Erratum to : "A piecewise deterministic model for a prey-predator community" An extinction-persistence issue.

Manon Costa *and Edouard Strickler †

21 November 2024

This note aims at correcting the statement and the proof of Theorem 2.2 in [4] concerning the ergodicity of a piecewise deterministic process Z_t defined on $\mathbb{N}^* \times [0, \infty[$. The result states the convergence of the law of (Z_t) toward its unique invariant measure defined on $\mathbb{N}^* \times]0, \infty[$. This result is however wrong stated as it is and requires an additional assumption in order to prevent extinction of the second coordinate.
The proof fails when verifying an irreducibility - Lyapunov argument in the proof of Proposition 3.3. The article claims that the process is irreducible on $\mathbb{N}^* \times (0, \infty)$ (which is true) and verifies equation (3.2) with a compact set K of the form $\{1, \ldots, \max(n_0, n_1)\} \times [0, \max(h_0, h_1]$. The problem is twofold: K is not included in the irreducibility set and this can be not be corrected neither by reducing K since it would not be compact, nor by augmenting the state space, since irreducibility does not hold on $\mathbb{N}^* \times [0, \infty)$

To circumvent this issue, we shall make use of the theory of Stochastic Persistence as developed in [1] to prove that the result remains true under the additional assumption that a Lyapunov exponent is positive. Moreover, when this Lyapunov exponent is non positive, we prove that the second coordinate vanishes and therefore the process converges to an invariant measure supported by the boundary $\mathbb{N}^* \times \{0\}$.

1 Model and statement of the corrected result

1.1 Model

Let us recall the definition of the process Z_t introduced in [4]. The article considers a community of prey individuals and predators in which the predator dynamics is faster than the prey dynamics. The community is described at any time by a vector $Z_t = (N_t, H_t)$ where $N_t \in \mathbb{N}$ is the number of living prey individuals at time t and $H_t \in \mathbb{R}_+$ is the density of predators.

The prey population evolves according to a birth and death process where the individual birth rate is denoted by $b > 0$ and the individual death rate by $d \geq 0$. The logistic competition among the prey population is represented by a parameter $c > 0$. The predation intensity exerted at time

*Institut de Mathématiques de Toulouse, CNRS UMR 5219 , Université Toulouse III Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse cedex 09. manon.costa@math.univ-toulouse.fr
†Université de Lorraine, CNRS, Inria, IECL, Nancy, France. edouard.strickler@univ-lorraine.fr

t on each prey individual is BH_t.
The predators density follows a deterministic differential equation whose parameters depend on the prey population. The individual birth rate at time t is rBN_t. It is proportional to the amount of prey consumed by the predator. The parameter $r \in (0,1)$ represents the conversion efficiency of prey biomass into predator biomass. The predator individual death rate $D + CH_t$ includes logistic competition among predators ($D \geq 0$, $C > 0$).

The community dynamics is given by the differential equation

$$\frac{d}{dt}H_t = H_t(rBN_t - D - CH_t), \tag{1}$$

coupled with the jump mechanism

$$\begin{aligned}\mathbb{P}(N_{t+s} = j | N_t = n, H_t) &= bns + o(s) \qquad \text{if } j = n+1, n \geq 1,\\ &= n(d + cn + BH_t)s + o(s) \qquad \text{if } j = n-1, n \geq 2,\\ &= 1 - (b + d + cn + BH_t)ns + o(s) \qquad \text{if } n = j, n \geq 2,\\ &= 1 - bs + o(s) \qquad \text{if } n = j = 1,\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

The state space of the prey-predator process is denoted by

$$E = \mathbb{N}^* \times [0, \infty).$$

The process $Z_t = (N_t, H_t)_{t\geq 0}$ is a E-valued Markov process whose infinitesimal generator writes

$$\begin{aligned}\mathcal{L}f(n,h) = h(rBn - D - Ch)\partial_2 f(n,h) &+ \Big(f(n+1,h) - f(n,h)\Big)bn\\ &+ \Big(f(n-1,h) - f(n,h)\Big)n(d + cn + Bh)\mathbb{1}_{n\geq 2}.\end{aligned} \tag{2}$$

