Derrida–Retaux model: from discrete to continuous time

Michel Pain (ENS Paris / Sorbonne Université)
joint work with **Yueyun Hu** and **Bastien Mallein** (Paris 13)

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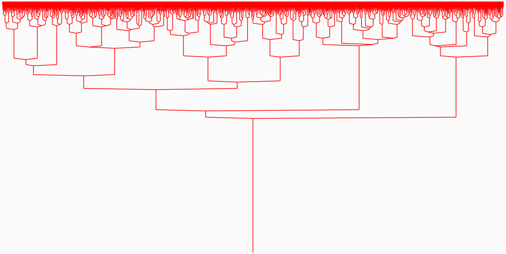

Discrete-time Derrida–Retaux model

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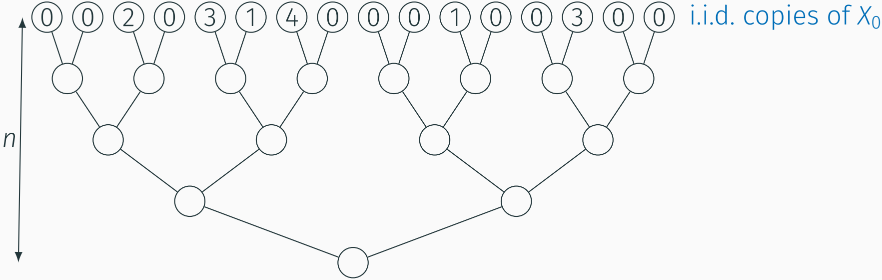
- **Definition:** Start with a nonnegative random variable X_0 and, for any $n \geq 0$,
$$X_{n+1} = \left(X_n + \widetilde{X}_n - 1\right)_+$$
where $\widetilde{X}_n$ is an independent copy of X_n.

Definition on a tree

Construction of X_n on a binary tree:



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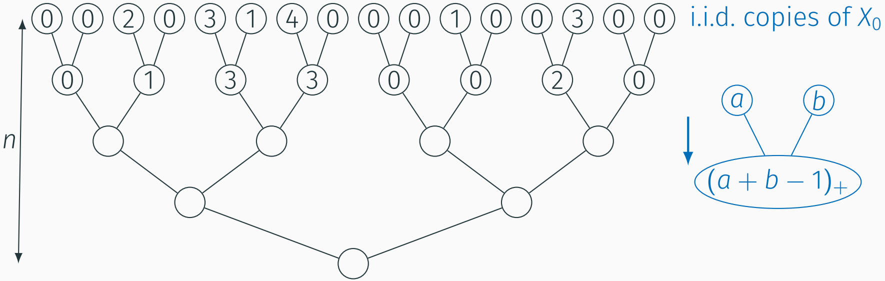

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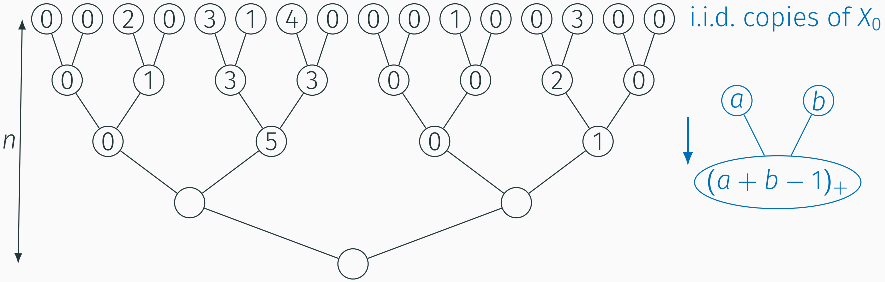

