# Asymptotic stability for a singular reaction-diffusion problem

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## 1 – Reaction-diffusion equations

The scalar extended FKPP equation writes

$$\partial_t \rho = \underbrace{-\varepsilon \partial_x^4 \rho + \partial_x^2 \rho}_{\text{diffusion term}} + \underbrace{\rho(1-\rho)}_{\text{reaction term}}, \qquad \text{(eFKPP)}$$

where $t > 0$, $x \in \mathbb{R}$ and the unknown $\rho(t,x) \in [0,1]$ is a concentration. Such an equation modelize a propagation phenomenon: tumor growth, invasion of species. When $\varepsilon = 0$, it reduces to the Fisher-KPP equation:

$$\partial_t \rho = \partial_x^2 \rho + \rho(1-\rho). \qquad \text{(FKPP)}$$

## 2 – Travelling front solutions

Both (FKPP) and (eFKPP) admits travelling front solutions of the form $\rho(t,x) = \bar{u}_c(x - ct)$ where the speed $c$ is greater than a critical speed $c_*(\varepsilon) = 2 - \varepsilon + o(\varepsilon)$.
Furthermore, $\bar{u}_c$ is monotonic, converges to 0 (resp. 1) at $+\infty$ (resp. $-\infty$), and satifies

$$\partial_t u = -\varepsilon \partial_x^4 u + \partial_x^2 u + c\partial_x u + u(1-u). \qquad (1)$$

For $\varepsilon = 0$, it was shown by KPP in 1937. For $\varepsilon > 0$ small enough, this result was proved by Rottschäfer and Wayne [3].

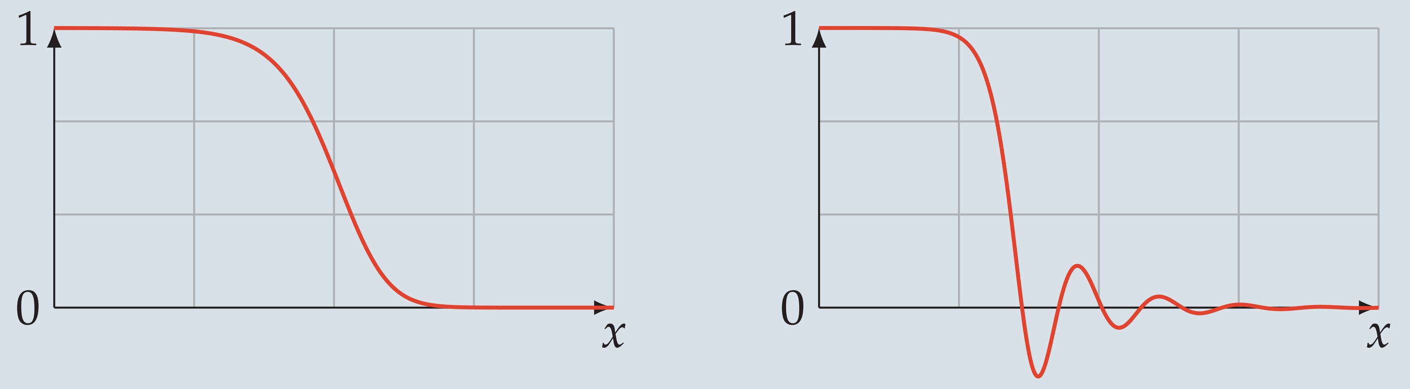


Figure 1: Fronts $\bar{u}_c(x)$ for $c \geq c_*$ at left and $c < c_*$ at right.

## 3 – Asymptotic stability

For the dynamic given by (1), $\bar{u}$ is an equilibrium. We want to study its asymptotic stability. With $u(t,x) = \bar{u}(x) + p(t,x)$, the perturbation $p$ satisfies

$$\partial_t p = Lp + \mathcal{N}(p), \qquad (2)$$

where $L : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a linear operator with domain $H^4(\mathbb{R})$, and $\mathcal{N}(p) = O(\|p\|^2)$.