Its domains contains functions $f : E \to \mathbb{R}$ bounded measurable, continuously differentiable with respect to their second variable with bounded derivative.
In Theorem 2.4 ii) in [4] the author proves that the process Z_t is a Feller process in the sense that for any $g : E \to \mathbb{R}$ continuous and bounded, the function $z \mapsto \mathbb{E}_z(g(Z_t))$ is continuous and bounded on E, $\forall t \geq 0$.

In [4] Theorem 3.1, it is proved that the process Z_t is $\sigma-$ irreducible on $E_+ = \mathbb{N}^* \times (0, \infty)$ for σ the product of the counting measure on $\mathbb{N}^*$ and the Lebesgue measure on (h_1^*, ∞), where $h_1^* = \max((rB - D)/C, 0)$ is the stable equilibrium of (1) when the number of prey is equal to 1. This means that for all Borel set $A \in \mathcal{B}(E_+)$, one has

$$\sigma(A) > 0 \quad \Rightarrow \quad \forall z \in E_+, \int_0^{+\infty} \mathbb{P}_z(Z_t \in A)dt > 0. \tag{3}$$

Note that when $h_1^* = 0$, Z_t is irreducible on all E_+, while when $h_1^* > 0$, the set $\mathbb{N}^* \times (0, h_1]$ is transient for the dynamics.

1.2 A persistence - extinction result

Let us remark that the state space E can be decomposed into two invariant sets

$$E = E_0 \cup E_+ = \mathbb{N}^* \times \{0\} \cup \mathbb{N}^* \times (0, \infty).$$

E_0 corresponds to the set of extinction of the predators, while E_+ corresponds to their persistence. The main goal here is to find a criterion for extinction or persistence of predators in the dynamics.

Behavior of the process on E_0 On E_0 there is no predator, the prey population N_t evolves as a logistic birth and death process with birth, death and competition represented by the parameters b, d and c. Here the death rate of preys in null when only 1 prey is left avoiding the extinction of preys. Hence, $(N_t)_{t\geq 0}$ admits a unique invariant distribution μ on $\mathbb{N}^*$ satisfying

$$\mu_n = \frac{b^{n-1}}{n\prod_{i=1}^{n}(d+ci)}\mu_1, \qquad \sum_{i=1}^{\infty}\mu_i = 1.$$

Remark 1. *The fact that the death rate is 0 when there is only one prey left prevents from an extinction in finite time of the population of preys. For a similar model but with extinction of the slow dynamic, and convergence of conditional law towards quasi-stationary distributions, the reader is referred to [3].*

In the absence of predators, at equilibrium, the mean number of prey is given by

$$\hat{n}_0 = \sum_{n=1}^{+\infty} n\mu_n.$$

If a small amount of predator is introduced in a population of prey at equilibrium, its growth rate will be (neglecting the nonlinear terms)

$$\Lambda = rB\hat{n}_0 - D. \tag{4}$$

Intuitively, it should happen that if $\Lambda > 0$, the predators are able to invade the prey population and to have a community equilibrium - that is, an invariant distribution that gives mass no mass to E_0. On the contrary, if $\Lambda \leq 0$, the predators should not be able to invade and thus go to extinction. This is indeed the case, as stated in the following result.

We let $\mathcal{C}_{exp}$ be the set of continuous functions $f : E \to \mathbb{R}$ such that for some $\varepsilon > 0$ and $\alpha > 0$, $f(n,h)e^{(-(2r)^{-1}+\varepsilon)h}e^{-\alpha n} \to 0$ as $(n,h) \to \infty$.

Theorem 2.