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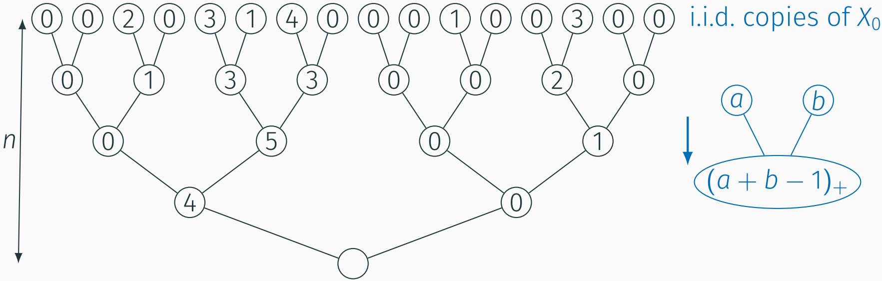

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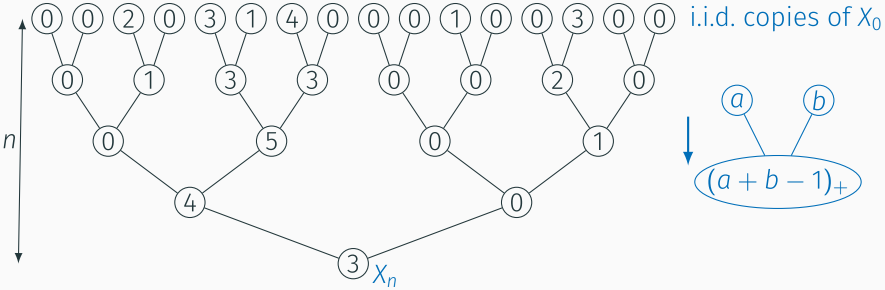

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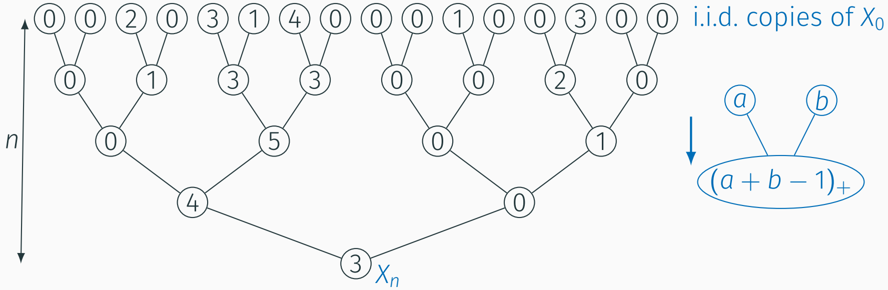

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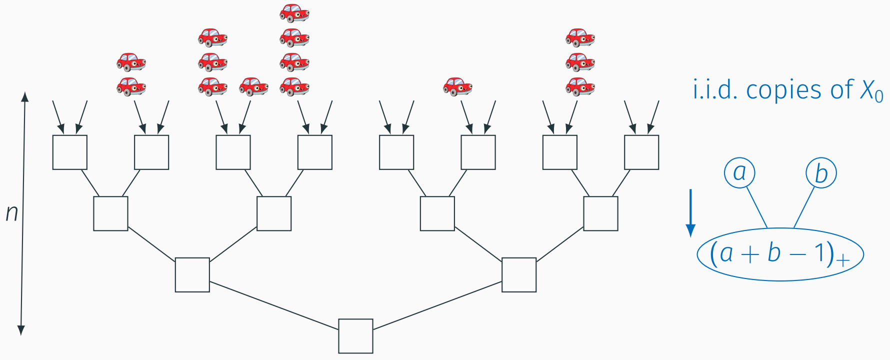

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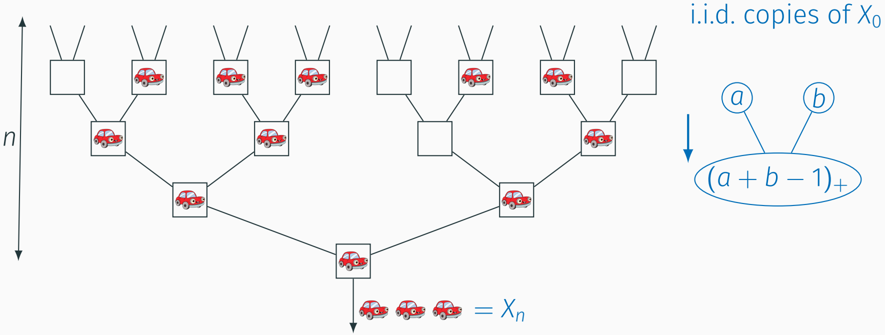

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Free energy: $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}[X_n]}{2^n} \in [0,\infty]$.

