Burger's equation

Internship report

Louis Garénaux

ENS Rennes

Universidad de Sevilla - España

Departemento EDAN

2016-2017

Contents

Introduction

I write this internship report with a precise idea in mind. This text must reflect my 2 months work with as much fidelity as possible. For this reason, and because I spoke Spanish when working, it seems important for me that – at least – one chapter is writen in Spanish.

∴

Escribo este reporte de practica con una idea precisa en la mente. Este texto debe ser el reflejo mas exacto de mis 2 meses de trabajo. Por eso, y porque hablé Español con los otros matemáticos, me parece importante que – por lo menos – uno capitulo sea escrito en Español.

This report is structured as follow. First we study the heat equation, mainly for smooth solutions. Then we talk about the characteristics method, a general results about existence and uniqueness of first order partial differential equation (PDE). In a third chapter we study the theory of conservation law. We restrict ourself to the scalar case. See [4] for systems of conservation laws. Finally we take a look at numerical resolution of this conservation laws. The transition between each part may be a little confuse, I wanted to summarize what interest me the most during this internship.

Notations

We will note

- $[\![a, b]\!]$ if $a, b \in \mathbb{Z}$, to stand for $\{n \in \mathbb{Z} : a \leq n \leq b\}$,
- $u \cdot v$ if $u, v \in \mathbb{R}^n$, to stand for the euclidean scalar product between u and v,
- $\lambda \times u$ if $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ to stand for the scalar multiplication of u by λ. We may note this by simply λu,
- u_i if $u \in \mathbb{R}^n$ and $i \in [\![1, n]\!]$, to stand for the i-th component of u,
- u_{x_i} if x_i is a variable of a function u, to stand for $\frac{\partial u}{\partial x_i}$, we may also use the notation $\partial_{x_i} u$,
- $Df(x)$ if $f : E \longrightarrow F$ is a function and $x \in E$, to stand for the differential of f at point x. As this object is a linear application, we will note its evaluation by $Df(x) \cdot h$, for $h \in E$,

$\nu(x)$ to stand for a unit normal at point x to a surface,

$\delta_{a,b}$ if $a, b \in \mathbb{Z}$ to stand for the Kronecker symbol: $\delta_{a,b} := 1$ if $a = b$, else $\delta_{a,b} := 0$,

Δu if u is a twice differentiable function (in the classical sens, or in the distribution sens) of x, to stand for $u_{x,x}$,

$\|x\|$ if $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ to stand for the euclidean norm $\sqrt{{x_1}^2 + \cdots + {x_n}^2}$,

$|x|$ if $x \in \mathbb{R}$ to stand for the absolute value of x,

$B(x, r)$ if $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+^*$ to stand for the open ball of center x and radius r,

$\mathcal{D}(\Omega)$ if Ω is open in an normed vector space, to stand for $\mathcal{C}_c^\infty(\Omega)$ the set of $\mathcal{C}^\infty$ functions with compact support.

Agradecimientos

Quiero agradecer a

Manuel Gonzáles Burgos por haber dirigido está practica,

la Universidad de Sevilla por acogerme en sus despachos y su biblioteca,

todos los estudiantes en doctorado del departamento EDAN,

Karine Beauchard por ponerme en contacto con Manuel Gonzáles Burgos.

Chapter 1

Ecuación del calor

Se puede dividir las EDP sencillas en tres grandes familias. Las parabólicas, elípticas e hiperbólicas. En este capitulo vamos a estudiar el ejemplo mas simple de ecuación parabólica. Esté estudio nos enseña técnicas simples para resolver otras ecuaciones similares.

La ecuacíon del calor que vamos a estudiar es un modelo simple de la evolución de la temperatura de un sistema cón una fuente de calentamiento. Veremos que las soluciones que vamos a obtener tienen comportamientos que no satisfacen leyes físicas.

Trabajemos en U abierto de $\mathbb{R}^n$ que representa el espacio, y en $\mathbb{R}_+$ para el tiempo. Suponemos que U no es acotado. Supongamos conocidos

- ⋆ una función $f : U\times \longrightarrow \mathbb{R}$, que representa una fuente de calor en cada punto x del espacio y para cada tiempo t ;
- ⋆ otra función $g : U \longrightarrow \mathbb{R}$, que representa la temperatura para el tiempo inicial – tomamos $t = 0$ – en cada punto x del espacio.

Notaremos entonces

$$\left\{ \begin{array}{ll} u_t - u_{x,x} = f & U \times \mathbb{R}_+^* \\ u(0,x) = g(x) & x \in U \end{array} \right. \tag{1.1}$$

la ecuación – y su condición inicial – estudiada.

1.1 Caso homogéneo

Para resolver la ecuación, empezamos por el caso donde $f = 0$. Vamos a buscar primero una solución particular. Supongamos que existen soluciones que conservan su estructura cuando dilatamos el tiempo y el espacio. Es decir que existe α, β tales que para todo $\lambda \in \mathbb{R}_+^*$,

$$u(t,x) = \lambda^\alpha u(\lambda t, \lambda^\beta x).$$

Para $\lambda := \frac{1}{t}$ – trabajamos aquí con $t > 0$ – obtenemos una expresión para u :

$$u(t,x) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right),$$

donde $v(x) := u(1, x)$, para $x \in U$. Si inyectemos esta expresión en (1.1), obtenemos

$$\alpha t^{-\alpha-1} v(y) + \beta t^{-\alpha-1} y \cdot Dv(y) + t^{-\alpha-2\beta} v_{x,x}(y) = 0$$

para $y := \frac{x}{t^\beta}$. Nos gustaremos simplificar esta ecuación, elegimos $\beta = \frac{1}{2}$. Así multiplicando por $t^{\alpha+1}$ obtenemos

$$\alpha v(y) + \frac{1}{2} y \cdot Dv(y) + \Delta v(y) = 0. \tag{1.2}$$

Para simplificar aún mas, pasamos en dimensión una. Para eso suponemos que la solución particular que buscamos es radial. Es decir que existe $w : \mathbb{R} \longrightarrow \mathbb{R}$ tal que $v(y) = w(\|y\|)$. Entonces (1.2) vuelve

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0.$$

Si tomamos $\alpha = \frac{n}{2}$, podemos poner esta ecuación bajo la forma

$$\left(r^{n-1} w'\right)' + \frac{1}{2} \left(r^n w\right)' = 0,$$

y entonces existe $a \in \mathbb{R}$ tal que

$$r^{n-1} w' + \frac{1}{2} r^n w = a. \tag{1.3}$$

Podemos suponer que nuestra solución desaparece rapidamente al infinito – mas rapidamente que cualquier polinomio – y entonces tomamos $a = 0$. Finalmente (1.3) da

$$w' = -\frac{1}{2} r w,$$

cuyas soluciones son de la forma

$$w(r) = b e^{-\frac{r^2}{4}}$$

para $b \in \mathbb{R}$. Eso nos permite decir que

$$u(t, x) := \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

es solucion de (1.1) para tiempos $t > 0$.

Definición – Solución fundamental Definemos, para $t, x \in \mathbb{R} \times \mathbb{R}^n$:

$$\phi(t, x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{4t}} & \text{si } t \in \mathbb{R}_+^*, \\ 0 & \text{sino.} \end{cases} \tag{1.4}$$

Llamamos solución fundamental de la ecuación (1.1) está función.

Hemos elegido una constante b particular para que el resultado siguiente aguanta.

Proposición Para todo $t \in \mathbb{R}_+^*$

(1.5) $$\int_{\mathbb{R}^n} \phi(t,x)\mathrm{d}x = 1.$$

Demostración. Hacemos un cambio de variable $y = \frac{x}{2\sqrt{t}}$:

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(t,x)\mathrm{d}x &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|y\|^2}\mathrm{d}y \\ &= \frac{1}{\pi^{\frac{n}{2}}} \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-y_j^2}\mathrm{d}y_j\end{aligned}$$

según Fubini-Tonelli. Cada una de estas integrales es una integral de Gauss, y vale $\sqrt{\pi}$. □

Para resolver totalmente (1.1), tenemos que encontrar una función que cumple también la condición de transborde.

1.1. Theorema – Solución en caso homogéneo Suponemos que $g \in \mathcal{C} \cap L^\infty$, y notamos, para $(t,x) \in \mathbb{R} \times \mathbb{R}^n$

(1.6) $$\psi(t,x) := \int_{\mathbb{R}^n} \phi(t,x-y)g(y)\mathrm{d}y.$$

Entonces, ψ satisface (1.1) en el caso homogéneo $f = 0$.

Demostración.
1. La definición de ψ no es clara. En efecto g solamente es definido para $y \in U$. Por eso en el teorema, y en resto de la sección, prolongamos g sobre $\mathbb{R}^n$ por 0 sobre $\mathbb{R}^n - U$.
2. Si para $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ notamos $\phi^t(x) := \phi(t,x)$, podemos notar que $\psi(t,x) = (\phi^t \star g)(x)$, donde $\star$ denota la convolución. Así, podemos poner las derivadas parciales sobre ϕ^t, o sea sobre ϕ. Sea $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n$, entonces

$$\psi_t(t,x) - \Delta\psi(t,x) = \int_{R^n} \left[\phi_t(t,x-y) - \Delta\phi(t,x-y)\right] g(y)\mathrm{d}y = 0,$$

puesto que ϕ satisface la ecuación del calor para $t > 0$.
3. Sea $x_0 \in \mathbb{R}^n$, y $\varepsilon > 0$. Por continuidad de g sobre el abierto U, elegimos $\delta \in \mathbb{R}_+^*$ tal que para todos $x \in \mathbb{R}^n$ tales que $\|x_0 - x\| < \delta$, tenemos

(1.7) $$|g(x_0) - g(x)| < \varepsilon.$$

Entonces para $x \in \mathbb{R}^n$ tal que $\|x_0 - x\| \leq \frac{\delta}{2}$, tenemos gracias a (1.5)

$$\begin{aligned}|\psi(t,x) - g(x_0)| &= \left|\int_{\mathbb{R}^n} \phi(t,x-y)\left[g(y) - g(x_0)\right]\mathrm{d}y\right| \\ &\leq \int_{B(x_0,\varepsilon)} \phi(t,x-y)\left|g(y) - g(x_0)\right|\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n - B(x_0,\varepsilon)} \phi(t,x-y)\left|g(y) - g(x_0)\right|\mathrm{d}y\end{aligned}$$

Notamos I y J estas dos integrales.

