_{kj} + e^{-i\theta/2}\sum_{k=p+1}^{r} x_k n_{kj}\right).$$

Since

$$\sum_{k=1}^{p} x_k n_{kj} = \frac{1}{2}\left(\sum_{i=1}^{r} x_i n_{ij} + \sum_{i=1}^{r}\epsilon(i)x_i n_{ij}\right),$$

and

$$\sum_{k=p+1}^{r} x_k n_{kj} = \frac{1}{2}\left(\sum_{i=1}^{r} x_i n_{ij} - \sum_{i=1}^{r}\epsilon(i)x_i n_{ij}\right),$$

the application of Lemma 3.5 gives the result. □

3.9. Lemma. The elements $\epsilon(i)\alpha_i,\ 1 \le i \le r$, belong to r different orbits of the action of c on R.

Proof. *Kostant proves in [K], Thm 8.1 and Thm 8.4, that exactly* r *negative roots, say* $\{\beta_1, ..., \beta_r\}$, *become positive under the action of* c *on* R *and they belong to* r *different orbits of this action.*

In the proof of Lemma 3.2, we have seen that

$$c = c_b c_w = \begin{pmatrix} -I + XY & X \\ -Y & -I \end{pmatrix} \text{ and } c^{-1} = c_w c_b = \begin{pmatrix} -I & -X \\ Y & YX - I \end{pmatrix},$$

where X *and* Y *are matrices with nonpositive entries, such that* $A = \begin{pmatrix} 2I_p & X \\ Y & 2I_{r-p} \end{pmatrix}$.

For $1 \le i \le p$, $c^{-1}(\alpha_i)$ *is a negative root, whence* $\alpha_i = c(\beta_k)$ *for some* k *in* $\{1, .., r\}$. *For* $p + 1 \le i \le r$, $c(-\alpha_i)$ *is a positive root, whence* $-\alpha_i = \beta_j$ *for some* j *in* $\{1, .., r\}$.

Thus, to each root $c(i)\alpha_i$, *we have associated a root* β_j *in the same orbit, and this is a one-to-one correspondence.* □

4 Cartan involution and Hermitian form

4.1. Recall the setup 1.1. Let us choose, with F.Bruhat [Br] and Kostant [K], p. 1003, a *Weyl basis* $\{e_\alpha \in \mathfrak{g}_\alpha\}$. By definition, this means that all $e_\alpha \neq 0$, $(e_\alpha, e_{-\alpha}) = 1$, and if we denote

$$[e_\alpha, e_\beta] = n_{\alpha\beta} e_{\alpha+\beta},$$

then $n_{\alpha\beta} = n_{-\alpha,-\beta}$. Here we have chosen the Kostant's normalization of the Weyl basis. We set $h_i := [e_{\alpha_i}, e_{-\alpha_i}],\ 1 \leq i \leq r$.

Let $\mathfrak{k} \subset \mathfrak{g}$ be the real Lie subalgebra with the base

$$e_\alpha - e_{-\alpha},\ i(e_\alpha + e_{-\alpha}),\ ih_j,\ \alpha \in R_+,\ 1 \leq j \leq r.$$

It is a *compact form* of $\mathfrak{g}$, which means by definition that

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k},$$

and the restriction of the Killing form to $\mathfrak{k}$ is negative definite.

Define, following Kostant, an involution $(\cdot)^* : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ by

$$(x + iy)^* = x - iy,\ x, y \in i\mathfrak{k}.$$

Then $(\lambda x)^* = \bar{\lambda} x, \lambda \in \mathbb{C}$, and

$$[x, y]^* = [y^*, x^*].$$

The sesquilinear form on $\mathfrak{g}$

$$H(x, y) = (x, y^*)$$

is Hermitian positive definite, cf. [Br], (21).