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Open question: Try to say something about the case where X_0 is not integer-valued.

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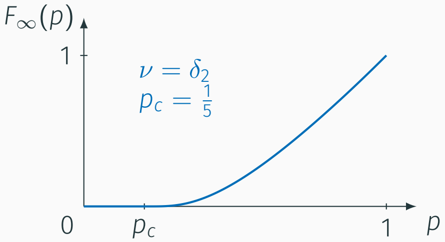

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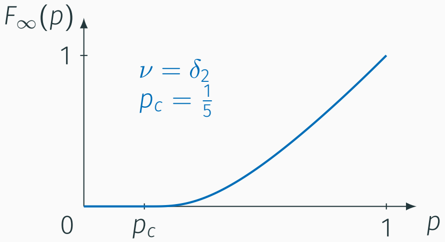

- If $X_0 \in \mathbb{N}$ a.s., then p_c is explicit by CEGM 1984.

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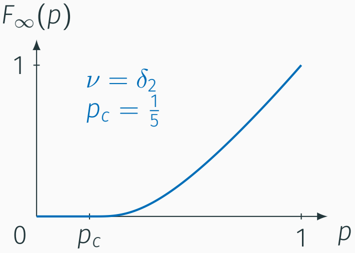
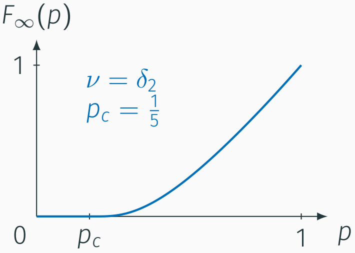
If $p_c > 0$, then as $p \downarrow p_c$

$$F_\infty(p) = \exp\left(-\frac{K + o(1)}{(p - p_c)^{1/2}}\right).$$





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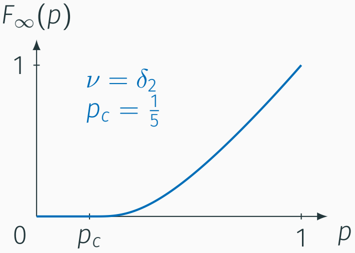

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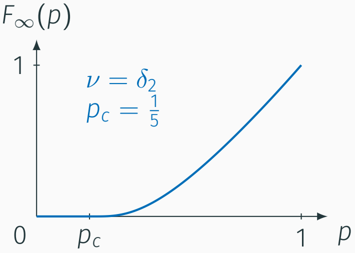

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- ▹ Hu–Shi 2018: case $p_c = 0$.

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$$\frac{c_1}{n} \leq \mathbb{E}\left[2^{X_n}\right] - 1 \leq \frac{c_2}{n}.$$

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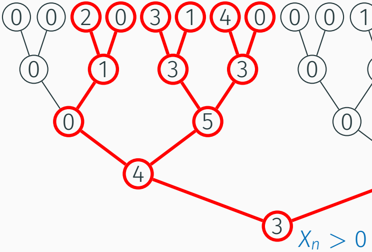
$$\mathbb{P}(X_n > 0) \sim \frac{4}{n^2}.$$

  Moreover, given $X_n > 0$, X_n *converges in law to a geometric distribution with parameter* $\frac{1}{2}$.

The red tree at criticality

Given that $X_n > 0$, we color in red the paths from a leaf to the root, where the operation "positive part" was not needed.



