Spectral stability and singular perturbation

Louis Garénaux
Institut de Mathématiques de Toulouse

Montie Avery
University of Minnesota

Séminaire EDP, IRMAR

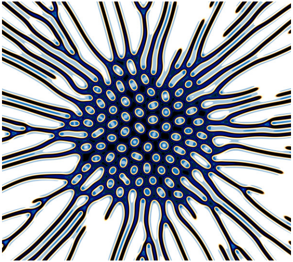
07/01/21

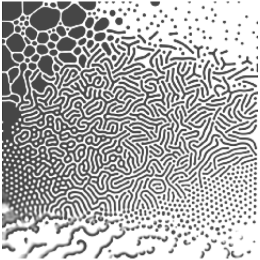
Reaction-difusion equation

$$\partial_t u = Du + R(u)$$

Examples: Gray-Scott equations:

$$\begin{cases} \partial_t u = \Delta u - uv^2 + a(1-u), \\ \partial_t v = d\Delta v + uv^2 + bv, \end{cases}$$





https://www.karlsims.com/rd.html

Fitzhugh-Nagumo, Allen-Cahn, bistable equations

A simple equation, and a perturbation

- The scalar, one dimensional FKPP equation writes:

$$\partial_t u = \partial_{xx} u + u(1-u), \tag{FKPP}$$

  where $u(t,x) \in \mathbb{R}$, $t > 0$ and $x \in \mathbb{R}$. First studied in 1937, it models the spreading of a population (bacteria, muskrat, chemical reaction).

- The extended FKPP equation writes:

$$\partial_t u = -\delta^2 \partial_{xxxx} u + \partial_{xx} u + u(1-u). \tag{eFKPP}$$

  It can be obtained as an amplitude equation for the solutions of a reaction-diffusion system, that undergoes a (co-dimension 2) bifurcation. In our case, it is a good model to see how properties depend on the equation.

For both problem, there exists front-like solutions that travel at constant speed c, greater than the critical speed $c_* = 2 + \mathrm{O}(\delta)$. See KPP 1937 and Rottschäfer–Wayne 2001.

Asymptotic stability of the front

We see the critical front q_* as an equilibrium, and study it stability with respect to perturbations $p \in H^4(\mathbb{R})$.

There exists $\varepsilon > 0$ such that for any $p_0 \in X$ that satisfy $\|p_0\| \leq \varepsilon$, the solution $u(t,x) = q_*(x) + p(t,x)$ is defined for all time, and the perturbation satisfy

$$\|p(t,\cdot)\| \leq f(t)\|p_0\|,$$

where $f(t) \longrightarrow 0$ when $t \to +\infty$.

1976 Sattinger: the supercritical fronts for FKPP are exponentially stable in weighted spaces.

1994 Gallay: the critical front for FKPP is algebraically stable in a weighted space: $f(t) = \frac{1}{(1+t)^{3/2}}$. Furthermore, the decay is optimal.

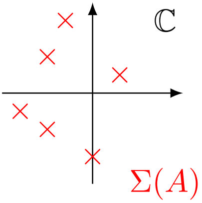
2018 Faye–Holzer: re-obtain the algebraic decay of Gallay by more robust method. The weak decay of the front $q_*'(x) \sim xe^{-x}$ at $+\infty$ is essential.

2001 Rottschäfer–Wayne: The critical front for eFKPP is stable in weighted space. No rate of convergence.

A dynamical point of view

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^n$ such that $F(\bar{y}) = 0$. The dynamic near the equilibrium is given by the spectrum of the matrix $A = dF(\bar{y})$. In our case, we want to study

$$\mathcal{A}(\delta) = -\delta^2\partial_{xxxx} + \partial_{xx} + c_*(\delta)\partial_x + (1 - 2q_*(\delta, x)).$$



Theorem (Avery–Scheel 2020)

If the linear dynamic satisfy the spectral hypotheses 1 through 4, then for any initial condition p_0 with $\|\omega p_0\|_{H^1_r(\mathbb{R})}$ small enough, the solution $u(t,x) = q_*(x) + p(t,x)$ of the nonlinear problem satisfy

$$\|\omega p(t,\cdot)\|_{H^1_{-r}} \leq \frac{C}{(1+t)^{3/2}} \|\omega p_0\|_{H^1_r}.$$

Furthermore the rate is optimal. There exists ψ that does not depends on p_0 such that $\|\omega p(t,\cdot) - t^{-3/2}\psi\|_{H^1_{-r}} \leq \frac{C}{(1+t)^2}\|\omega p_0\|_{H^1_r}$.

