Turing bifurcation behind a monostable front, stability

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1.1. A reaction-diffusion system

We consider a system that couples a Kolmogorov-Petrovski-Piskounov equation (KPP) together with a Swift-Hohenberg equation (SH).

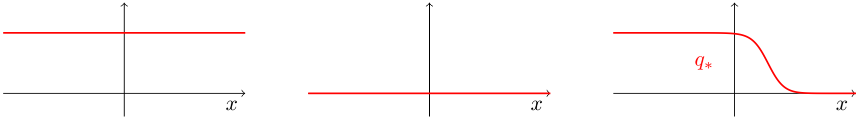
$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-u) + \beta v, \\ \partial_t v = -(1+\partial_{xx})^2 v + \mu v - \sigma v^3 - \gamma v(1-u), \end{cases} \quad (1)$$

with $t > 0$, $x \in \mathbb{R}$, and $u(t,x), v(t,x) \in \mathbb{R}$. It models the evolution of a species u, that invade space while undergoing a Turing instability – modeled by v – at his back.

The KPP equation

$$\partial_t u = \partial_{xx} u + u(1-u)$$

is a typical model for front propagation. Important solutions are the two steady states 1 and 0, together with a (two parameter) family of propagating fronts.

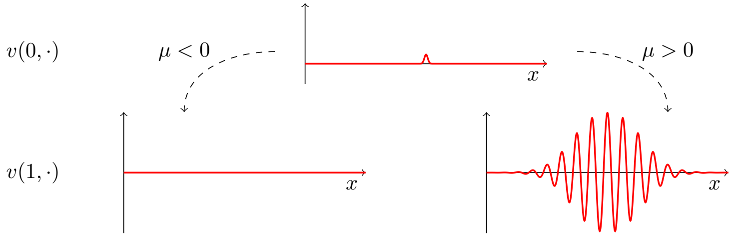


1.1. A reaction-diffusion system

The SH equation

$$\partial_t v = -(1 + \partial_{xx})^2 v + \mu v - v^3$$

is often used to describe the creation of oscillating profile through a Turing bifurcation. It admits the steady state 0, stable or unstable depending on the sign of μ.



For our system (1), we consider the propagating front

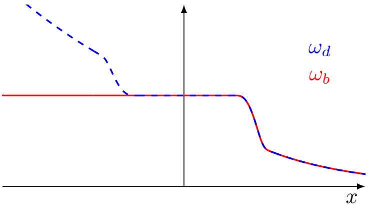
$$Q_*(x - ct) := (q_*(x - ct), 0)^{\mathrm{T}},$$

and investigate its stability when $\mu > 0$. In this setting, remark that both steady states $(1, 0)^{\mathrm{T}}$ and $(0, 0)^{\mathrm{T}}$ are unstable for the dynamic (1) *c.f.* [NPT11].

1.2. Main result

We note $U = (u, v)^{\mathrm{T}}$, and decompose it as:

$$U(t, x) = Q_*(x - ct) + P(t, x - ct).$$



We introduce two weights with exponential behaviors: ω_d and ω_b, and write the perturbation as

$$P(t, x) = \omega_b(x) P_b(t, x) = \omega_d(x) P_d(t, x).$$

Theorem 1

There exists δ, μ_0, C positive constants such that for all $0 < \mu < \mu_0$, if $\|P_d(0, \cdot)\|_X \leq \delta$ and $\|P_b(0, \cdot)\|_Y \leq \delta$, then the solution to (1) with initial condition $U_0 = Q_* + \omega_b P_b(0) = Q_* + \omega_d P_d(0)$ is defined for all times and satisfies:

$$\|P_d(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C \frac{\|P_d(0)\|_X}{(1+t)^{3/2}}, \qquad \|P_b(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C\sqrt{\mu}.$$

The front and the Turing pattern coexist, but are not coherent. It seems that there is no 'modulated front' to study.

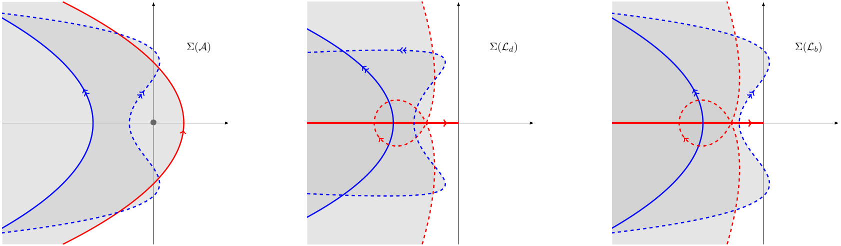
2.1. Choice of weights

The perturbation P satisfies

$$\partial_t P = \mathcal{A}P + N(P),$$

where the linear operator $\mathcal{A} : H^2(\mathbb{R}) \times H^4(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ has unstable spectrum: we can not hope for a stability result in this setting. However, writing $P = \omega_i P_i$ leads to

$$\partial_t P_i = (\omega_i^{-1}\mathcal{A}\omega_i)P_i + \frac{1}{\omega_i}N(\omega_i P_i).$$



When $x \geq 1$, the only possible choice is $\omega_d(x) = \omega_b(x) = q_*'(x)$. It erases the eigenmode associated to $\lambda = 0$. The Turing pattern is seen as a combination of the unstable 'eigenmodes' close to $x \mapsto e^{\pm ix}$.