Primero por (1.7) y (1.5), tenemos

$$I \leq \varepsilon \int_{B(x_0,\varepsilon)} \phi(t, x-y)\mathrm{d}y \leq \varepsilon. \tag{1.8}$$

Por otra parte, en J tenemos $\|x_0 - x\| \leq \frac{\delta}{2}$ y $\delta \leq \|y - x_0\|$. Por eso

$$\|y - x_0\| \leq \|y - x\| + \frac{\delta}{2} \leq \|y - x\| + \frac{1}{2}\|y - x_0\|,$$

y entonces $\frac{1}{2}\|y - x_0\| \leq \|y - x\|$. Por eso

$$\begin{aligned} J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x_0,\varepsilon)} \phi(t, x-y)\mathrm{d}y \\ &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B(x_0,\varepsilon)} e^{-\frac{\|y-x_0\|^2}{16t}}\mathrm{d}y. \end{aligned}$$

Si hacemos otro cambio de variable $z = \frac{y-x_0}{4\sqrt{t}}$, obtenemos

$$J \leq \tilde{C} \int_{\mathbb{R}^n - B\left(0, \frac{\delta}{4\sqrt{t}}\right)} e^{-z^2}\mathrm{d}z,$$

y para $t \to 0$, $J \to 0$. Entonces para t bastante pequeño, y x cerca de x_0 tenemos $|\psi(t,x) - g(x_0)| \leq 2\varepsilon$. Es decir que ψ satisface la condición de transbordo. $\square$

Para obtener una solución de nuestra problema, basta restringir ψ sobre U. Mostraremos luego que es la única solución del caso homogéneo.

1.2 Caso de una condición inicial nula

Estudiamos ahora el caso donde $g = 0$. f puede ser no-nula.

La idea es imaginar que la fuente de calor $f(.,s)$ a tiempo s es el dato inicial para el tiempo inicial s. Se puede también verlo al revés : en el caso precedente, se podía ver la condición inicial g como una impulsión que se dara al sistema a tiempo 0. Es decir que una condición inicial parece a una fuente de calor a tiempo 0.

Luego sumaremos las contribuciónes de todos tiempos $s \in [0,t]$ (la ecuación es lineal), es el principio de Duhamel. Para $x \in \mathbb{R}^n$, $t, s \in \mathbb{R}$, se nota

$$v(t, x; s) := \int_{\mathbb{R}^n} \phi(t-s, x-y) f(s,y)\mathrm{d}y. \tag{1.9}$$

Fijamos $s \in \mathbb{R}$. Al ver las soluciones (1.6) en el caso homogéneo, $v(.;s)$ está solucion del problema siguiente :

$$\left\{ \begin{array}{ll} v_t - \Delta u = 0 & \mathbb{R}_+ \times \mathbb{R}^n \\ v(0, \cdot; s) = f(s, \cdot) & \mathbb{R}^n \end{array} \right. \tag{1.10}$$

Como dicho antes, sumamos cada una de esta soluciones : para $t, x \in \mathbb{R}_+^* \times \mathbb{R}^n$ notamos

$$u(t,x) = \int_0^t v(t, x; s)\mathrm{d}s. \tag{1.11}$$

1.2. Theorema – Solución para condición inicial nula Definimos u por (1.11). Suponemos que $f \in \mathcal{C}_1^2(\mathbb{R}_+ \times \mathbb{R}^n)$ tiene soporte compacto. Entonces u satisface (1.1) para $g = 0$.

Demostración. La notación $\mathcal{C}_1^2$ significa que f es $\mathcal{C}^2$ en espacio y $\mathcal{C}^1$ en tiempo

1. Empezamos por un cambio de variable

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} \phi(s,y) f(t-s, x-y) \mathrm{d}y \mathrm{d}s.$$

Derivamos con respeto al tiempo :

$$u_t(t,x) = \int_0^t \int_{\mathbb{R}^n} \phi(s,y) f_t(t-s, x-y) \mathrm{d}y \mathrm{d}s + \int_{\mathbb{R}^n} \phi(t,y) f(0, x-y) \mathrm{d}y, \tag{1.12}$$

y dos veces en espacio :

$$u_{x_i,x_j}(t,x) = \int_0^t \int_{\mathbb{R}^n} \phi(s,y) f_{x_i,x_j}(t-s, x-y) \mathrm{d}y \mathrm{d}s \tag{1.13}$$

Para mas comodidad, introducimos el operador $\natural$: $\natural u = u_t - \Delta u$. (1.12) y (1.13) dan

$$\begin{aligned}
\natural u(t,x) &= \int_0^t \int_{\mathbb{R}^n} \phi(s,y) \natural f(t-s, x-y) \mathrm{d}y \mathrm{d}s + \int_{\mathbb{R}^n} \phi(t,y) f(0, x-y) \mathrm{d}y, \\
&= \int_0^\varepsilon \int_{\mathbb{R}^n} \phi(s,y) \natural f(t-s, x-y) \mathrm{d}y \mathrm{d}s \\
&\quad + \int_\varepsilon^t \int_{\mathbb{R}^n} \phi(s,y) \natural f(t-s, x-y) \mathrm{d}y \mathrm{d}s \\
&\quad + \int_{\mathbb{R}^n} \phi(t,y) f(0, x-y) \mathrm{d}y, \\
&=: I_\varepsilon + J_\varepsilon + K.
\end{aligned}$$

2. I_ε se acota fácilmente por $C\varepsilon$, donde $C \in \mathbb{R}_+^*$. Para J_ε, se pone las derivadas parciales sobre ϕ (integración por partes) que satisface la ecuación del calor. Así se encuentra

$$J_\varepsilon + K = \int_{R^n} \phi(\varepsilon, y) f(t-\varepsilon, x-y) \mathrm{d}y. \tag{1.14}$$

Para acotar la parte derecha en (1.14), basta utilizar una técnica similar a la etapa *3* de la prueva del theorema 1.1. Para mas detalle, ver [4] p. 50. □

1.3 Caso general y unicidad

Como la ecuación es linear, se puede sumar las soluciones de las dos partidas precedentes. En el caso general

$$u(t,x) = \int_{\mathbb{R}^n} \phi(t, x-y) g(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \phi(t-s, x-y) f(s,y) \mathrm{d}y \mathrm{d}s$$

es una solución.

¿ Quid la unicidad ? Vamos a mirar primero el caso donde U no es $\mathbb{R}^n$.

U acotado

Nos restringimos a un dominio $U \subset \mathbb{R}^n$ acotado y conexo. Necesitamos plantear un nuevo problema. Para $T \in \mathbb{R}_+^*$, notamos $U_T := U \times]0,T]$ y $\Gamma_T := \bar{U}_T - U_T$. Trabajamos entonces con el problema

$$(1.15) \qquad \begin{cases} u_t - u_{x,x} = f & U_T \\ u(0,\cdot) = g & \Gamma_T \end{cases}$$

Observación Fijase que se impone condiciones de transbordo para tiempos estrictamente positivos : $\Gamma_T = U \times \{0\} \bigcup \partial U \times]0,T[$. Por eso no podemos utilizar las soluciones ya descubiertas.

Este problema tiene una única solución. Eso viene del principio siguiente (prueba p. 53 en [4]) :

1.3. Theorema – Principio del máximo Fijemos $T \in \mathbb{R}_+^*$. Suponemos definido $u \in \mathcal{C}_1^2(U_T) \cap \mathcal{C}(\bar{U}_T)$ que satisface (1.15).

(*i*) Tenemos

$$\max_{\bar{U}_T} u = \sup_{\Gamma_T} u.$$

(*ii*) Si ademas U es conexo y si existe $(t_0, x_0) \in U_T$ tal que

$$u(x_0, t_0) = \max_{\bar{U}_T} u,$$

entonces u es constante sobre U_{t_0}.

Eso da la unicidad de solución al problema (1.15). Sean $u, \tilde{u}$ soluciones del problema. Basta aplicar el primero punto del principio del máximo a $u - \tilde{u}$ y $\tilde{u} - u$.

$U = \mathbb{R}^n$

En este caso, tenemos resultados similares. Solamente necesitamos acotar las soluciones en espacio.

1.4. Theorema Suponemos $f \in \mathcal{C}([0,T] \times \mathbb{R}^n), g \in \mathcal{C}(\mathbb{R}^n)$, y fijamos $a, b \in \mathbb{R}_+^*$. Entonces hay por lo mas una solución u de

$$\begin{cases} u_t - u_{x,x} = f & [0,T] \times \mathbb{R}^n \\ u(.,0) = g & \mathbb{R}^n \end{cases}$$

tal que para todo $x \in \mathbb{R}^n$, $t \in [0,T]$,

$$|u(t,x)| \leq a e^{b|x|^2}.$$

La prueba se construye de la misma manera : se muestra primero un principio del máximo para soluciones acotadas.

1.4 Otra ecuación del calor

Esta ecuación tiene un efecto regularizante muy fuerte : aún que g sea solamente $\mathcal{C}^0$, la convolución con ϕ hace que para todo tiempo $t > 0$, la solución es totalmente regular. Esta regularización no es físicamente aceptable (la información viaja con una velocidad infinita).

Otro fenómeno de transporte es el siguiente. Suponemos que $g \geq 0$ y que g no es idénticamente nula. Entonces para $t > 0$, la expresión de ϕ (1.6) nos asegura en el caso homogéneo que para todo $x \in U$, $\phi(t, x) > 0$. Para ver eso basta elegir un $x_0 \in U$ tal que $g(x_0) \neq 0$, y por continuidad de g sobre U, existe ε tal que

$$\int_{B(x_0,\varepsilon)} \phi(t, x-y)g(y)\mathrm{d}y > 0.$$

Esto vale para todo $t > 0$, tan pequeño sea. Es decir que la solución se vuelve inmediatamente no-nula en todo U, aun que g sea nula en algún lugar. Entra en contradicción con el echo físico de que la información solamente puede propagarse a velocidad finita.