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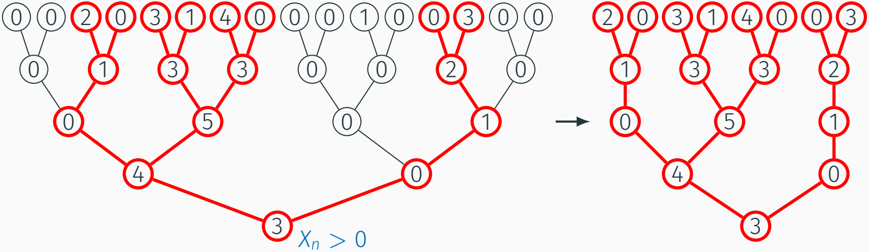
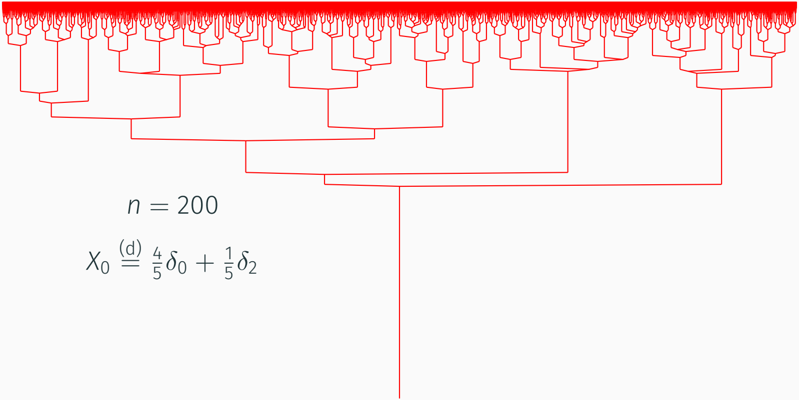


The red vertices form a subtree, called the *red tree*.

Questions concerning the red tree

Question: *Given $X_n > 0$, what does the red tree look like for large n?*



$n = 200$

$X_0 \overset{(d)}{=} \frac{4}{5}\delta_0 + \frac{1}{5}\delta_2$

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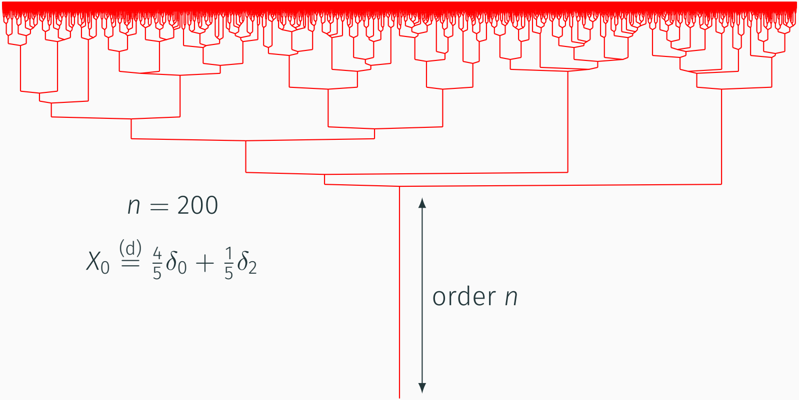

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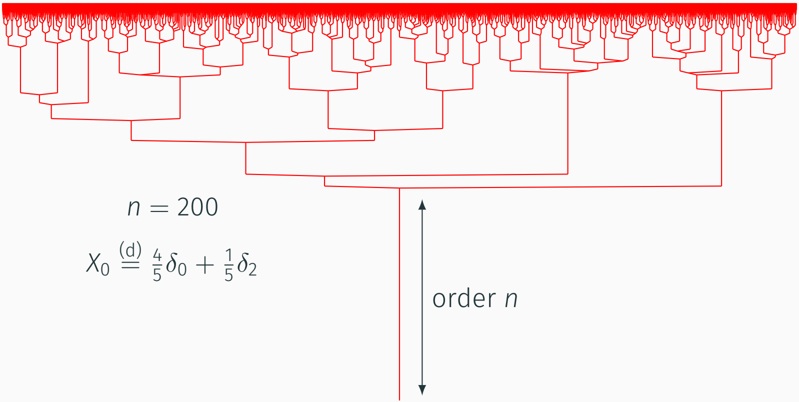

