Convective stability of a propagation front connecting two unstable states

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General framework

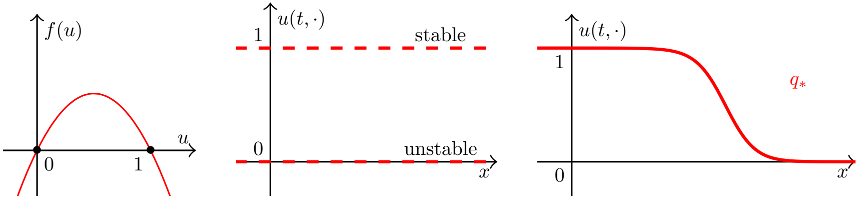
Reaction diffusion equations:

$$\partial_t u = Du + R(u)$$

Examples: Monostable (FKPP) equation in one space dimension: $t > 0$, $x \in \mathbb{R}$, u is scalar and

$$\partial_t u = \partial_{xx} u + u(1-u)$$

Two equilibria. For all speeds c larger than $c_* \stackrel{\text{def}}{=} 2$, there exists a front-like solution $u(t,x) = q_c(x - ct)$.



General framework

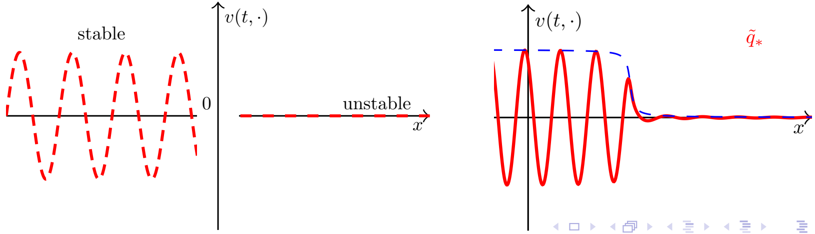
Reaction diffusion equations:

$$\partial_t u = Du + R(u)$$

Examples: Swift-Hohenberg equation in one space dimension: $t > 0$, $x \in \mathbb{R}$, v is scalar and

$$\partial_t v = -(1 + \partial_{xx})^2 v + \mu v - v^3$$

We only focus on the case μ positive and small. One constant equilibrium and a periodic equilibrium. For all speeds c larger than $\tilde{c}_* \stackrel{\text{def}}{=} 4\sqrt{\mu}$, there exists a front solution $v(t,x) = \tilde{q}_c(x - ct, x)$, see *Eckmann Wayne 91*.



Main system

Now couple the previous two equations:

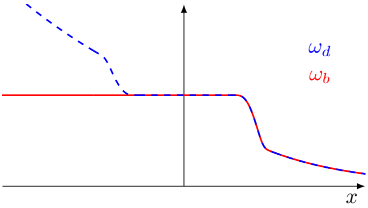
$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-u) + v, \\ \partial_t v = -(1+\partial_{xx})^2 v + \mu v - v^3 - v(1-u). \end{cases} \tag{1}$$

Remark that $Q(x-ct) \overset{\text{def}}{=} (q_*(x-ct), 0)^{\mathrm{T}}$ is still a solution. It connects the unstable state $(1,0)$ to the unstable state $(0,0)$. **Is Q stable?**

Note $U = (u,v)^{\mathrm{T}}$, assume it writes as: $U(t,x) = Q(x-ct) + P(t, x-ct)$.

To correct instabilities ahead and behind the front, we introduce two weights with exponential behaviors: ω_d and ω_b, and write the perturbation as

$$P(t,y) = \omega_b(y) P_b(t,y) = \omega_d(y) P_d(t,y).$$



We measure perturbations according to

$$\|\omega_b^{-1} P(t,\cdot)\|_{L^\infty(\mathbb{R})}, \qquad \|\omega_d^{-1} P(t,\cdot)\|_{L^\infty(\mathbb{R})}.$$

Main result

Theorem 1

There exists δ, μ_0, C positive constants such that for all $0 < \mu < \mu_0$, if $\|P_d(0,\cdot)\|_{W^{4,\infty}(\mathbb{R})} \leq \delta$ and $\|P_b(0,\cdot)\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$, then the solution to (1) with initial condition $U_0 = Q + \omega_b P_b(0) = Q + \omega_d P_d(0)$ is defined for all times and satisfies:

$$\|P_d(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C \frac{\|P_d(0)\|_{W^{4,\infty}(\mathbb{R})}}{(1+t)^{3/2}}, \qquad \|P_b(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\sqrt{\mu}.$$

The front and the Turing pattern coexist, but are not coherent: they travel at different speeds. The front without oscillations is stable.

A similar result holds true for bistable fronts with the state behind that is Turing unstable: see *Sandstede Scheel 01*, *Beck Ghazaryan Sandstede 09*.