Para obtener un modelo mas realista, se puede cambiar el termino de conductividad $u_{x,x}$. Para establecer la ecuación clásica a partir de leyes físicas, se utiliza la ley de Fourier

$$q(t, x) = -\lambda \mathrm{grad}(T(t, x))$$

como expresión del flujo de calor, donde λ es una constante. Gurtin y Pipkin – ver [5] – han utilizado otra formula para el flujo de energía

$$q(t, x) = -T(t, x)^\gamma \int_0^t K(t-\tau)T_x(\tau, x)\mathrm{d}\tau, \tag{1.16}$$

donde $\gamma \geq -1$ y K es el núcleo de relajación. La integral entre 0 y t representa un efecto de memoria. La idea es que el calor no se transmite inmediatamente. Para conocer la temperatura en x a tiempo t, es entonces necesario saber lo que ha pasado para los tiempos $\tau \leq t$ (por eso hay una convolución, una integral) en los puntos del espacio cercanos de x (por eso se utiliza T_x). Utilizar (1.16) en vez de la ley de Fourier lleva a la ecuación del calor con memoria

$$u_t - \frac{\partial}{\partial x} \int_0^t K(t-\tau)\left(T^\gamma T_x\right)(\tau, x)\mathrm{d}\tau = 0. \tag{1.17}$$

Las soluciones de (1.17) ya no tienen una velocidad infinita para el transporte de información. Si se utiliza para K un dirac [1] y $\gamma = 0$, se encuentra la ecuación del calor clásica.

[1] Hay que tomar $K = \rho_\varepsilon$ una aproximación de la unidad, y hacer $\varepsilon \to 0$

Chapter 2

Characteristics method

This chapter contains a general result about PDE.

We will work on nonlinear first-order PDE. In the general case, we can note the studied equation as

$$(2.1) \qquad F\left(Du(x), u(x), x\right) = 0.$$

For this we fix in the hole chapter $U \subset \mathbb{R}^N$ open, the set of all x. Then $u : U \longrightarrow \mathbb{R}$ represent the unknown function and $F : \mathbb{R}^N \times \mathbb{R} \times U \longrightarrow \mathbb{R}$ is a function whose variables will be noted later as p, z, x.

We will also need some boundary condition. We suppose that ∂U is smooth, and we fix $\Gamma \subset \partial U$. Then the bondary condition is noted

$$(2.2) \qquad u = g \qquad \text{on } \Gamma,$$

with $g : \Gamma \longrightarrow \mathbb{R}$.

We intend to prove here that for $x_0 \in U$, there exist V an open neighborhood of x_0, such that our problem is well-posed on V, *i.e.* has exactly one solution. We will suppose for the rest of this chapter that F, which defines our equation, is a $\mathcal{C}^1$ function, and that the boundary condition g is a $\mathcal{C}^2$ function. If this conditions do not hold, the existence and uniqueness are not guaranteed (see proof of 2.2).

For those who have studied the resolution of transport equation, the idea will be similar: we try to find curves along which the solutions of our general equation are easy to compute.

2.1 Changing coordinates

In order to make our problem easier, we will suppose that $\Gamma \subset \partial U$ is flat. We show we can do this without loss of generality. U is supposed to be $\mathcal{C}^1$, hence for $x_0 \in U$ we can find $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a $\mathcal{C}^1$ bijection that straightens Γ near x_0. We suppose for example that $\Phi(U) \subset \mathbb{R}^{N-1} \times \{0\}$.

Let u be a $\mathcal{C}^1$ solution of (2.1), (2.2). We note $\Psi := \Phi^{-1}$, and $v := u \circ \Psi$, such that

$$(2.3) \qquad u = v \circ \Phi.$$

By differentiate (2.3), we obtain

$$Du(x) = Dv(\Phi(x)) \cdot D\Phi(x),$$

and hence (2.1) gives

$$G(Dv(y), v(y), y) := F(Dv(y) \cdot D\Phi(\Psi(y)), v(y), \Psi(y)) = 0, \tag{2.4}$$

with $y := \Phi(x)$. We finaly note

$$h(y) := g \circ \Psi(y), \tag{2.5}$$

and our initial problem has now the form

$$\begin{cases} G(Dv(y), v(y), y) = 0 & \text{on } \Phi(U) \\ v = h & \text{on } \Phi(\Gamma) \end{cases}$$

with G, h defined by (2.4), (2.5). This new problem is similar to the initial problem, and we can assume for the following that Γ is flat near some special point x_0.

2.2 Reduction to an ODE

We first assume that we dispose of $u \in \mathcal{C}^2(U, \mathbb{R})$ a solution of (2.1), in order to find a system of ordinary differential equations (ODE) satisfied by u. We hope this ODE system to be easier to solve.

Let's note x a curve of U:

$$x : \begin{array}{ccc} \mathbb{R} & \longrightarrow & U \subset \mathbb{R}^N \\ s & \longmapsto & x(s) \end{array} .$$

Then we define from this curve

$$z(s) := u(x(s)), \qquad\qquad p(s) := (Du)(x(s)), \tag{2.6}$$

for all $s \in \mathbb{R}$. For $i \in [\![1, N]\!]$, if we differentiate (2.1) with respect to x_i, we find with the chain rule:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_j x_i} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0. \tag{2.7}$$

The firsts terms comprise second order derivative which could be complicate to deal with. But as

$$p_i(s) = u_{x_i}(x(s)), \tag{2.8}$$

we compute

$$\dot{p}_i(s) = \sum_{j=1}^{n} u_{x_j x_i}(x(s)) \dot{x}_j(s). \tag{2.9}$$

So if we suppose that $\forall j \in [\![1, N]\!]$

$$\frac{\partial F}{\partial p_j}(p(s), z(s), x(s)) = \dot{x}_j(s), \tag{2.10}$$

then we can use (2.9) to get rid of this second order terms in (2.7).

We assume for the moment that (2.10) hold, and see what we can deduce. First, we compute that

$$\dot{z}(s) = \sum_{j=1}^{n} u_{x_j}(x(s))\dot{x}_j(s),$$

which gives with (2.8)

$$\dot{z}(s) = \dot{x}(s) \cdot p(s). \tag{2.11}$$

Then, evaluate (2.7) at $x(s)$ and using (2.9), (2.10) as we said earlier, we obtain

$$\dot{p}_i(s) = -\frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) - \frac{\partial F}{\partial x_i}(p(s), z(s), x(s)). \tag{2.12}$$

(2.10), (2.11) and (2.12) are the *characteristic equations*. If we write them in vectorial form, they looks like

$$\begin{cases} (a) & \dot{x}(s) &= D_pF(p(s), z(s), x(s)), \\ (b) & \dot{z}(s) &= D_pF(p(s), z(s), x(s)) \cdot p(s), \\ (c) & \dot{p}(s) &= -D_xF(p(s), z(s), x(s)) - D_zF(p(s), z(s), x(s)) \times p(s). \end{cases} \tag{2.13}$$

This is a a system of $2N + 1$ first order ODE. We can resume what we have done by

2.1. Theorem – Structure of characteristics equation Let u be a solution of (2.1). Let $s \in \mathbb{R} \mapsto x(s)$ be a curve of $\mathbb{R}^N$. If we define z and p by (2.6), and assume that (p, z, x) satisfies (1.10.a), then (p, z, x) satisfies also (1.10.b) and (1.10.c).

We have successfully obtained an ODE system for any curves of U.

2.3 Conditions on the boundary

In the previous paragraph we toke an arbitrary curve x, and supposed we had a solution u to construct z and p. In the next paragraph we will chose 3-tuples (p, z, x) that satisfy the characteristic equations (2.13) and some initial conditions, and will use them to construct u a local solution of (2.1) (intuitively, we want to take $u(x(s)) := z(s)$, for a well-chosen 3-tuple). Here we talk about this initial conditions.

Let's take $x_0 \in \Gamma$. We choose a 3-tuple such that $x(0) = x_0$. As we want (2.2) to be satisfied by our future local solution u, we can assume that

$$z(0) = g(x_0). \tag{2.14}$$

What about $p = (p_1, \cdots, p_N)$? As $\Gamma \subset \mathbb{R}^{N-1} \times \{0\}$ near x_0, differentiate (2.2) with respect to x_i (for $i \in [\![1, N-1]\!]$) gives

$$u_{x_i}(x_0) = g_{x_i}(x_0).$$

Hence we can assume that for $i \in [\![1, N-1]\!]$, our 3-tuple satisfies

$$p_i(0) = g_{x_i}(x_0). \tag{2.15}$$

We obtain an N-th equation for p_0 with

$$F(p_0, z_0, x_0) = 0 \tag{2.16}$$

– where p_0 and z_0 refer to $p(0)$ and $z(0)$ – as (2.1) must hold at x_0.

(2.14), (2.15) and (2.16) are the **compatibility conditions**. A 3-tuple (p, z, x) with $x_0 := x(0) \in \Gamma$ that satisfies them will be qualified as **admissible**.

We want to solve the ODE system near x_0, and control its solution with respect to the initial condition. The following result ensure we can find 3-tuples that are admissible.

2.2. Theorem – Non-characteristic problem We suppose that

$$F_{p_N}(p_0, z_0, x_0) \neq 0, \tag{2.17}$$

and that our problem is regular: F is a $\mathcal{C}^1$ function, and g is a $\mathcal{C}^2$ function.

Then there exist V an open neighborhood of x_0, and a unique solution $q : V \longrightarrow \mathbb{R}^N$ of the following problem.

$$\begin{cases} q_i(y) = g_{x_i}(y) & \text{for } i \in [\![1, N-1]\!] \\ F(q(y), g(y), y) = 0 \end{cases} \tag{2.18}$$

Furthermore, $q \in \mathcal{C}^1$.