With respect to this form

$$(\mathrm{ad}_x)^* = \mathrm{ad}_{x^*}$$

One computes that

$$e_\alpha^* = e_{-\alpha},\ \alpha \in R. \tag{4.1.1}$$

4.2. Now let us return to the setup of Sections 1 and 2. However now we will work with specifically chosen Weyl vectors e_α, instead of arbitrary root vectors E_α.

Fix nonzero complex numbers m_i such that $m_i\bar{m}_i = n_i$, $1 \leq i \leq r$. Let

$$e = \sum_{i=1}^{r} m_i e_{\alpha_i} + e_{-\theta}$$

be the cyclic element.

By (4.1.1),

$$e^* = \sum_{i=1}^{r} \bar{m}_i e_{-\alpha_i} + e_\theta.$$

Recall that $[e, e^*] = 0$.

Let $\mathfrak{h}' = Z(e) = Z(e^*)$ be the corresponding Cartan subalgebra in apposition to $\mathfrak{h}$, as in 1.3.

For all $m \in \mathbb{Z}/h\mathbb{Z}$

$$* : \mathfrak{g}_m \cap \mathfrak{h}' \xrightarrow{\sim} \mathfrak{g}_{-m} \cap \mathfrak{h}'$$

4.3. Let us apply Theorem 2.4 to $y = x^*$. With $a_i \in \mathfrak{h}$ and $\gamma_i \in R' \subset \mathfrak{h}'^*$ as in 2.3, we obtain

$$\mathrm{ad}_x\, \mathrm{ad}_{x^*}(a_i) = \gamma_i(x)\gamma_i(x^*)a_i,\ \ 1 \leq i \leq r.$$

4.4. Lemma. $\gamma_i(x^*) = \overline{\gamma_i(x)}$.

Proof. *Consider the equality (2.3.1):*

$$\mathrm{ad}_x(z) = \zeta^{-km}\gamma_i(x)z$$

where we set for brevity $z = e_{c^k(\gamma_i)}$. *It follows that*

$$H(\mathrm{ad}_x(z), z) = \zeta^{-km}\gamma_i(x)H(z, z).$$

Similarly,

$$\mathrm{ad}_{x^*}(z) = \zeta^{km}\gamma_i(x^*)z,$$

whence

$$H(\mathrm{ad}_{x^*}(z), z) = \zeta^{km}\gamma_i(x^*)H(z, z).$$

The assertion follows now from the adjointness of the operators ad_x *and* ad_{x^*}, *since* $H(z, z) \neq 0$. □

5 Main theorem

5.1. Let us start with a Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}$, whence the root system $R' \subset \mathfrak{h}'^*$. Choose a base of simple roots $\{\alpha'_i\} \subset R'$ and a bicolouring of the Dynkin graph as in §3. Thus, α'_i with $1 \leq i \leq p$ (resp. with $p+1 \leq i \leq r$) will denote the *black* (resp. *white*) simple roots. Let

$$c' = c'_b c'_w, \ c'_b = \prod_{i=1}^{p} s'_i, \ c'_w = \prod_{i=p+1}^{r} s'_i,$$

where $s'_i := s_{\alpha'_i}$ be the corresponding Coxeter element.

Let G be the adjoint group of $\mathfrak{g}$, $T' \subset G$ the maximal torus with $\mathrm{Lie}(T') = \mathfrak{h}'$, so that the Weyl group $W' \subset \mathrm{Aut}(R')$ will be identified with $N_G(T')/T'$. Let $P' \in N_G(T')$ be an element that projects to c'. Set

$$\mathfrak{h} = \mathfrak{g}^{P'}.$$

Then $\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{h}'$ is in apposition to $\mathfrak{h}$ with respect to P', cf [K], Theorem 8.6[4].

Consider the principal gradation generated by P', $\mathfrak{g} = \oplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the ζ^i-eigenspace of $\mathrm{Ad}_{P'}$, as in §1, and one-dimensional spaces $\mathfrak{h}'^{(i)} := \mathfrak{h}' \cap \mathfrak{g}_{k_i}$, $1 \leq i \leq r$.