Continuous-time Derrida–Retaux model

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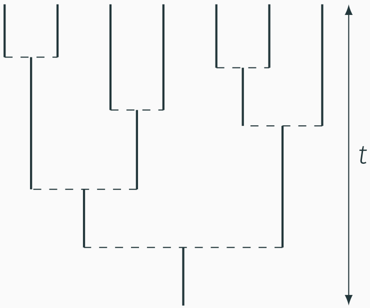

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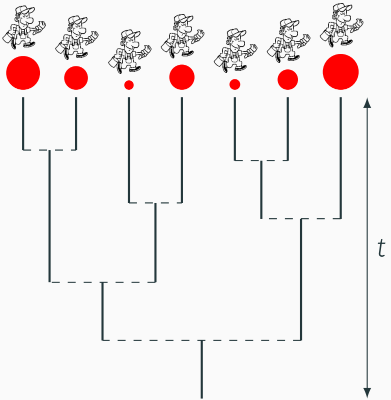

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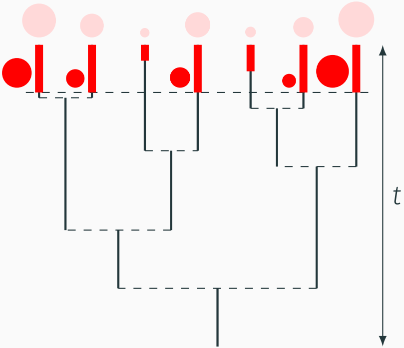

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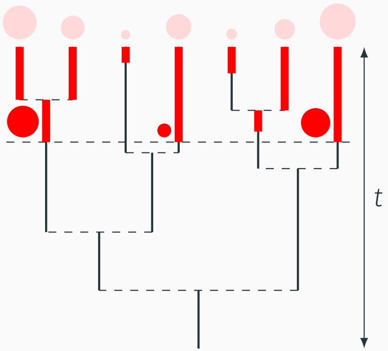

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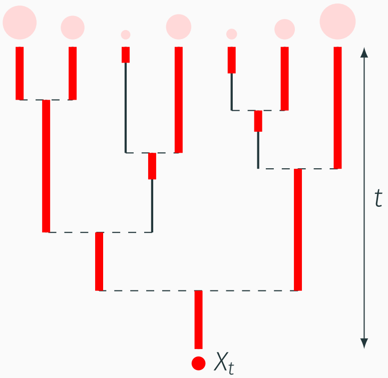
- X_t is the remaining paint at the root.



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- **Open question:** If $F_\infty = 0$, then prove that $X_t \xrightarrow[t\to\infty]{\text{probability}} 0$.
- **Proposition:** *Let* μ_t *denote the distribution of* X_t *for each* $t \geq 0$*. Then,* $(\mu_t)_{t\geq 0}$ *is the unique family of positive measures on* $\mathbb{R}$ *solution (in the weak sense) of the PDE*

$$\partial_t \mu_t = \partial_x(\mathbb{1}_{\{x>0\}}\mu_t) + \mu_t * \mu_t - \mu_t,$$

with initial condition μ_0.

An exactly solvable family of solutions

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- **Proposition:** *For any $t \geq 0$, $\mu_t = p(t)\delta_0(dx) + (1-p(t))\lambda(t)\mathrm{e}^{-\lambda(t)x}\,dx$, where $p\colon \mathbb{R}_+ \to [0,1]$ and $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ are the unique solutions of the ODE*

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- $H := \dfrac{p(t)}{\lambda(t)} + \log \lambda(t)$ is an invariant of the dynamics.