Proof. This result is obtained using the implicit function theorem (see appendix A) to $G : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined for $p, y \in R^N$ by

$$\begin{cases} G_i(p, y) := p_i - g_{x_i}(y) & i \in [\![1, N-1]\!], \\ G_N(p, y) := F(p, g(y), y). \end{cases}$$

G is $\mathcal{C}^1$ because of the regularity of F and g ; $G(p_0, x_0) = 0$ thanks to compatibility conditions (2.15) and (2.16) ; finally we have $\det(D_p G(p_0, x_0)) \neq 0$ by calculus:

$$D_p G(p_0, x_0) = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ F_{p_1}(p_0, z_0, x_0) & \cdots & F_{p_{N-1}}(p_0, z_0, x_0) & F_{p_N}(p_0, z_0, x_0) \end{pmatrix}.$$

Hence there exist $q : V \longrightarrow W$ that satisfies the equivalence given by the implicit function theorem. Then for $x \in V$, we have $G(q(x), x) = 0$, we are done for existence. For uniqueness, note that the $(x, q(x))$ are the only couples that can satisfy this last equality. Suppose we have r a solution of (2.18). Then for $x \in V$, $G(r(x), x) = 0$ and so $r(x) = q(x)$. □

2.4 Construct solution

We now use the ODE system to locally solve (2.1), (2.2). We assume that (2.17) hold.

Let $x_0 \in \Gamma$. Let V, q be a neighborhood of x_0 and a function $V \longrightarrow \mathbb{R}^N$ given by theorem 2.2. Then for each $y := (y_1, \cdots, y_{N-1}, 0) \in V \cap \Gamma$, we solve the characteristics equations

$$\begin{cases} \dot{x} = D_p F(p, z, x) \\ \dot{z} = D_p F(p, z, x) \cdot p \\ \dot{p} = -D_x F(p, z, x) - D_z F(p, z, x) \times p \end{cases}$$

with initial condition

$$x(0) = y, \qquad z(0) = g(y), \qquad p(0) = q(y). \tag{2.19}$$

This forms a Cauchy problem, we take its maximal solutions. We then note $(\mathbf{p}, \mathbf{z}, \mathbf{x})$ the flow. For example $\mathbf{x}(y, s)$ is the maximal solution at time s of the problem with initial conditions $(p, z, x)(0) = (q(y), g(y), y)$. For $y = x_0$, this quantity is noted (p_0, z_0, x_0).

This process gives us a bunch of curves (one for each $y \in V \cap \Gamma$). We have to check that the union of this curves contain at least a open neighborhood of x_0, in order to use them to define our solution u.

2.3. Theorem – Inversion of characteristics Assume we have

$$F_{p_N}(p_0, z_0, x_0) \neq 0,$$

Then there exist $I \subset \mathbb{R}$ intervall that contain 0, $\tilde{W} \subset \Gamma$ open neighborhood of x_0 in $\mathbb{R}^{N-1}$, and $\tilde{V} \subset V$ open neighborhood of x_0 in $\mathbb{R}^N$, such that

$$\forall x \in \tilde{V}, \exists! (y, s) \in \tilde{W} \times I \qquad \mathbf{x}(y, s) = x. \tag{2.20}$$

Furthermore, $x \mapsto (y, s)$ is a $\mathcal{C}^2$ function.

Proof.

1. We apply the inverse function theorem (see A p. 33) to $(y, s) \mapsto \mathbf{x}(y, s)$. First characteristic equations give for all $j \in [\![1, N]\!]$

$$\frac{\partial \mathbf{x}_j}{\partial s}(x_0, 0) = F_{p_j}(p_0, z_0, x_0).$$

Also, for all $y \in V \cap \Gamma$,

$$\mathbf{x}(y, 0) = y.$$

This two equalities give

$$D\mathbf{x}(x_0, 0) = \begin{pmatrix} 1 & & 0 & F_{p_1}(p_0, z_0, x_0) \\ & \ddots & & \vdots \\ 0 & & 1 & F_{p_{N-1}}(p_0, z_0, x_0) \\ 0 & \cdots & 0 & F_{p_N}(p_0, z_0, x_0) \end{pmatrix},$$

and so $\det(D\mathbf{x}(x_0, 0)) \neq 0$.
2. According to the inverse function theorem, it is enough to have regularity on $\mathbf{x}$ to obtain the regularity on $x \mapsto (y, s)$. Since $\mathbf{x}$ is obtained by solving a Cauchy problem, we are done if we suppose that F is – at least locally – smooth. ☐

From now on we note $V := \tilde{V}$ and $W := \tilde{W}$ the open neighborhoods given by theorem 2.3. Then for $x \in V$, we define

$$u(x) := \mathbf{z}(y, s), \qquad p(x) := \mathbf{p}(y, s) \tag{2.21}$$

where (y, s) is given by 2.20.

2.4. Theorem The function $u : V \longrightarrow \mathbb{R}$ defined by (2.21) satisfies

$$F(Du(x), u(x), x) = 0$$

for $x \in V$, and

$$u(y) = g(y)$$

for $y \in V \cap \Gamma$.

Proof.
1. We begin by the second part of the theorem: let $y \in V \cap \Gamma$, we show $u(y) = g(y)$. The theorem 2.3 of inversion of characteristics and the definition of u (2.21) ensure that $u(y) = z(y, 0)$. We conclude by the definition of $\mathbf{z}$ (2.19).
2. Let $y \in V \cap \Gamma$. For $s \in I$ – see theorem 2.3 for notations – we define

$$f(y, s) := F(\mathbf{p}(y, s), \mathbf{z}(y, s), \mathbf{x}(y, s)),$$

and prove this function is constantly null. Let $y \in V \cap \Gamma$. Since $p(y, 0) = q(y)$, and $(q(y), z(y), y)$ is admissible – see (2.16) – we have

$$f(y, 0) = 0. \tag{2.22}$$

Furthermore,

$$\begin{aligned} f_s(y, s) &= \sum_{j=1}^{N} \dot{\mathbf{p}}_j f_{p_j} + \dot{\mathbf{z}} f_z + \sum_{j=1}^{N} \dot{\mathbf{x}}_j f_{x_j} \\ &= \sum_{j=1}^{N} \left(-f_{x_j} - f_z \mathbf{p}_j\right) f_{p_j} + \left(\sum_{j=1}^{N} f_{p_j} \mathbf{p}_j\right) f_z + \sum_{j=1}^{N} \left(f_{p_j}\right) f_{x_j} \\ &= 0 \end{aligned}$$

The second equality come from the fact that $(\mathbf{p}, \mathbf{z}, \mathbf{x})$ satisfies (2.13). This calculation and (2.22) show that $s \mapsto f(y, s)$ is solution of a Cauchy problem, as well as the null function. Uniqueness of the solution to this problem conclude.
3. In *2* we proved – see (2.21) – that for $x \in V$

$$F(p(x), u(x), x) = 0.$$

To finish up the proof it's enough to show that for $x \in V$

$$p(x) = Du(x). \tag{2.23}$$

For this, we first show

$$\mathbf{z}_s(y,s) = \sum_{j=1}^{N} \mathbf{p}_j(y,s) \frac{\partial \mathbf{x}_j}{\partial s}(y,s) \tag{2.24}$$

for all $(y,s) \in W \times I$, and

$$\mathbf{z}_{y_i}(y,s) = \sum_{j=1}^{N} \mathbf{p}_j(y,s) \frac{\partial \mathbf{x}_j}{\partial y_i}(y,s) \tag{2.25}$$

for all $(y,s) \in W \times I$, and $i \in [\![1, N-1]\!]$.

The first one of these equalities come from the characteristics equations that $(\mathbf{p}, \mathbf{z}, \mathbf{x})$ satisfies. For the second one we fix $y \in V \cap \Gamma$. We note, for $i \in [\![1, N-1]\!]$ and $s \in I$

$$r_i(s) := \mathbf{z}_{y_i}(y,s) - \sum_{j=1}^{N} \mathbf{p}_j(y,s) \frac{\partial \mathbf{x}_j}{\partial y_i}(y,s), \tag{2.26}$$

and prove this function is null.

Let $i \in [\![1, N-1]\!]$. By definition of $\mathbf{z}$,

$$\mathbf{z}(y,0) = g(y).$$

Derivating this expression with respect to y_i gives $\mathbf{z}_{y_i}(y,0) = g_{x_i}(y)$. A similar method leads to $r_i(0) = g_{x_i}(y) - q_i(y)$. As $(q(y), g(y), y)$ is admissible (q satisfies problem (2.18)), this last quantity is null.

On the other hand, for $i \in [\![1, N-1]\!]$

$$\dot{r}_i(s) = \mathbf{z}_{y_i,s} - \sum_{j=1}^{N} \left(\frac{\partial \mathbf{p}_j}{\partial s} \frac{\partial \mathbf{x}_j}{\partial y_i} + \mathbf{p}_j \frac{\partial^2 \mathbf{x}_j}{\partial y_i \partial s}(y,s) \right). \tag{2.27}$$

We simplify this expression by differentiating (2.24) with respect to y_i:

$$\mathbf{z}_{s,y_i} = \sum_{j=1}^{N} \left[\frac{\partial \mathbf{p}_j}{\partial y_i} \frac{\partial \mathbf{x}_j}{\partial s} + \mathbf{p}_j \frac{\partial^2 \mathbf{x}_j}{\partial s \partial y_i} \right],$$

wich gives with (2.27)

$$\dot{r}_i(s) = \sum_{j=1}^{N} \left(\frac{\partial \mathbf{p}_j}{\partial y_i} \frac{\partial \mathbf{x}_j}{\partial s} - \frac{\partial \mathbf{p}_j}{\partial s} \frac{\partial \mathbf{x}_j}{\partial y_i} \right).$$

As $(\mathbf{p}, \mathbf{z}, \mathbf{x})$ satisfies the characteristic equations, we obtain

$$\dot{r}_i(s) = \sum_{j=1}^{N} \left[\frac{\partial \mathbf{p}_j}{\partial y_i} F_{p_j} - \left(-F_{x_j} - \mathbf{p}_j F_z \right) \frac{\partial \mathbf{x}_j}{\partial y_i} \right]. \tag{2.28}$$

Finally, differentiate $f(y,s)=0$ with respect to y_i

$$\sum_{j=1}^{N}\frac{\partial \mathbf{p}_j}{\partial y_i}F_{p_j}+\mathbf{z}_{y_i}F_z+\sum_{j=1}^{N}\frac{\partial \mathbf{x}_j}{\partial y_i}F_{x_j}=0$$

and combine it with (2.28) leads to

$$\dot{r}_i(s)=F_z\left(\sum_{j=1}^{N}\mathbf{p}_j\frac{\partial \mathbf{x}_j}{\partial y_i}-\mathbf{z}_{y_i}\right)=-F_z\,\dot{r}_i(s).$$

This, with $r_i(0)=0$, is a Cauchy problem satisfied by the null function. Hence $r_i=0$, and (2.25) hold.