1. *If $\Lambda > 0$, then the community process $(Z_t)_{t\geq 0}$ admits a unique stationary distribution π such that $\pi(E_+) = 1$. Moreover, it converges exponentially fast towards it stationary distribution: there exist $R, \theta > 0$ and $\rho \in (0,1)$ such that for all $z = (n,h) \in E_+$ and $t \geq 0$,*

$$\|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{TV} \leq R\rho^t \left(1 + \left(\frac{1+rn+h}{h}\right)^{\theta}\right) \tag{5}$$

   where $\|\cdot\|_{TV}$ stands for the total variation norm.

2. *If $\Lambda < 0$, the predators go to extinction exponentially fast: for all $z \in E_+$,*

$$\mathbb{P}_z\left(\limsup_{t\to\infty} \frac{1}{t}\log(H_t) \leq \Lambda\right) = 1. \tag{6}$$

   In addition, $\mathbb{P}_z$ - almost surely, for all functions $f \in \mathcal{C}_{exp}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)ds = (\mu \otimes \delta_0)f.$$

3. If $\Lambda = 0$, then for all $z \in E_+$, $\mathbb{P}_z$ - almost surely, for all functions $f \in \mathcal{C}_{exp}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)ds = (\mu \otimes \delta_0)f.$$

In particular,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t H_s ds = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t N_s ds = \hat{n}_0.$$

Note that here the condition for persistence of the predator population $\Lambda > 0$ does not require that $h_1^* > 0$. This implies that for some small number of preys, the predator population could not survive, but the global survival of predators is due to the stochastic dynamics of the prey population size.

2 Proof of the corrected result

As announced, in order to prove Theorem 2, we use results from Stochastic Persistence theory as presented in [1]. We introduce some definitions taken from [1].

Definition 1. *Let $\mathcal{E}$ be one of the sets E, E_+ or E_0. We say that a couple of continuous functions (f, g) defined on $\mathcal{E}$ with value in $\mathbb{R}$ belongs to the extended domain of the generator, and denote $(f, g) \in \mathcal{D}^{ext}(\mathcal{E})$ if for all $x \in \mathcal{E}$, the process*

$$M_t^f = f(Z_t) - f(z) - \int_0^t g(Z_s)ds \tag{7}$$

is a local martingale with respect to the natural filtration of $(Z_t)_{t\geq 0}$.

The domain of the extended generator (2) has been characterized by Davis (Theorem 26.14 in [5]) in the case of piecewise deterministic processes. In particular we have that for all functions $f : \mathcal{E} \to \mathbb{R}$ measurable with respect to their first coordinate and $\mathcal{C}^1$ in their second, then $(f, \mathcal{L}f)$ belongs to $\mathcal{D}^{ext}(\mathcal{E})$.

Let us now introduce the operator Γ acting on measurable functions $f : E \to \mathbb{R}$ as

$$\Gamma f(n,h) = \Big(f(n+1,h) - f(n,h)\Big)^2 bn + \Big(f(n-1,h) - f(n,h)\Big)^2 n(d+cn+Bh)\mathbb{1}_{n\geq 2}.$$

Note that by Lemma 9.1 in [1], for all function f bounded measurable, $\mathcal{C}_b^1$ in their second variable, then Γ coincide with the *carré du champ* and

$$(M_t^f)^2 - \int_0^t \Gamma f(Z_s)ds$$

is a squared integrable martingale, where $M_t^f = f(Z_t) - f(Z_0) - \int_0^t \mathcal{L}f(Z_s)ds$.

We now extend this definition.

Definition 2. *We define $\mathcal{D}_2^{ext}(\mathcal{E})$ as the subset of $\mathcal{D}^{ext}(\mathcal{E})$ of functions f such that the process M_t^f defined by (7) is a square integrable martingale and for which there exists a continuous function h such that*

$$(M_t^f)^2 - \int_0^t h(Z_s)ds \tag{8}$$

is a martingale.