Idea of proof: use simultaneously the two dynamics:

$$\partial_t P_d = \mathcal{L}_d P_d + \mathcal{N}_d(P_d), \qquad \partial_t P_b = \mathcal{L}_b P_b + \mathcal{N}_b(P_b).$$

Stability of fronts: a dynamical approach

Main idea: adapt the techniques from ODE (finite dimension).

$$\partial_t p = \mathcal{L}p + \mathcal{N}(p).$$

First, show that solutions of the linear problem decay in time:

$$\|e^{t\mathcal{L}} p_0\| \leq h(t)\|p_0\|.$$

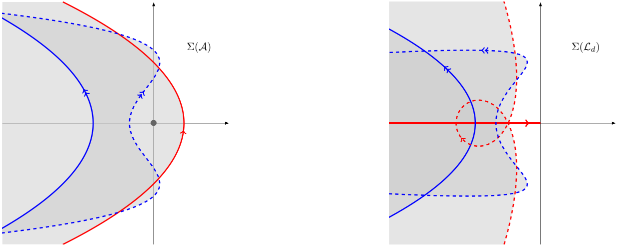
Second, obtain decay at non-linear level from the Duhamel formula:

$$p(t,x) = e^{t\mathcal{L}} p_0(x) + \int_0^t e^{(t-\tau)\mathcal{L}} \mathcal{N}(p(\tau,x))\mathrm{d}\tau.$$

Control of P_d: linear dynamic

Idea: study the spectrum of $\mathcal{L}_d$ and $\mathcal{L}_b$. If $\Sigma(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq \eta\}$, then

$$\|e^{t\mathcal{L}} p_0\| \leq e^{\eta t} \|p_0\|.$$



To study $e^{t\mathcal{L}_d}$, use pointwise bounds, *c.f.* *Faye Holzer 18* :

$$|e^{t\mathcal{L}_d} p_0(x)| \leq \int_{\mathbb{R}} \frac{1 + |x - x'|}{(1+t)^{3/2}} e^{-\frac{|x-x'|^2}{t}} |p_0(x')| \mathrm{d}x'.$$

Which then leads to

$$\|e^{t\mathcal{L}_d} p_0\|_{L^\infty(\mathbb{R})} \leq \frac{C}{(1+t)^{3/2}} \|x \mapsto (1+|x|)^2 p_0(x)\|_{L^1(\mathbb{R})}.$$

Control of P_d: non-linear dynamic

To stabilize all curves, it is necessary to take ω_d unbounded with respect to x.
Problem: then the coefficients of $\mathcal{N}_d(P_d)$ are unbounded:

$$\text{If } \mathcal{N}(P) = P^k, \text{ then } \mathcal{N}_d(P_d) = \omega_d^{k-1} P_d^k.$$

The non-linear terms are not Lipschitz, there may not be existence of solutions.

Solution: use the weaker weight to define solutions: If $\mathcal{N}(P) = P^k$, then

$$\begin{aligned} \mathcal{N}_d(P_d) &= (\omega_d P_d)^{k-1} P_d, \\ &= (\omega_b P_b)^{k-1} P_d, \\ &\leq C P_d. \end{aligned}$$

Control of P_b: amplitude equation

First idea: P_b is driven by the dynamic at $-\infty$:

$$\partial_t P_b = \mathcal{L}_b^- P_b + \mathcal{N}_b^-(P_b) + \mathcal{H},$$

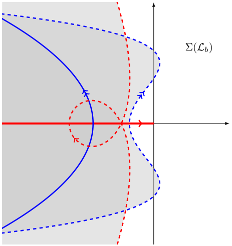
with a forcing term that decays fast enough:

$$\|\mathcal{H}(t)\|_{W^{1,\infty}(\mathbb{R})} \leq \|P_d(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{(1+t)^{3/2}}.$$

Second idea: the linear part is unstable, but nonlinear terms stabilize the dynamic. Note $\varepsilon \overset{\text{def}}{=} \sqrt{\mu}$, and assume that

$$P_b(t,x) = \mathrm{Re}\left(e^{ix} A(\varepsilon^2 t, \varepsilon x)\right).$$

Then A satisfies a "complex KPP" equation (*i.e.* a Ginzburg-Landau equation).
Follows *Schneider 94*.



References

Thanks for your attention!

M. Beck, A. Ghazaryan, and B. Sandstede. Nonlinear convective stability of travelling fronts near turing and hopf instabilities. *Journal of Differential Equations*, 246(11):4371 – 4390, 2009.

J.-P. Eckmann and C. E. Wayne. Propagating fronts and the center manifold theorem. *Communications in Mathematical Physics*, 136(2):285–307, Feb 1991.

G. Faye and M. Holzer. Asymptotic stability of the critical Fisher–KPP front using pointwise estimates. *Zeitschrift für angewandte Mathematik und Physik*, 70(1), Nov 2018.

G. Schneider. A new estimate for the Ginzburg-Landau approximation on the real axis. *Journal of Nonlinear Science*, 4(1):23–34, Dec 1994.

B. Sandstede and A. Scheel. Essential instabilities of fronts: bifurcation, and bifurcation failure. *Dynamical Systems*, 16(1):1–28, 2001.