The phase transition

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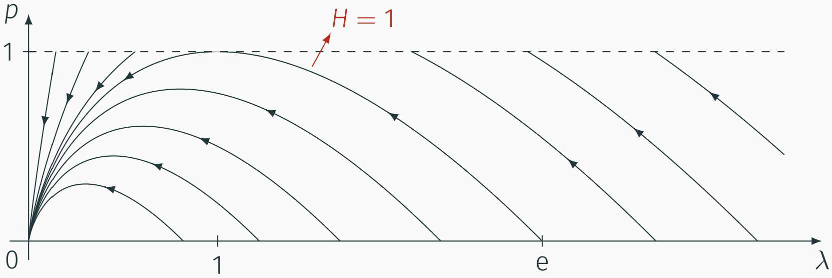

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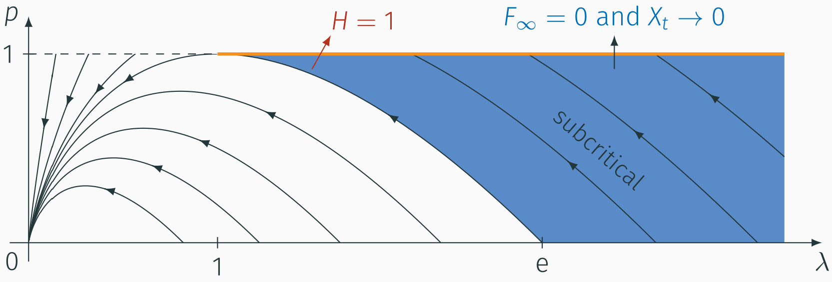

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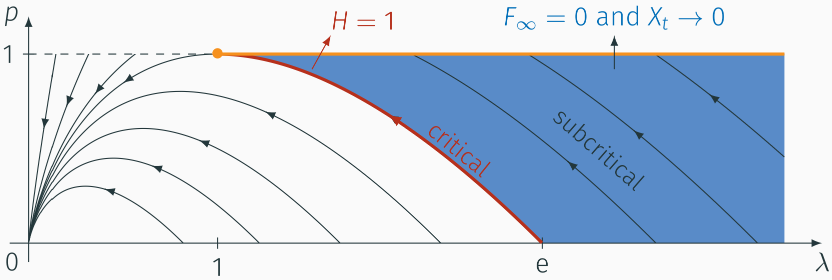

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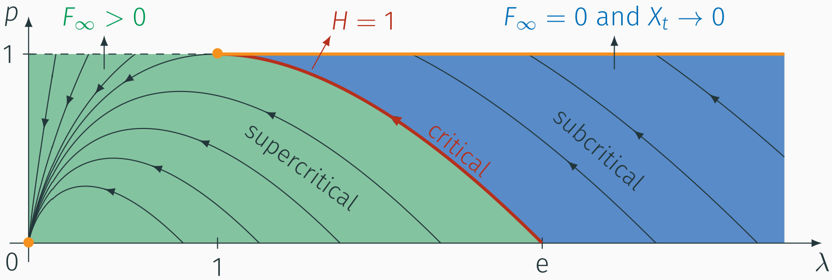

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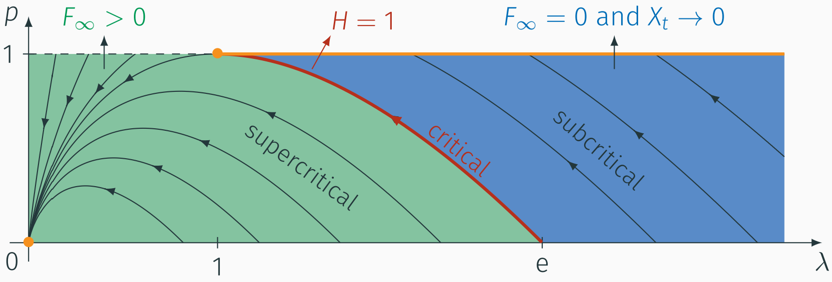

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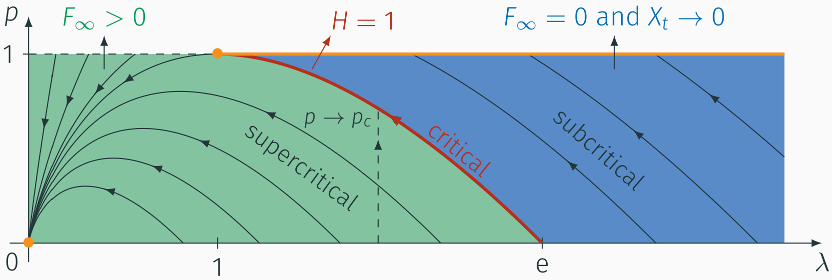