In the proofs, we will use a localization argument given by Lemma 8.12 in [6] in order to identify functions in $\mathcal{D}_2^{ext}$, and we will observe that the function h in (8) equals to Γf.

We say that a map $W : \mathcal{E} \to \mathbb{R}_+$ is *proper* if it is continuous and if for all $R > 0$, the set $\{W \leq R\}$ is compact in $\mathcal{E}$. We denote by $\mathcal{P}_{inv}(\mathcal{E})$ the (possibly empty) set of invariant distributions of $(Z_t)_{t\geq 0}$ such that $\nu(\mathcal{E}) = 1$. We define the empirical occupation measure as

$$\Pi_t = \frac{1}{t}\int_0^t \delta_{Z_s} ds.$$

We will make use at several places of the following result, which is a consequence of [1, Theorem 2.2 and Lemma 9.4].

Lemma 3. *Assume that there exists a proper map $W : E \to \mathbb{R}_+$ such that*

$$\mathcal{L}W \leq -\lambda W + C$$

for some constants $\lambda, C > 0$. Then, for all $z \in E$, the sequence $(\Pi_t)_{t>0}$ is $\mathbb{P}_z$ - almost surely tight. Moreover, all weak-limit points of $(\Pi_t)_{t>0}$ are in $\mathcal{P}_{inv}(E)$, and if for some sequence $t_n \to \infty$, $(\Pi_{t_n})_{n\geq 0}$ converges to some measure ν, then, for all continuous function f such that $\frac{W}{1+|f|}$ is proper, we have

$$\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(Z_s)ds = \nu f.$$

2.1 Proof of extinction when $\Lambda < 0$

In order to prove extinction, we will use Theorem 3.5 in [6]. According to this Theorem, we have the convergence in (6) if Assumptions 1 to 5 in [6] are satisfied and if the set E_0 is accessible from E_+ in the following sense: for all $z \in E_+$, there exists M such that, for all $D > 0$, there exists $t > 0$ such that $\mathbb{P}_z(Z_t \in \{V > D\} \cap \{W < M\}) > 0$.

Assumptions 1 and 2 in [6] are satisfied since $E = E_0 \cup E_+$ where E_0 and E_+ are invariant, E_0 are closed, and the process $(Z_t)_{t\geq 0}$ is $\mathcal{C}_b$ - Feller.

The Assumptions 3 to 5 requires to construct different Lyapunov functions in order to handle the behaviour of the process both at infinity and near E_0.

For some $\alpha, \beta > 0$, we define the function

$$W(n,h) = e^{\alpha n + \beta h} \tag{9}$$

Lemma 4. *Set $U = W^2$. Then, if $2\beta r < 1 - e^{-2\alpha}$, there exist constants $K, \lambda > 0$ and a continuous positive map U' such that*

$$\mathcal{L}W \leq K - \lambda W,$$

$$\mathcal{L}U \leq K - U',$$

$$\Gamma W \leq KU'.$$

In addition, $(W, \mathcal{L}W) \in \mathcal{D}_2^{ext}(E)$*.*

Proof. We can easily see that

$$\mathcal{L}W(n,h) = W(n,h)\psi(n,h),$$

where

$$\begin{aligned}\psi(n,h) &= \beta h(rBn - Ch - D) + bn(e^{\alpha} - 1) + n(d + cn + Bh)\mathbb{1}_{n\geq 2}(e^{-\alpha} - 1) \\ &= -\beta h^2 - nhB[1 - e^{-\alpha} - \beta r] - n^2 c(1 - e^{-\alpha}) - \beta hD + bn(e^{\alpha} - 1) \\ &\quad + (e^{-\alpha} - 1)(nd\mathbb{1}_{n\geq 2} - Bh\mathbb{1}_{n=1})\end{aligned}$$