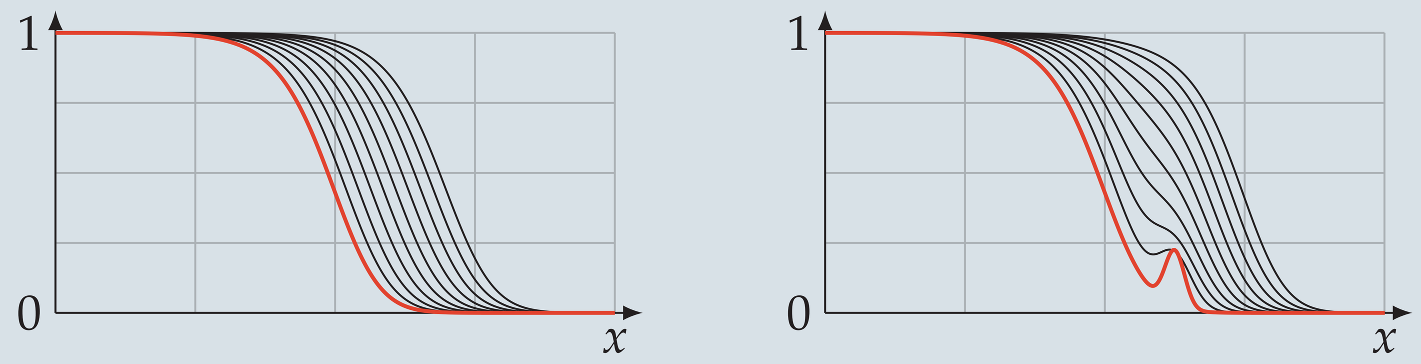


Figure 2: Numerical solutions of equation (1) : $\bar{u}(x)$ at left, and $u(t,x) = \bar{u}(x) + p(t,x)$ at right. The initial condition $u_0$ is plotted in red.

## 7 – Slow-fast dynamic

Writting the ODE satisfied by the front as both the slow and the fast system:

$$\begin{cases} u_0' = u_1, \\ u_1' = u_2, \\ \sqrt{\varepsilon} u_2' = u_3, \\ \sqrt{\varepsilon} u_3' = u_2 + cu_1 + u_0(1-u_0), \end{cases} \qquad \begin{cases} \dot{u}_0 = \sqrt{\varepsilon} u_1, \\ \dot{u}_1 = \sqrt{\varepsilon} u_2, \\ \dot{u}_2 = u_3, \\ \dot{u}_3 = u_2 + cu_1 + u_0(1-u_0), \end{cases}$$

[3] showed that $\bar{u}$ is smooth in $\varepsilon$.

## 8 – References

[1] G. FAYE and M. HOLZER. Asymptotic stability of the critical fisher–kpp front using pointwise estimates. *Zeitschrift für angewandte Mathematik und Physik*, 2018.

[2] P. HOWARD and C. HU. Pointwise Green's function estimates toward stability for multidimensional fourth-order viscous shock fronts. *Journal of Differential Equations*, 2005.

[3] V. ROTTSCHÄFER and C. E. WAYNE. Existence and stability of traveling fronts in the extended Fisher-Kolmogorov equation. *Journal of Differential Equations*, 2001.

## 4 – Stabilisation of linear spectrum

The linear dynamic of (2) is unstable (see figure 3). We restrict ourselves to perturbations that write

$$p(t,x) = \omega(x)\, q(t,x),$$

with $q \in H^4(\mathbb{R})$ and a weight $\omega$ (smooth positive function that decays exponentially). The equation satisfied by $q$ writes

$$\partial_t q = L_\omega q + \mathcal{N}_\omega(q), \qquad (3)$$

where $L_\omega = \omega^{-1} L \omega$. For a particular exponentially decaying weight $\omega$, the essential spectrum of $L_\omega$ lies at the left of the imaginary axis.