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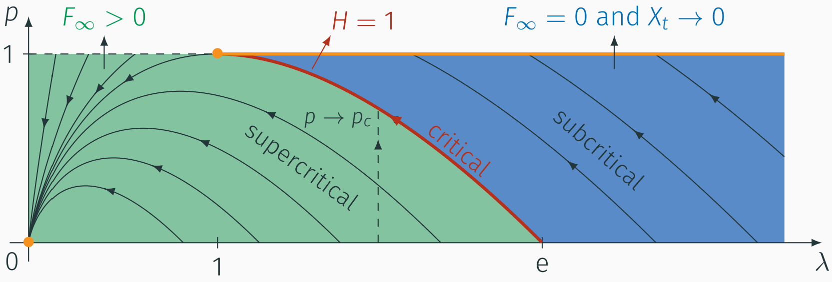

One can make explicit computations:

- Infinite order transition for the free energy with exponent $\frac{1}{2}$.
- Precise asymptotic behavior of $p(t)$ and $\lambda(t)$ in each phase.

Behavior at criticality

Theorem: *With a critical initial condition ($\lambda_0 > 1$ and $p_0 = \lambda_0 - \lambda_0 \log \lambda_0$),*

$$\mathbb{P}(X_t > 0) = 1 - p(t) = \frac{2}{t^2} + \frac{16 \log t}{3t^3} + o\left(\frac{\log t}{t^3}\right).$$

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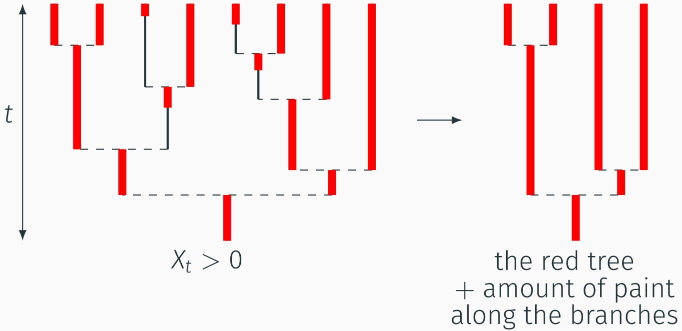
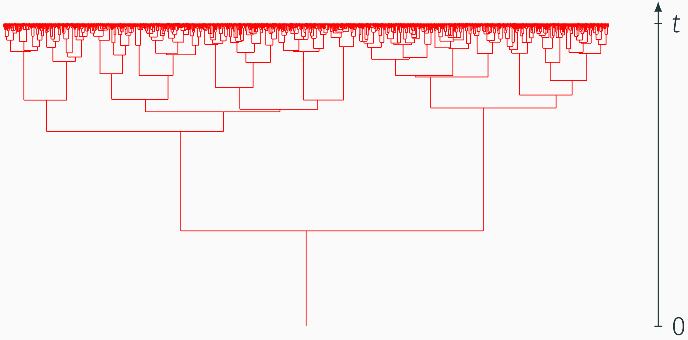
Our goal: Given $X_t > 0$, what does the subtree bringing paint to the root look like?