We can choose an involution $*$ as in §4 in such a way that it leaves $\mathfrak{h}'$ invariant, for every $x \in \mathfrak{h}'$ the operator ad_{x^*} is Hermitian conjugate to ad_x, and $(\mathfrak{h}'^{(i)})^* = \mathfrak{h}'^{(r-i)}$.

Indeed, this is true for the gradation induced by the principal element $P = P_0$ as defined in 1.2, and the involution (let us denote it $*_0$) defined as in §4 starting from $\mathfrak{h}$. Afterwards one can use the conjugacy Theorem 7.3 from [K] to define the desired involution for the principal gradation induced by P'.

5.2. Theorem. Let i be an integer, $1 \leq i \leq r$. Let $e^{(i)}$ be a nonzero vector in $\mathfrak{h}'^{(i)}$, whence $e^{(i)*} \in \mathfrak{h}'^{(r-i)}$. Consider a selfadjoint nonnegative operator

$$\tilde{M}^{(i)} := \mathrm{ad}_{e^{(i)}} \mathrm{ad}_{e^{(i)*}} : \ \mathfrak{h} \longrightarrow \mathfrak{h}.$$

Let $\tilde{\mu}_1^{(i)}, \ldots, \tilde{\mu}_r^{(i)}$ denote its eigenvalues.

[4]The couple of Cartan subalgebras $(\mathfrak{h}, \mathfrak{h}')$ from [K], §6 becomes $(\tilde{\mathfrak{h}}, \mathfrak{h})$ in *op. cit.*, 8.6. Two occurences of $\mathfrak{h}$ in [K], p. 1023, 2nd line, should be replaced by $\tilde{\mathfrak{h}}$.

There exists an (essentially unique) operator $M^{(i)} \in \mathfrak{gl}(\mathfrak{h})$ whose square is equal to $\tilde{M}^{(i)}$ such that the column vector of its eigenvalues in the approriate numbering

$$\mu^{(i)} := (\mu_1^{(i)}, \ldots, \mu_r^{(i)})^t \tag{5.2.1}$$

is an eigenvector of the Cartan matrix A with eigenvalue

$$\lambda_i := 2(1 - \cos(2k_i\pi/h)).$$

In particular, for $i = 1$ there exists an eigenvector of A^t with all coordinates positive (a **Perron − Frobenius vector**), and we may take as $M^{(1)}$ the positive square root of $\tilde{M}^{(1)}$.

The operators $M^{(1)}, \ldots, M^{(r)}$ commute with each other.

Proof. *All the necessary tools have been already prepared. Let* $x = (x_1, \ldots, x_r)^t$ *be an eigenvector of* A^t *(sic!) with eigenvalue* $\lambda = \lambda_i$. *Starting from it, define an eigenvector* y *of the Coxeter element* c', *with eigenvalue* $e^{\sqrt{-1}\theta}$, $\theta = k_i\pi/h$, *cf. Corollary 3.7. By Lemma 3.8, we have*

$$(y, \alpha'_j) = \sqrt{-1}\epsilon(j)e^{-\sqrt{-1}\epsilon(j)\theta/2}\sin\theta \cdot (\alpha'_j, \alpha'_j)x_j, \ 1 \le j \le r. \tag{5.2.2}$$

On the other hand, we know from §4 the eigenvalues of the operator $\tilde{M} = \tilde{M}^{(i)}$ *: as follows from 4.3 and 4.4, they are* $|\gamma_j(e^{(i)})|^2, 1 \le j \le r$ *where* $\gamma_j \in \mathfrak{h}'^*$, $1 \le j \le r$, *are arbitrary representatives of different orbits of the action of* c *on* R',

Let us identify $\mathfrak{h}'$ *with* $\mathfrak{h}'^*$ *using the scalar product* $(,)$, *so that we can consider* γ_j *as vectors belonging to* $\mathfrak{h}'$, *and* $|\gamma_j(e^{(i)})|^2 = |(e^{(i)}, \gamma_j)|^2$.