4. Finally, we show that (2.24) and (2.25) implies (2.23). Let $k\in[\![1,N]\!]$. Then by definition of u (2.21)

$$u_{x_k}=\sum_{i=1}^{N-1}\frac{\partial y_i}{\partial x_k}\mathbf{z}_{y_i}+s_{x_k}\mathbf{z}_s.$$

We use (2.25) and (2.24)

$$u_{x_k}=\sum_{i=1}^{N-1}\frac{\partial y_i}{\partial x_k}\left(\sum_{j=1}^{N}\mathbf{p}_j\frac{\partial \mathbf{x}_j}{\partial y_i}\right)+s_{x_k}\left(\sum_{j=1}^{N}\mathbf{p}_j\frac{\partial \mathbf{x}_j}{\partial s}\right),$$

and we rearrange

$$u_{x_k}=\sum_{j=1}^{N}\mathbf{p}_j\left(\sum_{i=1}^{N-1}\frac{\partial y_i}{\partial x_k}\frac{\partial \mathbf{x}_j}{\partial y_i}+s_{x_k}\frac{\partial \mathbf{x}_j}{\partial s}\right).$$

We can recognize here – see for example (2.20) –

$$u_{x_k}=\sum_{j=1}^{N}\mathbf{p}_j\frac{\partial \mathbf{x}_j}{\partial x_k}=\sum_{j=1}^{N}\mathbf{p}_j\delta_{j,k}=p_k.$$

This conclude the proof. □

Nota In the special case where our equation is **quasilinear**, *i.e.* has the form

$$\sum_{j=1}^{N}f_j(u)u_{x_j}=0,$$

with f_j functions, we can note that (2.13) gives $\dot{z}=0$ and so the solutions we construct are constant along the characteristics curves. because of this we do not need the theorem 2.2 (p. 14) and we can use F and g functions that are less regular.

Nota In the general case where Γ is not flat, the noncharacteristic condition write

$$D_pF(p_0, z_0, x_0) \cdot \nu(x_0) \neq 0, \tag{2.29}$$

where $\nu(x_0)$ denote a unit normal to Γ at x_0.

We can finally sumarize what we have done in this chapter with the following result.

2.5. Theorem Let $x_0 \in \Gamma$ and W be a neighborhood of x_0 such that $F \in \mathcal{C}^1$, and $g \in \mathcal{C}^2$ on W. Let $(p_0, z_0) \in \mathbb{R}^n \times \mathbb{R}$.

If (2.29) hold at (p_0, z_0, x_0), then there exist $V \subset W$ open neighborood of x_0 such that there exist a unique u solution of (2.1) and (2.2) on V and such that $u(x_0) = z_0$ and $\nabla u(x_0) = p_0$.

For example the equation

$$u_t + uu_x = 0$$

in $U := \mathbb{R}_+^* \times \mathbb{R}$ can be represented by the following application.

$$F : \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R} \times U & \longrightarrow & \mathbb{R} \\ (p_1, p_2, z, t, x) & \longmapsto & p_1 + zp_2 \end{array}$$

Chapter 3

Conservation laws

Here we study a particular equation. Let $F : \mathbb{R} \to \mathbb{R}^n$ and $g : R^n \to \mathbb{R}$ be functions. We suppose $g \in \mathrm{L}^1$. We want to solve the following problem:

$$\begin{cases} u_t(t,x) + \mathrm{div}(F \circ u)(t,x) = 0 & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n \\ u(0,x) = g & x \in \mathbb{R}^n \end{cases} \tag{3.1}$$

where div denote the divergence with respect to space variable: $\mathrm{div}(F \circ u) = \sum_{j=1}^n F_j(u)_{x_j}$. The interesting fact about this equation is that the integral of solutions is conserved as the time evolves.

Let u be a solution of (3.1). For simplicity we suppose in this paragraph that u is continuous and has compact support. We note for $t \in \mathbb{R}_+^*$

$$I(t) := \int_{\mathbb{R}^n} u(t,x)\mathrm{d}x$$

and show that for all $t \in \mathbb{R}_+^*$, $I(t) = I(0)$. We compute that

$$\begin{aligned} I'(t) &= \int_{\mathbb{R}^n} u_t(t,x)\mathrm{d}x \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} (F_j \circ u)_{x_j} \mathrm{d}x \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^{n-1}} [F_j \circ u]_{-\infty}^{+\infty} \mathrm{d}\bar{x}_j, \end{aligned}$$

where $\mathrm{d}\bar{x}_j$ means $\mathrm{d}x_1 \cdots \mathrm{d}x_{j-1}\mathrm{d}x_{j+1} \cdots \mathrm{d}x_n$.
As u has compact support, for each $j \in [\![1,n]\!]$ we have $[F_j \circ u]_{-\infty}^{+\infty} = F(0) - F(0) = 0$, and so $I'(t) = 0$. This calculus show that the integral of solution doesn't change with time. For this reason we call problem (3.1) a **conservation law**.

In [4] chapter 11 it is studied systems of conservations laws. In this case $u : \mathbb{R}_+^* \times \mathbb{R}^n \to \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}^{m \times n}$. This correspond to a system where m physical quantities are conserved as time evolve, we do not talk about it.

This equations can models fluid flow: for $F(u) = uV(u)$, with u representing the fluid density and $V(u)$ the fluid speed. It can for example be used to describe a traffic flow – though not perfectly, a driver react essentially to what's happen in front of him, meanwhile a fluid particle is sensible to front and back environment – see [2].

3.1 Characteristics method, weak formulation

We want to apply the characteristics method we have exposed earlier. We note

$$G(q, z, y) = q_0 + F'(z) \cdot p$$

with $y = (t, x)$, and $q = (p_0, p) = (p_0, p_1, \cdots, p_n)$. This is the right form to apply characteristics method. As

$$D_q G(q, z, y) = (1, F'(z))$$

and a unit normal to $\partial U = \{0\} \times \mathbb{R}^n$ is given by $(1, 0, \cdots, 0)$, the non characteristic condition hold and there is a local unique smooth solution to our problem.

Furthermore, $D_y G = 0$ and so characteristics equation ensure that $\dot{z} = 0$. This mean that every smooth solution of a conservation law is constant along the characteristics curves.

What about non-regular solutions? If we multiply by $v \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ and integrate, an integration by part gives the **weak formulation** of this equation (3.1):

$$\int_0^{+\infty} \int_{\mathbb{R}^n} u v_t + f(u) \operatorname{div}_x v \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^n} g(x) v(0, x) \mathrm{d}x = 0 \tag{3.2}$$

Hence we say that u satisfy our problem in the weak sense if (3.2) hold for all $v \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$.

3.2 Case of dimension one

We begin by interesting ourself to the case $n = 1$. Then the problem write

$$\begin{cases} u_t + f(u)_x = 0 & \mathbb{R}_+^* \times \mathbb{R} \\ u(0, \cdot) = g & \mathbb{R} \end{cases} \tag{3.3}$$

Nota If we take $f(x) := \frac{1}{2}x^2$, we obtain the Burger's equation

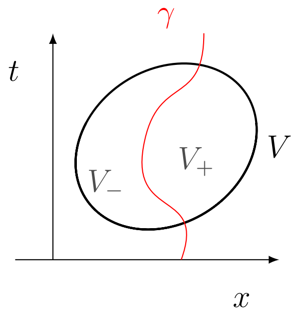
$$u_t + u u_x = 0$$

we have already see at the end of chapter 2.

In this section we suppose that f is strictly convex.

Rankine-Hugoniot condition

Let suppose we have u a solution that is $\mathcal{C}^1$ on V open subset of $\mathbb{R}_+^* \times \mathbb{R}$, up to a curve γ that goes through V. We assume γ cut V into two distinct regions V_- and V_+.