Since $2\beta r \leq 1 - e^{-2\alpha}$, we have $\beta r \leq 1 - e^{-\alpha}$ and thus $\psi(n,h) \to -\infty$ as $(n,h) \to \infty$. In particular, for all $\lambda > 0$, there exists a constant $K > 0$ such that

$$\mathcal{L}W \leq K - \lambda W.$$

Similarly, we obtain that

$$\mathcal{L}U(n,h) = U(n,h)\psi_2(n,h),$$

where

$$\psi_2(n,h) = 2\beta h(rBn - Ch - D) + bn(e^{2\alpha} - 1) + n(d + cn + Bh)\mathbb{1}_{n\geq 2}(e^{-2\alpha} - 1),$$

and since $2\beta r \leq 1 - e^{-2\alpha}$, $\psi_2(n,h) \to -\infty$ as $(n,h) \to \infty$. In particular, there exist a constant K large enough such that $U' = K - \mathcal{L}U \geq 0$, and by definition

$$\mathcal{L}U = K - U'$$

For the last inequality, let us remark that

$$\Gamma W = U^2 \psi_3(n,h)$$

with

$$\psi_3(n,h) = (e^{\alpha} - 1)^2 bn + (e^{-\alpha} - 1)^2 n(d + cn + Bh)\mathbb{1}_{n\geq 2}.$$

Then, in order to have $\Gamma W \leq KU'$, it is sufficient to choose K such that

$$U\psi_3 \leq K^2 - K\mathcal{L}U.$$

To conclude, let us note that ψ_3 is bounded on the compact set $\{\mathcal{L}U \geq -\varepsilon\}$. Outside this compact set if suffices to prove that for some constant K,

$$\frac{\psi_3}{\psi_2} \geq -K + \frac{K^2}{\mathcal{L}U} \geq -K,$$

which is possible since ψ_3 is a positive quadratic function, ψ_2 is a quadratic function bounded by above and thus $\frac{\psi_3}{\psi_2}$ is bounded by below on $\{\mathcal{L}U \geq -\varepsilon\}$.

It remains to prove that $W \in \mathcal{D}_2^{ext}(E)$. We use the localization argument of Lemma 8.12 in [6]. Since any continuous bounded function f, continuously differentiable with respect to its second coordinate with bounded derivative, f and f^2 are in the domain of the infinitesimal generator, points 1., 2. and 3. of [6, Lemma 8.12] are easily satisfied by setting for all $m \in \mathbb{N}^*$ $W_m = W\phi_m$ for a smooth function ϕ_m such that $\phi_m(n,h) = 1$ if $\|(n,h)\| \leq m$, $\phi(m,h) = 0$ if $\|(n,h)\| > m+1$ and derivatives of ϕ with respect to h are bounded. Point 4. of [6, Lemma 8.12] has just been proved above by the choice of K and U'. □

Our Lemma 4 above implies Assumption 3 in [6]. To check Assumption 4, we define on E_+ the function

$$V_1(n,h) = -\log(h),$$

and set, for all $(n,h) \in E$,

$$\mathbf{H}_1(n,h) = -(rBn - D - Ch).$$

It is easily seen that $\mathbf{H}_1$ coincide with $\mathcal{L}V_1$ on E_+ and is continuous on all E. Since V_1 depends only on h, it is easily seen that M_t^V, the local martingale defined in (7) is null, so that $(V, \mathcal{L}V) \in \mathcal{D}_2^{ext}(E_+)$ and $\Gamma V = 0$.

Moreover, by definition of Λ, we have

$$\mu \mathbf{H}_1 = -\Lambda > 0.$$

Note that for all sequence $z_n \in E_+$, $V(z_n) \to \infty$ if and only if $z_n \to E_0$. In addition, $\lim_{z\to\infty} \frac{H_1}{W} = 0$. This proves that [6, Assumption 4] holds true in our model. Regarding [6, Assumption 5], it is a direct consequence of Lemma 4 and the fact that $\Gamma V_1 = 0$.