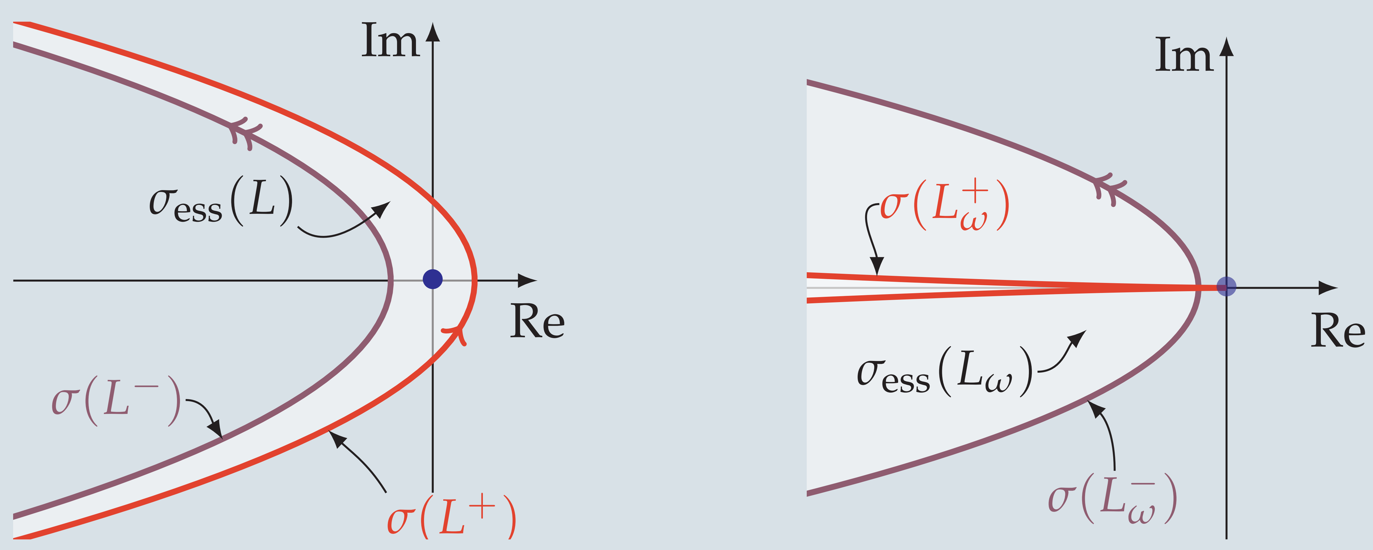


Figure 3: Essential spectrum of the linear operators. It is unstable for $L$ (at left) and stable for $L_\omega$ (at right).

## 5 – Known results

In the general case, [3] showed an asymptotic stability result thanks to energy estimates.

**Theorem – Asymptotic stability, $\varepsilon > 0$**

Take $0 < \varepsilon < \varepsilon_0$. There exists $\delta > 0$ such that for all $q_0 \in H^5(\mathbb{R})$ with $\|q_0\|_{H^3} \leq \delta$, equation (3) with initial condition $q_0$ admits a unique solution $q(t,x)$ that vanishes as $t \to +\infty$:

$$\|\omega\, q(t,\cdot)\|_{L^\infty} \longrightarrow 0.$$

In the case $\varepsilon = 0$, Faye and Holzer [1] showed a more precise result.

**Theorem – Explicit algebraic decay, $\varepsilon = 0$**

Take $\varepsilon = 0$. There exists $\delta > 0$ such that for all $q_0 \in H^2(\mathbb{R})$ with

$$\|q_0\|_{L^\infty} + \|(1+|x|)\, q_0(x)\|_{L^1} \leq \delta,$$

equation (3) with initial condition $q_0$ admits a unique solution $q(t,\cdot)$ that decays algebraically with time:

$$\left\| \frac{1}{1+|x|} q(t,x) \right\|_{L^\infty} \leq \frac{C\delta}{(1+t)^{3/2}}.$$

This algebraic decay was already obtained by Gallay. I aim to adapt the proof of this latter theorem in the case $\varepsilon > 0$.

## 6 – Laplace transform

We note $G(t,x,y)$ the Green kernel of the linear dynamic and $G_\lambda(x,y)$ its Laplace transform. Then for a suitable integration contour $\Gamma \subset \mathbb{C}$,

$$G(t,x,y) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} G_\lambda(x,y) \mathrm{d}\lambda. \qquad (4)$$

**Proposition – Control of spectral Green function**

For $|\lambda| \ll 1$ and $\lambda \notin \sigma(L_\omega)$ we have:

$$|G_\lambda(x,y)| \leq C(\varepsilon) e^{-\operatorname{Re}\sqrt{\lambda}|x-y|}.$$

Using a contour $\Gamma$ that depends on $t$, I obtained a short time control of the linear dynamic, similarly to what is done in [2].

**Proposition – Control of temporal Green function**

There exists $K, \eta > 0$ such that if $t < 1$, or if $|x-y| \geq K\, t\, \varepsilon^{1/4}$, we have

$$|G(t,x,y)| \leq \frac{C}{(\varepsilon\, t)^{1/4}} e^{-\eta \frac{|x-y|^{4/3}}{(\varepsilon t)^{1/3}}}.$$

The next goal is to control $G(t,x,y)$ for large time.