We chose v a test function with compact support in V_-. Then the weak formulation gives

$$\int_{V_-} uv_t + f(u)v_x \mathrm{d}x\mathrm{d}t = 0$$

as v vanishes near ∂V. u is then smooth on V_- and we can integrate by part

$$\int_{V_-} u_t v + f(u)_x v \mathrm{d}x\mathrm{d}t = 0.$$

This equality hold for all $v \in \mathcal{D}(V_-)$, so u satisfy

$$u_t + f(u)_x = 0 \qquad V_-. \tag{3.4}$$

We obtain the equation on V_+ with similar method. Now taking $v \in \mathcal{D}(V)$, we obtain from the weak formulation that

$$0 = \int_{V_-} uv_t + f(u)v_x \mathrm{d}x\mathrm{d}t + \int_{V_+} uv_t + f(u)v_x \mathrm{d}x\mathrm{d}t.$$

Integrating by part[1] and with (3.4), this becomes

$$0 = \int_\gamma [u_- n_1 + f(u_-)n_2]\, v\mathrm{d}s - \int_\gamma [u_+ n_1 + f(u_r)n_2]\, v\mathrm{d}s. \tag{3.5}$$

Here $n = (n_1, n_2)$ is the unit normal to γ pointing from V_- to V_+, and $u_-(\gamma(s))$ is the limit of $u(t,x)$ as (t,x) goes to $\gamma(s)$ and staying in V_-. (3.5) hold for all $v \in \mathcal{D}(V)$, and so

$$[u_- - u_+]\, n_1 + [f(u_-) - f(u_+)]\, n_2 = 0 \qquad \text{on } \gamma. \tag{3.6}$$

We now make the hypothesis that γ can be write as

$$\gamma = \{(t, h(t)) : t \in W_{t_0}(\varepsilon)\},$$

where $W_{t_0}(\varepsilon) :=]t_0 - \varepsilon, t_0 + \varepsilon[$. We'll see later that this is not a big restriction. Hence we have $(n_1, n_2) = \alpha \cdot (-h', 1)$, with $\alpha = (1 + h')^{-\frac{1}{2}}$. Using this expression in (3.6) finally gives the Rankine-Hugoniot condition

$$f(u_-) - f(u_+) = h'\,(u_- - u_+) \qquad \text{on } V \cap \gamma. \tag{3.7}$$

For simplicity, we note from now $[\![u]\!](t,x) := u_-(t,x) - u_+(t,x)$ the **jump** in u across γ at point (t,x). The condition above write

$$[\![f \circ u]\!] = \sigma[\![u]\!] \qquad \text{on } V \cap \gamma, \tag{3.8}$$

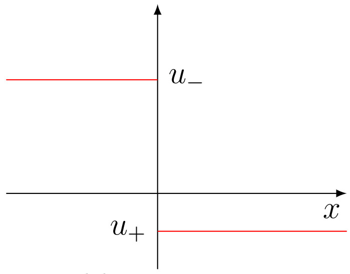
with $\sigma := h'$ is the **speed** of γ.

[1] See the Gauss-Green theorem p. 33.

Riemann's problem

With the previous paragraph in mind, we try to find solutions of our problem (3.3), in the case where

$$g(x) := \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x > 0. \end{cases}$$



With this initial condition, we call (3.3) a Riemann problem.

First, we uses the characteristics method. We already saw that a solution of a Riemann problem is constant along the characteristics curves, which are defined by

$$\dot{t}(s) = 1 \tag{3.9}$$

and

$$\dot{x}(s) = f'(z(s)) = f'(z_0)$$

as z is constant along the characteristics curves. Hence

$$x(t) = tf'(g(x_0)) + x(0).$$

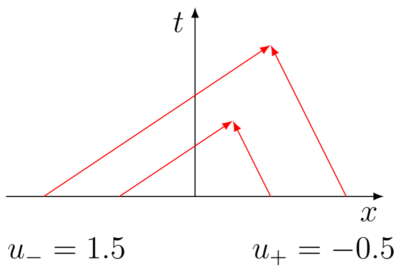
Note that (3.9) show $t(s) = s$. This is consistent with the hypothesis we made p. 22 on the structure of γ.

Shock

Let suppose first that

$$u_- > u_+$$

As f' is increasing we have $f'(u_-) > f'(u_+)$. Then there exist some $t \in \mathbb{R}_+^*$ for which two characteristics curves will meet, and our solution is no longer smooth. From this time on, we can not use the characteristic method.



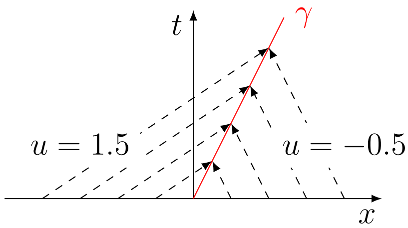
The left and right curves will meet, we say there is a **shock**. We use the Rankine-Hugoniot condition to dicover what happen after this shock. We have a curve of discontinuity – where the shocks takes place – that has to satisfy (3.8). If we take for example the Burger's equation – *i.e.* $f(x) := \frac{1}{2}x^2$ – with $u_- = 1.5$ and $u_+ = -0.5$, we obtain that $[\![f(u)]\!] = 1$ and $[\![u]\!] = 2$. Hence $\sigma = \frac{1}{2}$ and the shock is the curve given by

$$\gamma = \left\{ \left(t, \frac{t}{2}\right) : t \in \mathbb{R}_+ \right\}.$$

This cut $U := \mathbb{R}_+ \times \mathbb{R}$ into $U_- := \{(t,x) : x < 2t\}$ and $U_+ := \{(t,x) : x > 2t\}$. Hence we can construct a solution

$$u(t,x) := \begin{cases} u_- & \text{if } (t,x) \in U_- \\ u_+ & \text{if } (t,x) \in U_+ \end{cases}$$

of Burger's equation.

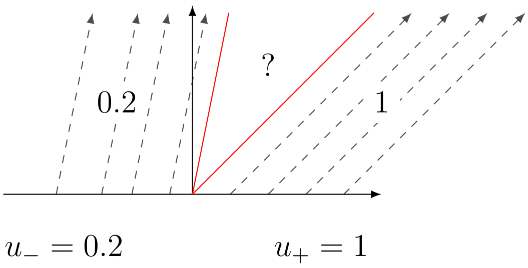


Rarefaction wave

We now suppose that

$$u_- < u_+,$$

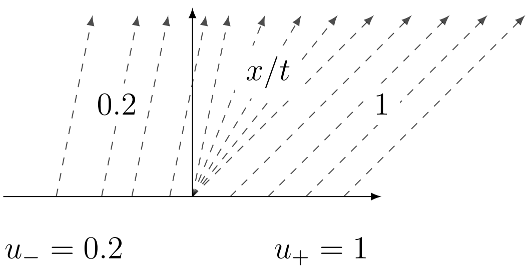
hence $f(u_-) < f(u_+)$. In this case, the characteristics curves never cross, the main problem here is that there exist point which will never be joined by this curves. This mean we have to find the correct value of u in this area.



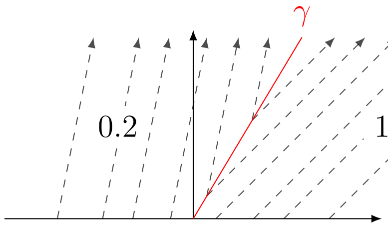
The first possibility is to choose a solution continuous everywhere – except at $(0,0)$ – we set for (t,x) in the empty area:

(3.10) $$u(t,x) = x/t,$$

which define a continuous solution – called a **rarefaction wave** – that indeed satisfy the Rankine-Hugoniot condition.



But we can also define a *non physical shock* solution, by setting $u = 0.2$ in $U_- := \{(x,t) : x < \alpha t\}$ and $u = 1$ in $\{(x,t) : x > \alpha t\}$. This define a discontinuity curve γ defined by $\{\ (x,t) : x = \alpha t\}$ which satisfy the Rankine-Hugoniot condition for $\alpha = \frac{f(1)-f(0.2)}{1-0.2} = 0.6$



As this second solution seems non-physical, we would like to find a second conditions that the rarefaction wave satisfy, but not this last solution. The problem in the shock is that there is no collision: characteristics curves go away from the discontinuity. Hence it does not satisfy the **entropy condition** for a shock with speed σ:

$$f'(u_-) > \sigma > f'(u_+).$$

As f is supposed strictly convex, this condition becomes $u_- > u_+$ at each discontinuity.

Other initial conditions

Here we suppose that $g \in \mathrm{L}^1 \cap \mathrm{L}^\infty$, and that F is smooth, and $F(0) = 0$. This last condition is not a problem, as F is defined up to a constant.

There is a lot of things we know about solutions in this case. There exist an expression for weak solutions that uses f^* the Legendre transformation[2] of f, and $(f')^{-1}$ – which is defined as long as f is strictly convex. It is the *Lax-Oleinik formula.*

It is also possible to refine the entropy condition, such that we have uniqueness of weak solutions that satisfy the jump condition and this new entropy condition – see [4] p. 150.

The behavior of solution is also known for large t. There is a decay in $t^{-\frac{1}{2}}$ of $\|u\|_\infty$, and a limit for $\|\cdot\|_{\mathrm{L}^1}$. We define, for $p, q, d > 0$ and $\sigma \in \mathbb{R}$ some constants, the N-wave

$$N(t,x) := \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & \text{if } -\sqrt{tpd} < x - \sigma t < \sqrt{tqd}, \\ 0 & \text{otherwise.} \end{cases}$$

For well-chosen constants p, q, d, σ – see [4] p. 159 – we have the following result.

3.1. Theorem If g has compact support, the entropy solution u of (3.3) satisfy

[2] See appendix B.

for $t \in \mathbb{R}_+^*$

$$\int_{\mathbb{R}} |u(t,x) - N(t,x)| \, \mathrm{d}x \leq \frac{C}{\sqrt{t}}$$

for some constant C.

We have numerically check this theorem for simple g. We represent u in function of t and x, see figures 3.1, 3.2 and 3.3.

3.3 Case of dimension two

We now work on the $n = 2$ case. Here the problem wright

$$\left\{ \begin{array}{ll} u_t + f_1(u)_x + f_2(u)_y = 0 & \text{on } U := \mathbb{R}_+^* \times \mathbb{R}^2, \\ u(0,x) = g(x) & \text{on } \mathbb{R}^2. \end{array} \right. \tag{3.11}$$

As in the case $n = 1$, we can establish a Rankine-Hugoniot condition. Now $V \subset \mathbb{R}_+^* \times \mathbb{R}^2$, and we note S a surface – of dimension 2 – that cut V into two parts. Using the same method as in case $n = 1$, we obtain an equivalent to identity (3.6):

$$[\![G(u)]\!] \cdot n = 0 \qquad \text{on } S, \tag{3.12}$$

with $G = (\mathrm{id}_{\mathbb{R}}, f_1, f_2)$. As in the previous case, we now suppose that S can be write as

$$S = \{(h(\xi,\eta), \xi, \eta) : \xi \in W_{\xi_0}(\varepsilon), \eta \in W_{\eta_0}(\varepsilon)\}.$$

Then

$$v_1 = (h_\xi, 1, 0) \qquad \text{and} \qquad v_2 = (h_\eta, 0, 1)$$

are two tangent vectors to S, hence n is collinear to

$$v_1 \wedge v_2 = (1, -h_\xi, -h_\eta).$$

With this expression of n, (3.12) becomes

$$[\![u]\!] = h_\xi [\![f_1(u)]\!] + h_\eta [\![f_2(u)]\!] = \nabla h \cdot [\![F(u)]\!] \qquad \text{on } S, \tag{3.13}$$

with $F = (f_1, f_2)$.