Recall that the process Z is $\sigma-$irreducible on E_+. Applying (3) with $A = \{V > D\} \cap \{W < M\}$, there exists $t > 0$ such that $\mathbb{P}_z(Z_t \in A) > 0$ and the accessibility condition is satisfied. This concludes the proof of (6) by [6, Theorem 3.5].

2.2 Proof of extinction when $\Lambda = 0$

We prove by contradiction that when $\Lambda = 0$, $\mathcal{P}_{inv}(E_+)$ is empty, which will give the announced convergence result by Lemma 3 and 4. Note that since the process $(Z_t)_{t\geq 0}$ is σ - irreducible on E_+, it admits at most one invariant distribution on E_+. Therefore, if we assume that $\mathcal{P}_{inv}(E_+)$ is non-empty, we have exactly one invariant, hence ergodic, distribution on E_+, that we denote μ_*. By Birkhoff ergodic Theorem, for all $z \in E_+$, we have $\mathbb{P}_z$ - almost surely that Π_t converges, in the sense of distribution, towards μ_*. Moreover, by Lemma 3 and 4, for all $z \in E_+$, $\mathbb{P}_z$ - almost surely,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(Z_s)ds = \mu_* f, \tag{10}$$

for all $f \in \mathcal{C}_{exp}$ Applying this to $f(n,h) = n$, we find

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t N_s ds = \int_{E_+} n\mu_*(dn, dh) := \hat{n}_*.$$

Let $z = (n,h) \in E_+$ and $z_0 = (n,0) \in E_0$ (wih the same n). We can couple the processes $Z_t^z = (N_t^z, H_t^z)$ and $Z_t^{z_0} = (N_t^{z_0}, 0)$, with respective initial conditions z and z_0 such that $N_t^z \leq N_t^{z_0}$ $a.s.$ for all $t \geq 0$. This is possible thanks to the fact that N_t^z and $N_t^{z_0}$ are birth and death processes, and when $N_t^z = N_t^{z_0}$, they have the same birth rate, while the death rate of N_t^z is greater than those of $N_t^{z_0}$, due to the presence of predators in the first process. This implies that (using once again Lemma 3)

$$\hat{n}_0 = \lim_{t\to\infty} \frac{1}{t}\int_0^t N_s^{z_0} ds \geq \lim_{t\to\infty} \frac{1}{t}\int_0^t N_s^{z} ds = \hat{n}_*.$$

On the other hand, since $\mu^*(E_+) = 1$, we have by Lemma 4.6 in [1] that $\mu^* \mathbf{H}_1 = 0$ (note that the proof of this fact does not require H- persistence nor positivity of V_1). Hence,

$$\int_{E_+} (rBn - D - Ch)\, \mu_*(dn, dh) = 0,$$

which gives

$$\hat{n}_* = \frac{D}{rB} + \frac{C}{rB}\hat{h}_*,$$

where $\hat{h}_* = \int h d\mu_*(n,h)$ is strictly positive because $\mu_*(E_+) = 1$. But

$$0 = \Lambda = rB\hat{n}_0 - D,$$

which means that

$$\hat{n}_0 = \frac{D}{rB} < \hat{n}_*,$$

and leads to a contradiction. Therefore, $\mathcal{P}_{inv}(E_+)$ is empty, and finally, $\mathcal{P}_{inv}(E) = \{\mu \otimes \delta_0\}$.

2.3 Proof of persistence when $\Lambda > 0$

We would like to apply [1, Theorem 4.13]. To do so, we need to verify Hypothesis 1 to 5 from the article.

Hypothesis 1 and 2 only require that E_0 is closed and Z_t is $\mathcal{C}_b$ - Feller which is the case here.

Hypothesis 3 concerns the Lyapunov function W introduced in (9). Using the results of Lemma 4 it remains to prove that the martingale M_t^W associated with W through (7) satisfies a law of large numbers. From Lemma 4 we can apply [6, Corollary 4.2] and deduce that W has "linear bounded quadratic variation", which allows to apply [6, Lemma 2.12] and conclude.