2.2. Related problems

- *Decoupled system:* The slowest front q_* is stable for KPP dynamic with same algebraic decay rate $t^{-3/2}$ [FH18].
- *Bistable system:*
  Much works have been done for bistable fronts, *i.e.* replace $f(u) = u(1-u)$ by $g(u) = u(1-u)(u-\theta)$ with $\theta \in (0, 1/2)$, and consider a front connecting 1 to 0.

  When Turing bifurcation occurs behind the front, a similar result is obtained: the non-modulated front is stable. [SS01], [BGS09]. Furthermore, there does not exists a coherent structure that links the Turing pattern and the front.

  When Turing bifurcation occurs above the front, there is existence and non-linear stability of a modulated front [SS01], [GSU04], [GS07].

  In this setting, the spectral situation is nicer, which allows to obtain stability result up to a phase.

3.1. Decay of P_d

Claim: If P_b is bounded, then P_d decays.
Remind that

$$\partial_t P_d = \mathcal{L}_d P_d + \omega_d^{-1} N(\omega_d P_d).$$

At linear level, the claimed algebraic decay

$$\|e^{t\mathcal{L}_d} P_d(0)\|_{L^\infty(\mathbb{R})} \leq C \frac{\|P_d(0)\|_X}{(1+t)^{3/2}},$$

is obtained using point-wise estimate of Green's kernel for the resolvent [FH18].
A major issue is that ω_d is unbounded with respect to space. We use that

$$\omega_d^{-1}(\omega_d P_d)^n = (\omega_d P_d)^{n-1} P_d = (\omega_b P_b)^{n-1} P_d \leq P_b^{n-1} P_d,$$

and recover the decay at non-linear level.
Furthermore, the constant of decay is independent on the bound of P_b.

3.2. P_b is bounded

Claim: If P_d decays, then P_b is bounded.
We write

$$\begin{aligned}\partial_t P_b &= \mathcal{L}_b P_b + \mathcal{N}_b(P_b),\\ &= \mathcal{L}_b^- P_b + \mathcal{N}_b^-(P_b) + \mathcal{S}(t,x).\end{aligned}$$

Using that $\|\mathcal{S}(t)\|_{L^\infty(\mathbb{R})} \leq \|\omega_d P_b\|_{L^\infty(\mathbb{R})} \leq \|P_d\|_{L^\infty(\mathbb{R})}$, we recover decay of $\mathcal{S}$, which we consider as a vanishing source term.

Since $\mathcal{L}_b^-$ is a constant coefficient operator, it acts as multiplication in Fourier space, we can separate critical and stable frequencies. Set $\varepsilon^2 = \mu$ and insert the ansatz

$$P_b(t,x) = \varepsilon e^{ix} A(\varepsilon t, \varepsilon^2 x) + \varepsilon e^{-ix} \bar{A}(\varepsilon t, \varepsilon^2 x) + \mathcal{O}(\varepsilon^2),$$

then the amplitude A satifies a Ginzburg-Landau equation:

$$\partial_T A = 4\partial_{XX} A + A + bA|A|^2.$$

When σ is large enough, the coefficient b is negative – bifurcation is said to be super-critical – and GL admits a global attractor that can be shadowed to obtain that P_b is bounded in time.

3.3. Proof of main result

Claim: for all times, P_b is bounded and P_d decays.

Since P_b is defined locally in time, so is P_d using section 3.1. Hence solution exists for small bounded times.

Fix C_d the universal constant of decay in section 3.1, and note $t_0 > 0$ the first time where the constant may be overpass:

$$t_0 = \inf \left\{ t > 0 : \|P_d\|_{L^\infty(\mathbb{R})} > C_d \|P_d(0)\| (1+t)^{-3/2} \right\}.$$

Then from section 3.2, $\|P_b\|_{L^\infty(\mathbb{R})} \leq C$ when $t < t_0$.

However, using small time existence there exist $t_1 > 0$ such that $\|P_b\|_{L^\infty(\mathbb{R})} \leq 2C$, and applying section 3.1 again, we recover the decay of P_d with universal constant for times $t < t_1$. This contradicts the definition of t_0.

We conclude that for all times, both P_d decays and P_b is bounded.

References

Thanks for your attention!

M. Beck, A. Ghazaryan, and B. Sandstede. Nonlinear convective stability of travelling fronts near turing and hopf instabilities. *Journal of Differential Equations*, 246(11):4371 – 4390, 2009.

G. Faye and M. Holzer. Asymptotic stability of the critical fisher–kpp front using pointwise estimates. *Zeitschrift für angewandte Mathematik und Physik*, 70(1), Nov 2018.

A. Ghazaryan and B. Sandstede. Nonlinear convective instability of turing unstable fronts near onset: A case study. *SIAM J. Applied Dynamical Systems*, 6:319–347, 01 2007.

T. Gallay, G. Schneider, and H. Uecker. Stable transport of information near essentially unstable localized structures. *Discrete & Continuous Dynamical Systems - B*, 4(1531-3492-2004-2-349):349, 2004.

G. Nadin, B. Perthame, and M. Tang. Can a traveling wave connect two unstable states? the case of the nonlocal fisher equation. *Comptes Rendus Mathematique*, 349(9):553–557, 2011.