Recall that $e^{(i)}$ *is an eigenvector of* c' *in* $\mathfrak{h}'$: $c'(e^{(i)}) = \lambda e^{(i)}$ *(*c' *acts as* $\mathrm{Ad}_{P'}$ *on* $\mathfrak{h}'$*), whence*

$$e^{(i)} = \mu y$$

for some $\mu \in \mathbb{C}^*$.

Let us rewrite (5.2.2) in the form

$$(y, \epsilon(j)\alpha'_j) = \sqrt{-1}e^{-\sqrt{-1}\epsilon(j)\theta/2}\sin\theta \cdot \tilde{x}_j, \ 1 \le j \le r, \tag{5.2.2}$$

where $\tilde{x}_j = (\alpha'_j, \alpha'_j)x_j$.

Note that $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_r)^t$ *is a* λ*-eigenvector of* A.

Due to Lemma 3.9, the vectors $\epsilon(j)\alpha'_j, 1 \le j \le r$, *are representatives of* r *orbits of* c'*-action on* R', *so we can set*

$$\gamma_j := \epsilon(j)\alpha'_j.$$

It follows that the eigenvalues of $\tilde{M} = \tilde{M}^{(i)}$ *are*

$$|(e^{(i)}, \gamma_j)|^2 = |\mu|^2 \sin^2\theta \cdot \tilde{x}_j^2,\ 1 \le j \le r$$

(note that $\tilde{x}_j$ *are real numbers, not necessarily positive).*

Thus, the sequence of eigenvalues of $\tilde{M}^{(i)}$ *is*

$$(|\mu|^2 \sin^2\theta \tilde{x}_1^2, \ldots, |\mu|^2 \sin^2\theta \tilde{x}_r^2).$$

On the other hand, a λ_i*-eigenvector of* A *is*

$$(\tilde{x}_1, \ldots, \tilde{x}_r)^t$$

Moreover, the operators $\tilde{M}^{(1)}, \ldots, \tilde{M}^{(r)}$ *mutually commute since the elements* $e^{(1)}, \ldots, e^{(r)} \in \mathfrak{h}'$ *mutually commute.*

Now, as an operator $M = M^{(i)}$*, we take (the unique) one of the* 2^r *square roots of* $\tilde{M}$ *whose* j*-th eigenvalue, if nonzero, has the same sign as that of* x_j*. The set of eigenvalues of* $M^{(i)}$ *will be*

$$(|\mu \sin\theta|\tilde{x}_1, \ldots, |\mu \sin\theta|\tilde{x}_r),$$

and this vector is a λ_i*-eigenvector of* A*.* □

5.3. We can start with an arbitrary pair of Cartan subalgebras $\mathfrak{h}, \mathfrak{h}'$ where $\mathfrak{h}'$ is in apposition to $\mathfrak{h}$ with respect to a principal element P. Defining operators $\tilde{M}^{(i)}$ as in 5.2, we arrive at the same conclusions for their spectra as in 5.2, due again to the Kostant's conjugacy theorem, [K], Theorem 7.3.

6 Affine Toda field equations

6.1. Affine Toda field theories. Consider a classical field theory whose fields are smooth functions $\phi : X \longrightarrow \mathfrak{h}$ where $X = \mathbb{R}^2$ *space - time*, with coordinates x_1, x_2.

The Lagrangian density of the theory depends on an element $e \in \mathfrak{h}'$ where $\mathfrak{h}'$ is a Cartan algebra in apposition to $\mathfrak{h}$, and is given by

$$\mathcal{L}_e(\phi) = \frac{1}{2}\sum_{a=1}^{2}(\partial_a\phi, \partial_a\phi) - m^2(\mathrm{Ad}_{\exp(\phi)}(e), e^*).$$

Here $\partial_a := \partial/\partial x_a$.