We now built a Lyapunov function V controlling the behavior of Z_t close to E_0, that is satisfying Hypothesis 4 and 5. Let us define

$$V(n,h) = -\varepsilon \log(h) + \log(1 + rn + h),$$

for $\varepsilon < 1$ such that $\forall (n,h) \in E_+$,

$$V(n,h) > 0.$$

Denote by

$$\mathbf{H}(n,h) = \mathcal{L}V(n,h).$$

Lemma 5. *There exists $\varepsilon > 0$ such that $(V, \mathbf{H}) \in \mathcal{D}^{ext}(E_+)$, such that*

i) V is proper,

ii) There exists a compact $C \subset E_+$ such that $\sup_{E_+ \setminus C} \mathbf{H} < 0$, *iii) There exists $q > 1$, such that $|\mathbf{H}|^q \leq cst(1+W)$,*

iv) The jumps for $V(Z_t)$ are almost surely bounded.

v) $\gamma = \sup\{\|\Gamma V_{|C}\|, C \subset E_+,$ compact$\} < \infty$.

where W and U' are defined in Lemma 4.
As a consequence, the process (Z_t) is H-persistent (strong version') with respect to E_0 and at infinity

Note that the Lyapunov function $V_1 = -\log(h)$ used for the extinction argument, does not satisfy Hypothesis 4 in [1] since it is not positive nor proper and $\mathbf{H}_1$ does not verify $ii)$.

We therefore modify V using an additional function $V_2 = \log(1 + rn + h)$ following the ideas developed by [7].

Proof. **Proof of i)** By definition $i)$ is satisfied.
Proof of ii) and iii) We have

$$\mathbf{H}(n,h) = -\varepsilon(rBn - D - Ch) + \log\left(1 + \frac{r}{1+rn+h}\right) bn \\ + \log\left(1 - \frac{r}{1+rn+h}\right) n(d + cn + Bh)\mathbb{1}n \geq 2 + \frac{h(rBn - D - Ch)}{1+rn+h}$$

We deduce $ii)$ using that $\log(1+x) \sim x$ as $x \to 0$, and using that $\varepsilon < 1$. Recall now that $W(n,h) = e^{\alpha n + \beta h}$, then $iii)$ is obvious for any $q > 1$.
Proof of iv) By definition, since the jumps of Z_t are of size $+1$ or -1, we have that for all time $t \geq 0$,

$$|V(Z_t) - V(Z_{t-})| = \log\left(1 + \frac{r(N_t - N_{t-})}{1 + rN_t + H_t}\right) \\ \leq \sup_{(n,h)\in E_+} \left\{\log\left(1 + \frac{r}{1+rn+h}\right), \log\left(1 - \frac{r}{1+rn+h}\right)\right\} < \infty.$$

Proof of v) By definition,

$$\Gamma V(n,h) = \log\left(1 + \frac{r}{1+rn+h}\right)^2 bn + \log\left(1 - \frac{r}{1+rn+h}\right)^2 n(d + cn + Bh)\mathbb{1}n \geq 2.$$

We easily observe that ΓV is a bounded function on E and thus condition $v)$ is satisfied.
Conclusion In order to conclude that $(V, \mathbf{H})$ satisfies Hypothesis 4 et 5 in [1], it remains to justify that the martingale M^V associated through (7) satisfies a strong law. To do so, we use similar arguments as for Hypothesis 3 above combining Lemma 2.12 and Corollary 4.2 in [6]. As a consequence, we only need to justify $\Gamma V \leq KU'$ where U' was defined in Lemma 4. Recall that $U' = K - U\psi_2$, and in particular, $U' \to \infty$ as $(n,h) \to \infty$. Therefore, v) ensures that for K large enough, $\Gamma V \leq KU'$. □

In order to apply Theorem 4.13 of [1], we finally need to prove that there exists point $z* = (n^*, h^*) \in E' = \mathbb{N}^* \times (h_1^*, \infty)$ such that for a neighbourhood $\mathcal{U}$ of z^* in E', a time t^*, and a non-zero measure ξ we have

$$\mathbb{P}_z(Z_{t^*} \in \cdot) \geq \xi(\cdot)$$

for all $z \in \mathcal{U}$. This point is called a Doeblin point and will necessary be accessible since Z_t is irreducible on E'.