Definitions

We want to study the spectrum of

$$\mathcal{A}(\delta) = -\delta^2\partial_{xxxx} + \partial_{xx} + c_*(\delta)\partial_x + (1 - 2q_*(\delta, x)),$$

which is defined on the whole real line: $\mathcal{A}(\delta) : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$.

$$\Sigma(\mathcal{T}) = \{\lambda \in \mathbb{C} : \lambda - \mathcal{T} \text{ is not bounded invertible}\}.$$

It is the disjoint union of the point spectrum $\Sigma_{\mathrm{pt}}(\mathcal{T})$:

$$\dim\ker(\lambda - \mathcal{T}) - \operatorname{codim}\operatorname{im}(\lambda - \mathcal{T}) = 0,$$

and the essential spectrum $\Sigma_{\mathrm{ess}}(\mathcal{T})$:

$$\dim\ker(\lambda - \mathcal{T}) - \operatorname{codim}\operatorname{im}(\lambda - \mathcal{T}) \neq 0.$$

Furthermore, we say that λ is:

- an eigenvalue if there exists $u \in H^4(\mathbb{R})$ such that $\mathcal{T}u = \lambda u$,
- a resonance if there exists $u \in W^{4,\infty}(\mathbb{R})$ such that $\mathcal{T}u = \lambda u$.

Examples

- Differential operator with constant coefficients:

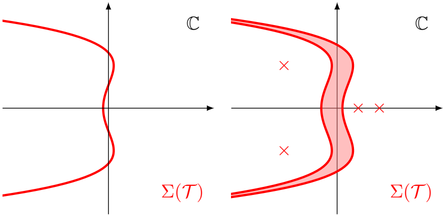
$$\mathcal{T} = a_m\partial_x^m + \cdots + a_1\partial_x + a_0 = P(\partial_x)$$

The essential spectrum is the curve $P(i\mathbb{R}) = \{P(ik) : k \in \mathbb{R}\}$. There is no point spectrum.

- Coefficients that converge at exponential speed to distinct limits at $\pm\infty$:

$$\mathcal{T}(x) = a_m(x)\partial_x^m + \cdots + a_1(x)\partial_x + a_0(x),$$

where both $\sup_{x>0}|a_i(x) - a_i^+|e^{\alpha|x|}$ and $\sup_{x<0}|a_i(x) - a_i^-|e^{\alpha|x|}$ are finite. Then the essential spectrum is contained in between the two curves $\Sigma(\mathcal{T}^+)$ and $\Sigma(\mathcal{T}^-)$. The point spectrum is not (necessarily) empty.



Sturm–Liouville theory

If $m = 2$, and if $a_i(x)$ are real, then $\Sigma_{\mathrm{pt}}(\mathcal{T})$ consists of a finite number of *simple* eigenvalues $\lambda_0 > \cdots > \lambda_N$, and the associated eigenfunction u_i vanishes exactly i-times.

Main Theorem

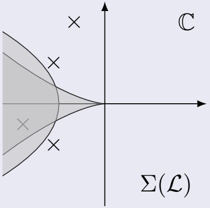
Theorem (Avery–G. 2020)

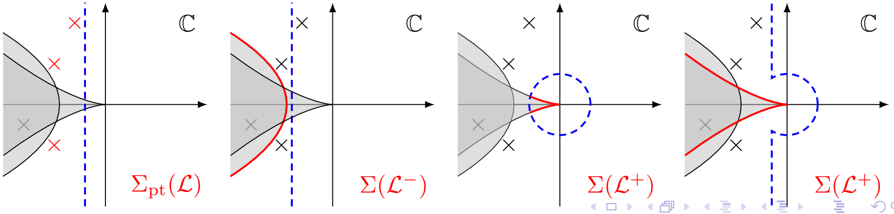
There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, the linear dynamic of (eFKPP) in a weighted space $\mathcal{L} = \omega \mathcal{A} \omega^{-1}$, satisfies the following spectral hypotheses.

1. Stable point spectrum and stable spectrum for $\mathcal{L}^-$.
2. Simple pinched double root at $(\lambda, k) = (0, 0)$ for $\mathcal{L}^+$:

$$P^+(ik) = -\alpha k^2 + \mathrm{O}(k^3).$$

3. Stable spectrum for $\mathcal{L}^+$, away from the origin.
4. No eigenvalue and no resonance at $\lambda = 0$.





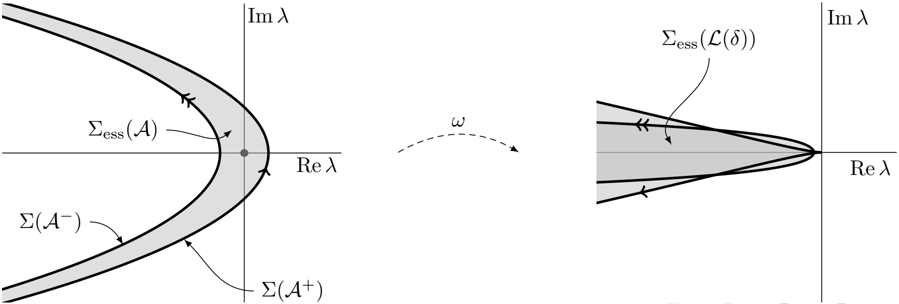
Essential spectrum for eFKPP

Proposition

There exists a weight ω such that $\mathcal{L} := \omega \mathcal{A} \omega^{-1}$ has marginally stable essential spectrum.