It is also possible to establish an entropy condition, see [8] p. 90.

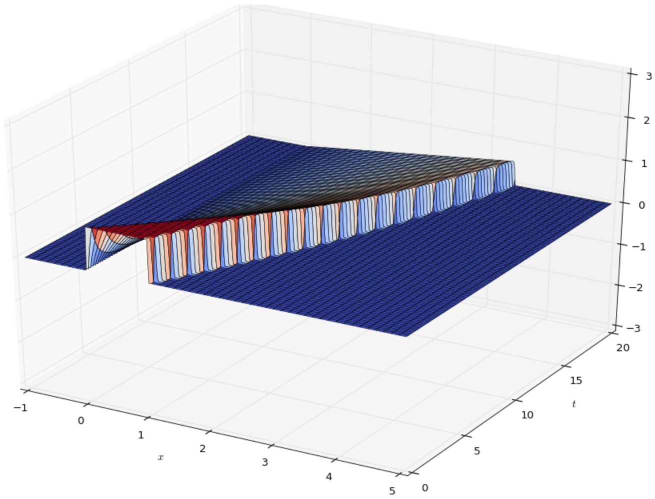


Figure 3.1: Numerical solution of Burger's equation with $g = 1_{[0,1]}$.

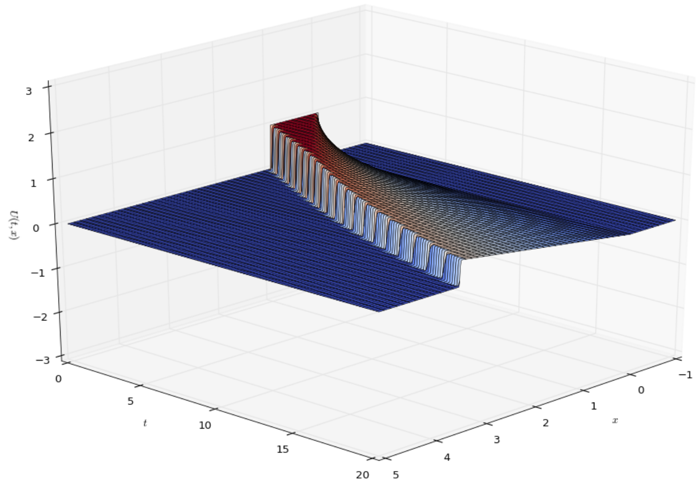


Figure 3.2: The solution goes to a N-wave for large times.

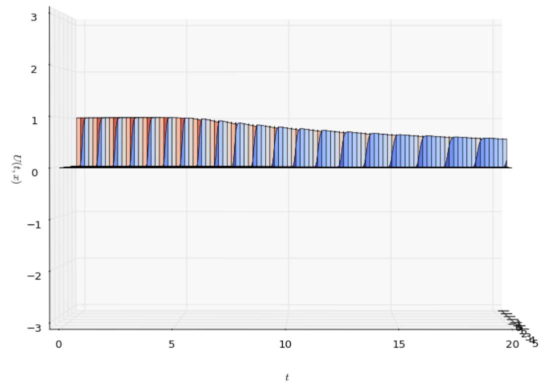


Figure 3.3: Decay in $t^{-\frac{1}{2}}$ of $\|u\|_\infty$.

Chapter 4

Numerical resolution

We now look at numerically solving the conservation laws. We chose to study the 1D Burger's equation that we have already meet in chapter 3.

4.1 Notations

We begin by fixing the usual notations. We discretize the space by constructing a mesh with constant step. For $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, we note

$$x_j := j\delta_x \qquad \text{and} \qquad t_n := n\delta_t.$$

The evaluation of the true solution u is noted

$$u_j^n := u(t_n, x_j).$$

For this particular equation, we will rather use

$$\bar{u}_j^n := \frac{1}{\delta_x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t_n, x)\mathrm{d}x,$$

as we already know that integral with respect to space is really stable. We hope that using this space average rather than a pointwise evaluation will lead us to better results. The computed solution will be write as U_j^n, and approximate either u_j^n or $\bar{u}_j^n$ in function of the used method.

The algorithm is to first construct $U^0 := \left(U_j^0\right)_{j\in\mathbb{Z}}$ by simply taking the initial condition we have, and then construct U^{n+1} in function of U^n by using the PDE.

Schemes

There is a lot of implicit or explicit method that can be used here, see for example [7] p. 101. We remind here only a few:

- ⋆ Upwind method: uses $\frac{U_j^{n+1}-U_j^n}{\delta_t}$ and $\frac{U_j^n-U_{j-1}^n}{\delta_x}$ to approximate u_t and u_x ;
- ⋆ Downwind method: uses $\frac{U_j^{n+1}-U_j^n}{\delta_t}$ and $\frac{U_j^n-U_{j-1}^n}{\delta_x}$ to approximate u_t and u_x ;
- ⋆ Lax-Friedrichs method: uses $\frac{1}{\delta_t}\left[U_j^{n+1} - \frac{U_{j-1}^n+U_{j+1}^n}{2}\right]$ and $\frac{1}{2\delta_x}\left[U_{j+1}^n - U_{j-1}^n\right]$ to approximate u_t and u_x.

Convergence

We want to know if our computation are useful. We introduce an error function:

$$E(t,x) = U_j^n - u(t,x) \qquad \text{if } (t,x) \in [t_n, t_{n+1}[\times \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right[.$$

or:

$$\bar{E}(t,x) = U_j^n - \bar{u}(t,x)$$

Remind that this error term depends of δ_x and δ_t. We then say that our solution converge if $\|E\|_{\mathrm{L}^1} \to 0$ as δ_t and δ_x goes to 0.

4.2 Conservative methods

The equation we study is nonlinear, with discontinuous solutions. For those reasons, results like the *Lax equivalence theorem* – see [7] p. 107 – doesn't hold, and some methods will converge – in the above sense – to non solution. For example, let's take a look at 1D Burger's equation with upwind scheme. We obtain

$$U_j^{n+1} = U_j^n - \frac{\delta_t}{\delta_x} U_j^n \left(U_j^n - U_j^{n-1}\right).$$

If we use

$$U_j^0 = \begin{cases} 1 & \text{if } j < 0, \\ 0 & \text{if } j \geq 0. \end{cases}$$

as initial condition, we can check that for all $n \in \mathbb{N}$, we have $U^n = U^0$. But this is not a solution of Burger's equation, the discontinuity doesn't satisfy the Rankine-Hugoniot condition.

To prevent this false solutions to appear, we can use method in **conservative form**:

$$U_j^{n+1} = U_j^n - \frac{\delta_t}{\delta_x}\left[G(U_j^n, U_{j+1}^n) - G(U_{j-1}^n, U_j^n)\right], \tag{4.1}$$

with G the **numerical flux function**. This form comes from the integral formulation of the conservation law.[1]

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t_{n+1}, x)\mathrm{d}x = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t_n, x)\mathrm{d}x + \int_{t_n}^{t_{n+1}} \left[u(t,\cdot)\right]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathrm{d}t.$$

Dividing by δ_x we obtain

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\delta_x}\int_{t_n}^{t_{n+1}} \left[f(u)(t,\cdot)\right]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathrm{d}t.$$

We have obtain (4.1) if G represent the average of the theoretical flux f through $x_{j+\frac{1}{2}}$, during $[t_n, t_{n+1}]$

$$G(U_j^n, U_{j+1}^n) \sim \frac{1}{\delta_t}\int_{t_n}^{t_{n+1}} f(u)(x_{j+\frac{1}{2}}, t)\mathrm{d}t.$$

[1] How to obtain this formulation from the weak one is described in appendix C.

Consistency, convergence

We introduce the concepts of consistency. We want to decide if a method is stable locally. Controlling the whole error can be hard, so we use a local error, defined as follow. We take u a weak solution of the equation we study, and we compute U^{n+1} by directly using $U_j^n := u(t_n, x_j)$. This way,

$$L(t,x) = \frac{1}{\delta_t}\left[u(t,x) - U_j^{n+1}\right] \qquad \text{if } (t,x) \in [t_n, t_{n+1}[\times \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right[$$

is the **local error** corresponding to the method used. Hence we can say this method is consistent if $L(t,x) \to 0$ as δ_t, δ_x go to 0. For conservatives method, we use the following definition of consistency.

Definition – Consistency We say that a conservative method is consistent if

- ⋆ for all constant function u, the numerical flux equal the theoretical flux

$$G(u,u) = f(u),$$

- ⋆ G is a continuous function of its two variables.

In conservative form, the upwind method becomes

$$U_j^{n+1} = U_j^n - \frac{\delta_t}{\delta_x}\left[\frac{(U_j^n)^2}{2} - \frac{\left(U_j^{n-1}\right)^2}{2}\right].$$

Nota This method isn't always stable, see figure 4.1. In the case of Burger's equation, it needs that $U_j^n \leq 0$. If we have U_j^n, it is more interesting to use the downwind method. If U_j^n take both sign, we must alternate the two method, or change for another one, like the Lax-Friedrichs scheme.