The Euler –Lagrange equations of motion are

$$\mathcal{D}_e(\phi) := \Delta\phi + m^2[\mathrm{Ad}_{\exp(\phi)}(e), e^*] = 0, \tag{6.1.1}$$

where $\Delta\phi = \sum_{a=1}^{2} \partial_a^2\phi$. It is a system of r nonlinear differential equations of the second order. To write them down explicitly one uses the formula (1.1.2).

The usual ATFT correponds to the choice of $e \in \mathfrak{g}^{(1)}$ as in 4.2, cf. [Fr].

The linear approximation to the nonlinear equation (6.1.1) is a Klein – Gordon equation

$$\Delta_e\phi := \Delta\phi + m^2\,\mathrm{ad}_e\,\mathrm{ad}_{e^*}(\phi) = 0 \tag{6.1.2}$$

It admits r *normal mode* solutions

$$\phi_j(x_1, x_2) = e^{i(k_j x_1 + \omega_j x_2)} y_j,\ \ k_j^2 + \omega_j^2 = m^2\mu_j^2,$$

$1 \leq j \leq r$, where μ_j^2 are the eigenvalues of the square mass operator

$$M_e^2 := \mathrm{ad}_e\,\mathrm{ad}_{e^*} :\ \mathfrak{h} \longrightarrow \mathfrak{h}$$

and y_j are the corresponding eigenvectors, cf. [H] (1.4), (1.5).

In other words, (6.1.2) decouples into r equations describing scalar particles of masses μ_j, which explains the name *masses* for them.

Due to commutativity of $\mathfrak{h}'$, for all $e, e' \in \mathfrak{h}'$,

$$[\Delta_e, \Delta_{e'}] = 0.$$

7 Factorization patterns in Cartan eigenvectors

7.1. We recall the eigenvectors $\mu^{(i)}$ from Theorem 5.2; they are in bijection with the exponents k_i, $1 \leq i \leq r$. In particular, $\mu^{(1)}$ is a Perron – Frobenius eigenvector.

The exponents come in pairs $k_i, k_{r-i} = h - k_i$. According to Lemma 3.5, the eigenvector $\mu^{(r-i)}$ is obtained from $\mu^{(i)}$ by multiplying the coordinates by the sequence $(\epsilon(1), \ldots, \epsilon(r))$, with $\epsilon(j) = \pm 1$.

Below we will use the following notation. For a vector $v = (x_1, \ldots, x_r)$ we denote $\tilde{v} = (|x_1|, \ldots, |x_r|)$. For $\sigma \in S_r$ (the symmetric group), $v_\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(r)})$.

The notation $\gcd(a,b)$ will mean the greatest common divisor of a and b.

Consider first the case $\mathfrak{g} = \mathfrak{sl}(m)$. A Perron–Frobenius vector for the Lie algebra $\mathfrak{sl}(m)$ has the form

$$\mu^{(1)} = v_{PF}(m) := (\sin(\pi/m), \ldots, \sin((m-1)\pi/m)$$

Let $\mathfrak{g} = \mathfrak{sl}(n)$, and let us describe the other eigenvectors $\mu^{(i)}$, $1 \leq i \leq r = n-1$.

Let $p(i) = \gcd(i, n)$, $n = p(i)q(i)$.

Consider first the case $p(i) = 1$. In this case it is not difficult to see that all the components of $\mu^{(i)}$ are nonzero, and form, up to a sign, a permutation of the components of $\mu^{(1)}$.

(The permutations involved will be described in 7.2 below.)

For an arbitrary $p(i)$, among the components of $\mu^{(i)}$ there are exactly $p(i) - 1$ zeros, and the remaning $p(i)(q(i) - 1)$ components may be decomposed into $p(i)$ groups, the numbers inside each group forming, up to a sign, a Perron – Frobenius eigenvector for $\mathfrak{sl}(q(i))$.