Let us fix $z^* \in E'$, and consider a time $t > 0$. We will restrict ourselves to trajectories where the prey population only jumps by $+1$ on $[0, t]$. We will denote by $t \mapsto \phi_n(h, t)$ the solution of (1) starting from h at $t = 0$. Let us consider an initial condition of the form $(n^*, h) \in E'$ and $A \subset (h_1^*, \infty)$, and denote by T_1 and T_2 the two first jump times of N_t then

$$\begin{aligned}\mathbb{P}_{(n^*,h)}(H_t \in A, N_t = n+1) &\geq \mathbb{P}_{(n^*,h)}(H_t \in A, T_1 \leq t < T_2, N_{T_1} = n+1) \\ &\geq \mathbb{P}_{(n^*,h)}\left(\phi_{n+1}(t - T_1, \phi_n(T_1, h)) \in A, T_1 \leq t < T_2, N_{T_1} = n+1\right)\end{aligned}$$

Let us first remark that for small t, the maps $s \mapsto \phi_{n+1}(t - s, \phi_n(s, h))$ is a bijection from $[0, t]$ to its image. Let us remark

$$\lim_{s,t \to 0} \partial_s \phi_{n+1}(t - s, \phi_n(s, h)) = -h(rB(n+1) - D - Ch) + h(rBn - D - Ch) < 0,$$

then we obtain the bijection using a continuity argument in t and h.
Furthermore, on the event $T_1 \leq t < T_2$, the jump rates of the prey population are bounded by above, and as a consequence the joint density of (T_1, T_2) is uniformly bounded by below by a positive constant with respect to h and $s \in [0, t]$. Combining these arguments, we can apply a reasoning similar as in [2] (based on a change of variables) to deduce that for a neighborhood $\{n^*\} \times I$ of z^* and a small enough t, there exists an interval $J \subset (h_1^*, \infty)$ such that

$$\mathbb{P}_{(n^*,h)}(H_t \in A, N_t \in B) \geq c\lambda(A \cap J)\delta_{n^*+1}(B), \qquad \forall h \in I$$

where λ is the Lebesgue measure on $\mathbb{R}$.

As a consequence, we have verified all the assumptions of Theorem 4.13 of [1] which conclude the proof of persistence.

References

[1] Michel Benaïm. Stochastic persistence. *arXiv preprint arXiv:1806.08450*, 2018.

[2] Michel Benaïm, Sthéphane Le Borgne, Florent Malrieu, and Pierre André Zitt. Qualitative properties of certain piecewise deterministic Markov processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(3):1040–1075, 2015.

[3] Bertrand Cloez and Coralie Fritsch. Quasi-stationary behavior for a piecewise deterministic markov model of chemostat: the crump–young model. *Annales Henri Lebesgue*, 6:1371–1427, 2023.

[4] Manon Costa. A piecewise deterministic model for a prey-predator community. *The Annals of Applied Probability*, pages 3491–3530, 2016.

[5] M. H. A. Davis. *Markov Models & Optimization*, volume 49. CRC Press, 1993.

[6] Juraj Foldes and Declan Stacy. Stochastic extinction, an average lyapunov function approach. *arXiv preprint arXiv:2407.19606*, 2024.

[7] Alexandru Hening and Dang H. Nguyen. Coexistence and extinction for stochastic Kolmogorov systems. *Ann. Appl. Probab.*, 28(3):1893–1942, 2018.