Figure 4.2 show this last scheme, the issues for negative values of U_j^n have disappear. The drawback of this method is that it introduce an oscillating behavior near discontinuity. For this kind of method, it is interesting to study a convergence in **total variation**, which also use the $\|\cdot\|_1$ of "derivative" of u.

$$TV_t(u) := \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} |u(t,x) - u(t,x-\varepsilon)| \,\mathrm{d}x.$$

More precisely, we say that $(U_\eta)_{\eta\in\mathbb{N}}$ a sequence of computed solutions converge to the weak solution u if

- ⋆ For all K compact of $\mathbb{R}_+^* \times \mathbb{R}$

$$\|U_\eta - u\|_{\mathrm{L}^1(K)} \longrightarrow 0$$

as η goes to $+\infty$;

- ⋆ For all $T \in \mathbb{R}_+^*$, there exist an $R > 0$ such that for all $t \in [0,T]$ and for all $\eta \in \mathbb{N}$ we have

$$TV_t(U_\eta) < R.$$

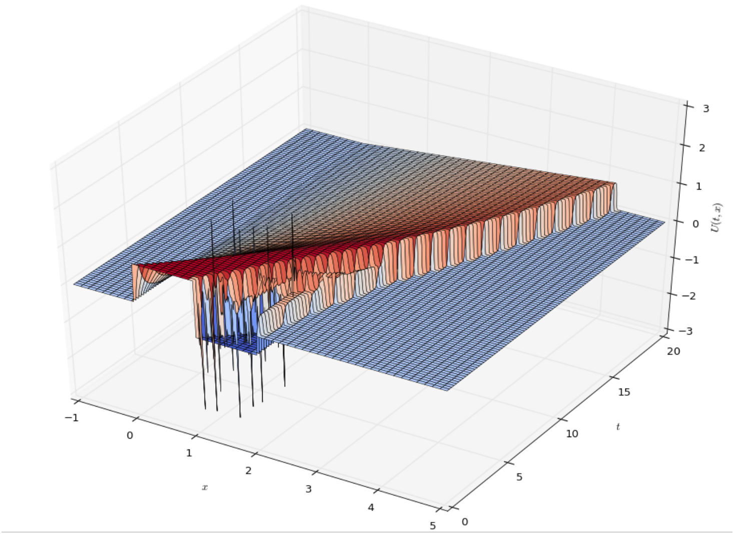


Figure 4.1: When upwind method is used, there is numerical issues where $U_j^n \leq 0$.

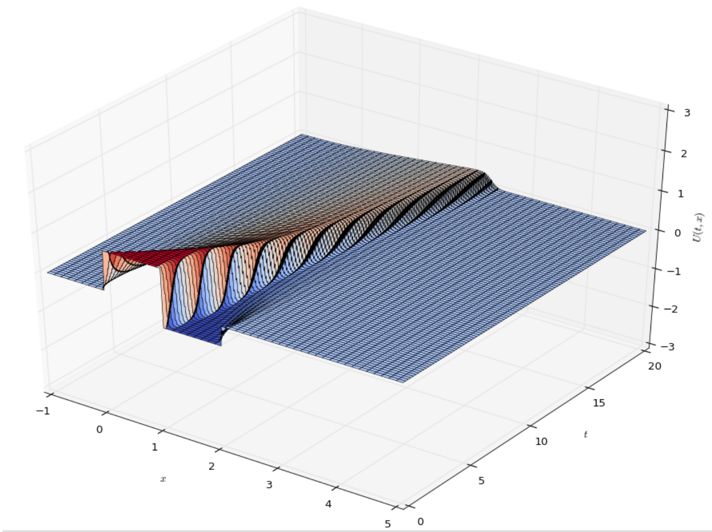


Figure 4.2: If we rather use the Lax-Friedrichs scheme, this issues disappear. However, an oscillating phenomenon appears near the discontinuity.

With this convergence, we have the following result.

4.1. Theorem – Lax-Wendroff We note $\eta \in \mathbb{N}$ a grid parameter, such that $\delta_t(\eta), \delta_x(\eta) \to 0$ as $\eta \to +\infty$. We then note $U_\eta(t, x)$ the numerical solution of the conservation law, computed with a consistent conservative method in the η-th grid.

If $(U_\eta)_\eta$ converges – in the above sense – to some function u, then u is a weak solution of the conservation law.

Proof. See [7] p. 131. □

Appendix

A Various results

A.1. Theorem – Implicit function Let E, F and G be normed vector spaces. $E \times F$ is a vector space with norm

$$\|(x,y)\| := \max \|x\|_E , \|y\|_F .$$

Let U be an open subset of $E \times F$. Let $(a,b) \in U$ and $f \in \mathcal{C}^1(U,G)$, such that $f(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b)$ be a bijection $F \longrightarrow G$.
Then there exist

- $\star$ V open neighborood of a in E,
- $\star$ W open neighborood of b in F such that $V \times W \subset U$,
- $\star$ $\varphi \in \mathcal{C}^1(V,W)$;

such that for all $(x,y) \in E \times F$,

$$(x,y) \in V \times W \text{ and } f(x,y) = 0 \qquad \textit{iff} \qquad x \in V \text{ and } \varphi(x) = y.$$

If furthermore $f \in \mathcal{C}^k(U,G)$, then $\varphi \in \mathcal{C}^k(V,W)$.

A.2. Theorem – Local inversion Lets E, F be Banach spaces, Ω open subset of E. Let $f \in \mathcal{C}^1(\Omega,F)$, $a \in \Omega$ such that $Df(a) : E \longrightarrow F$ is a bijection. Then, there exist

- $\star$ V, open neighborhood of a in Ω,
- $\star$ W, open neighborhood of $f(a)$ in F,

such that $f : V \longrightarrow W$ is bijective and $f^{-1} \in \mathcal{C}^1(W,V)$.
If furthermore $f \in \mathcal{C}^k(V,W)$, then $f^{-1} \in \mathcal{C}^k(W,V)$.

A.3. Theorem – Gauss-Green Let U be a open bounded subset of $\mathbb{R}^n$, such that ∂U is $\mathcal{C}^1$. Let $u \in \mathcal{C}^1(\bar{U})$, then for $i \in [\![1,n]\!]$

$$\int_U u_{x_i} \mathrm{d}x = \int_{\partial U} u n_i \mathrm{d}S,$$

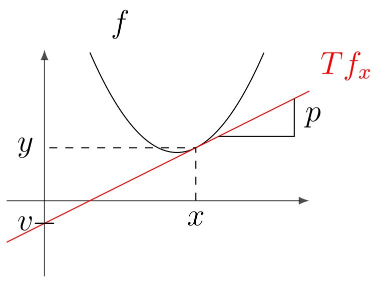
where n_i is the i-th component of a unit outward normal of ∂U.

If we apply the Green-Gauss theorem to uv, we obtain an *integration by parts* formula

$$\int_U uv_{x_i}\mathrm{d}x = -\int_U u_{x_i}v\mathrm{d}x + \int_{\partial U} uvn_i\mathrm{d}S.$$

B Legendre transform

The main idea in the Legendre transform is to describe a curve $y = f(x)$ by its tangents: $v = f^*(p)$. Here p is a the slope of Tf_x – the tangent of f at x – and v is such that $(0, v)$ is the intersection of Tf_x and the y-axis.



If f is strictly convex, we can effectively say that v is a function of p.

Definition – Legendre transform Let $I \subset \mathbb{R}$ interval, and let $f : I \to \mathbb{R}$ be a convex function. We define the Legendre transform of f as

$$f^*(y) := \sup_{x\in I} xy - f(x),$$

for all the $y \in \mathbb{R}$ such that the right term is finite. We note

$$I^* := \{y \in \mathbb{R} : f^*(y) < +\infty\}$$

this set.

See [1] for an exercise that uses Legendre transforms.

C From weak to integral formulation

Here we present an alternative to the weak formulation (3.2).

C.1. Theorem Suppose u is a weak solution of a conservation law with flux f. Then for all $t_1, t_2 \in \mathbb{R}_+^*$ and $x_1, x_2 \in \mathbb{R}$,

$$\int_{x_1}^{x_2} [u(t_2, x) - u(t_1, x)]\,\mathrm{d}x + \int_{t_1}^{t_2} [f(u)(t, x_2) - f(u)(t, x_1)]\,\mathrm{d}t = 0. \tag{4.2}$$

Proof. Let u be a weak solution of a conservation law. We begin fixing $x_1, x_2 \in \mathbb{R}$ and $t_1, t_2 \in \mathbb{R}_+^*$. Then we note for $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$

$$w(t, x) = 1_{[t_1,t_2]\times[x_1,x_2]}(t, x).$$

Then for $\varepsilon \in \mathbb{R}_+^*$, we note $w_\varepsilon = w \star \rho_\varepsilon$, with $(\rho_\varepsilon)_\varepsilon$ a unity approximation. Then $w_\varepsilon \in \mathcal{D}$, and we have

$$\int_0^{+\infty} \int_{\mathbb{R}} u\partial_t w_\varepsilon + f(u)\partial_x w_\varepsilon \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} g(x) w_\varepsilon(0,x)\mathrm{d}x = 0. \tag{4.3}$$

As ε goes to 0, the second integral goes to 0 because $t_1, t_2 > 0$. Furthermore,

$$\begin{aligned} \partial_t w_\varepsilon &= (\partial_t \rho_\varepsilon) \star w \\ &\longrightarrow_{\varepsilon \to 0} \delta_0^t w. \end{aligned}$$

where δ^t is the derivative of the Dirac distribution. Hence

$$\int_{\mathbb{R}_+^* \times \mathbb{R}} u\partial_t w_\varepsilon \mathrm{d}x\mathrm{d}t \longrightarrow \int_{x_1}^{x_2} [u(\cdot,x)]_{t_1}^{t_2} \mathrm{d}x.$$

We do the same with the second part of the first integral to conclude. □

Bibliography

[1] http://perso.eleves.ens-rennes.fr/~lgare802/documents/oral_concours_3a_corentin.pdf, 2017.

[2] A. Aw and M. Rascle. Resurrection of ”second order” models of traffic flow. Feb-Mar 2000.

[3] J. M. Burger. *A mathematical model illustrating the theory of turbulence – Advances in applied mechanics, Vol 1*. 1948.

[4] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, 1997.

[5] Pietro H. Harris and Roberto Garra. Nonlinear heat conduction equations with memory: Physical meaning and analytical results. May 2016.

[6] Frédéric Lagoutière. *Équation aux dérivées partielles et leurs approximations*.

[7] Randall J. LeVeque. *Numerical method for conservation laws*. Birkäuser, 1992.

[8] Yuxi Zheng. *System of conservation laws, Two-dimensional Riemann problem*. Springer science+Business media, LLC, 2001.