7.1.1. Example. For the Cartan matrix of $\mathfrak{sl}(12)$, the eigenvectors are $\mu^{(i)}, 1 \leq i \leq 11$, with $\tilde{\mu}^{(i)} = \tilde{\mu}^{(12-i)}$. Then we have
$\tilde{\mu}^{(2)} = (v_{PF}(6), 0, v_{PF}(6))$, $\tilde{\mu}^{(3)} = (v_{PF}(4), 0, v_{PF}(4), 0, v_{PF}(4))$
$\tilde{\mu}^{(4)} = (v_{PF}(3), 0, v_{PF}(3), 0, v_{PF}(3), 0, v_{PF}(3))$ $\tilde{\mu}^{(5)} = v_{PF}(12)_\sigma$, with $\sigma = (1\ 5)(7\ 11)$,
$\tilde{\mu}^{(6)} = (v_{PF}(2), 0, v_{PF}(2), 0, v_{PF}(2), 0, v_{PF}(2), 0, v_{PF}(2), 0, v_{PF}(2))$.

□

For an arbitrary $\mathfrak{g}$ we have a similar pattern. Let R be a finite reduced irreducible root system of rank r and the Coxeter number h, $1 \leq i \leq r$, k_i the corresponding exponent.

Note that according to [B], Chapitre VI, §1, Proposition 30, all numbers $1 \leq k \leq h-1$ prime to h are among the exponents.

Let $p(i) = \gcd(k_i, h)$.

7.2. Proposition. Suppose that R is simply laced. (a) The eigenvector $\mu^{(i)}$ has all components different from 0 iff $p(i) = 1$, and if this is the case, we have

$$\tilde{\mu}^{(i)} = \tilde{\mu}^{(1)}_{\sigma_i}$$

for some $\sigma_i \in S_r$.

There are $\phi(h)/2$ such permutations σ_i, and they form a group isomorphic to $U(\mathbb{Z}/h\mathbb{Z})/\{1,-1\}$.

(b) If $p(i)$ is arbitrary, then one can associate to such i a root subsystem $R_{p(i)} \subset R$ whose Coxeter number is $q(i) = h/p(i)$ in such a way that the nonzero components of $\mu^{(i)}$ are decomposed into $p(i)$ groups, each group being, up to signs, a permutation of the coordinates of a PF vector for $R_{p(i)}$.

These facts may be verified case-by-case, using the explicit formulas for the vectors $\mu^{(i)}$ given in [Do], Table 2 on p. 659.

However, it would be desirable to have a uniform proof of this.

We believe that the same holds true for non-simply laced R as well.

7.3. Example. For the root system of type E_8, we have $h = 30$, the exponents are
$1, 7, 11, 13, 17, 19, 23, 29$; they include exatly all prime numbers ≤ 30 not dividing 30 (and 1). Let us denote the corresponding eigenvectors
$v_1, \ldots, v_{29}$, so that v_k has eigenvalue $2(1 - \cos(k\pi/30))$. The first one
$v_1 = v_{PF}$ is a Perron – Frobenius vector. It is equal to

$v_{PF} =$

$$(1, \frac{1}{\mu}(\mu^2-1), \mu, \mu^2-1, \frac{1}{\mu}(\mu^4-3\mu^2+1), \mu^4-4\mu^2+2, \frac{1}{\mu}(\mu^6-5\mu^4+5\mu^2-1), \mu^6-6\mu^4+9\mu^2-3)$$

where $\mu = 2\cos(\pi/30)$.

Then we have $\tilde{v}_1 = \tilde{v}_{29},\ \tilde{v}_7 = \tilde{v}_{23}, \tilde{v}_{11} = \tilde{v}_{19}, \tilde{v}_{13} = \tilde{v}_{17}$ and

$$\tilde{v}_7 = (v_1)_\sigma, \tilde{v}_{11} = (v_1)_{\sigma^2}, \tilde{v}_{13} = (v_1)_{\sigma^3}$$

with $\sigma = (1742)(3658) \in S_8$. The cyclic subgroup $G = \{1, \sigma, \sigma^2, \sigma^3\} \subset S_8$ is isomorphic to $U(\mathbb{Z}/30\mathbb{Z})/\{1,-1\}$.

□

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