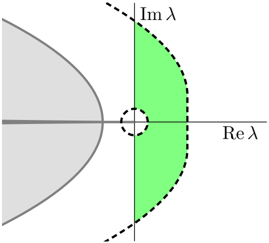
Proof: Choose ω of the form $\omega(x) = \begin{cases} e^{\eta_* x} & x \geq 1, \\ 1 & x \leq -1. \end{cases}$

Then the spectrum at $-\infty$ is unchanged: $\Sigma(\mathcal{L}^-) = \Sigma(\mathcal{A}^-) = P^-(i\mathbb{R})$, while the spectrum of $\mathcal{L}^+$ is the curve $P^+(-\eta_* + i\mathbb{R})$. □



Point spectrum for eFKPP – step 1

We decompose the study in three parts. Near the origin, outside of a compact set, and finally the intermediate domain.



Proposition

There exists a compact $K \subset \mathbb{C}$ such that $\mathcal{L}(\delta)$ has no unstable eigenvalues outside of K.

Proof: Use the ellipticity of $\mathcal{L}(\delta)$: assume that $\mathcal{L}(\delta)u = \lambda u$ and take the scalar product with u. It leads to estimates on $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$. □

Point spectrum for eFKPP – step 2

In the intermediate domain (green part), we treat $\mathcal{L}(\delta)$ as a perturbation of $\mathcal{L}(0)$. From Sturm-Liouville theory, the second order operator $\mathcal{L}(0)$ has no unstable point spectrum.

Proposition

There exists $\delta_0 > 0$ and $r(\delta_0) > 0$ such that if $0 < \delta < \delta_0$, then the operator $\mathcal{L}(\delta)$ has no eigenvalue inside $K \backslash B(0, r)$.

Proof: Remove the singularity by applying the preconditioner $(1 - \delta^2 \partial_{xx})^{-1}$. Computations leads to

$$(1 - \delta^2 \partial_{xx})^{-1} (\mathcal{L}(\delta) - \lambda) u = (\mathcal{L}(0) - \lambda) u + e(\delta, u),$$

where the error term $e(\delta, u)$ goes to 0 in $L^2(\mathbb{R})$ as $\delta \to 0$.
Since λ is away from the spectrum of $\mathcal{L}(0)$, we can invert $\mathcal{L}(0) - \lambda$. □

Point spectrum for eFKPP – step 3

When perturbing an operator, point spectrum can form near the essential spectrum. The previous step can not apply at the origin. Instead, we use a Lyapounov-Schmidt decomposition, as done in Pogan-Scheel 2010.

Proposition

There exists an holomorphic function $E : B(0, \delta_0) \times B(0, r_0) \longrightarrow \mathbb{C}$, such that the eigenproblem

$$\mathcal{L}(\delta)u = \lambda u$$

admits a bounded solution u if and only if $E(\delta, \sqrt{\lambda}) = 0$. Furthermore, $E(0,0) \neq 0$. In particular, there is no resonance at the origin.

Proof: Use implicit function theorem on a suitable formulation of the eigenproblem. Noting $v := (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \lambda)u$, the problem $v = 0$ is equivalent to

$$P_{\text{im}\, \mathcal{L}(0)} v = 0, \qquad \text{and} \qquad \langle v, \varphi \rangle = 0,$$

due to Fredholm property of $\mathcal{L}(0)$, and strong localization of u. We can solve the first equation locally, and use the second equation to define $E(\delta, \sqrt{\lambda})$. □

References

Thank's for your attention!

- M. Avery and L. Garénaux. Spectral stability of the critical front in the extended fisher-kpp equation, 2020.
- M. Avery and A. Scheel. Asymptotic stability of critical pulled fronts via resolvent expansions near the essential spectrum, 2020.
- G. Faye and M. Holzer. Asymptotic stability of the critical fisher–kpp front using pointwise estimates. *Zeitschrift für angewandte Mathematik und Physik*, 70(1), Nov 2018.
- T. Gallay. Local stability of critical fronts in nonlinear parabolic partial differential equations. *Nonlinearity*, 1994.
- A. Pogan and A. Scheel. Instability of spikes in the presence of conservation laws. *Zeitschrift für angewandte Mathematik und Physik*, 61:979–998, 12 2010.
- V. Rottschäfer and C. E. Wayne. Existence and stability of traveling fronts in the extended Fisher-Kolmogorov equation. *Journal of Differential Equations*, 2001.