Description of the red tree



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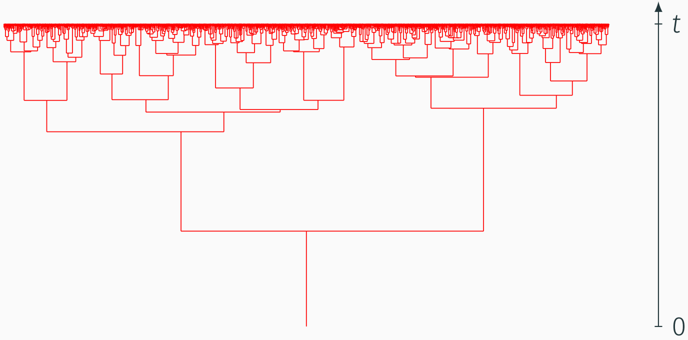

Given that $X_t = x$, the red tree of height t is a time-inhomogeneous branching Markov process defined on $[0, t]$ such that:

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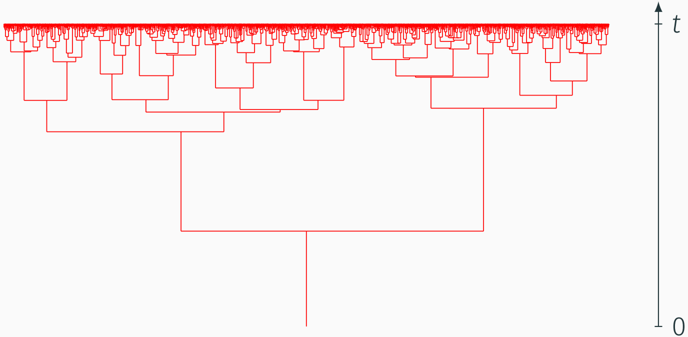

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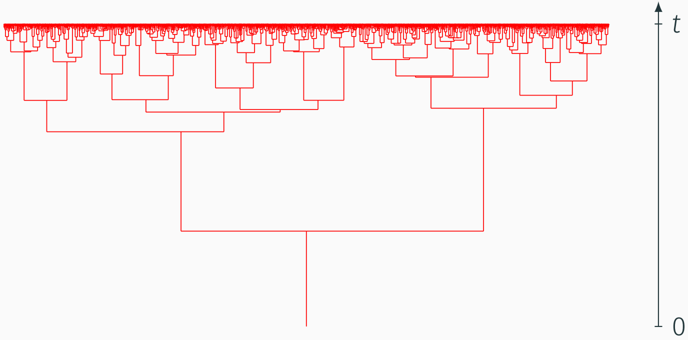

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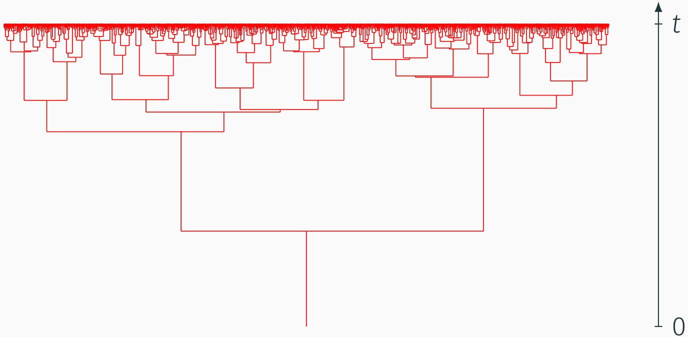

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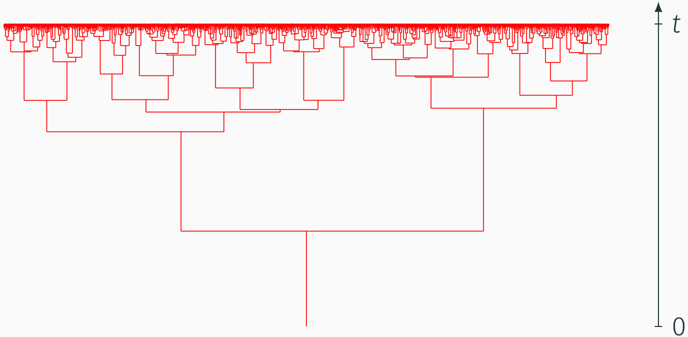

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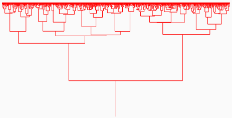
The scaling limit of the red tree

Let $(x_t)_{t \geq 0}$ be positive numbers such that $\frac{x_t}{t} \to x \geq 0$.



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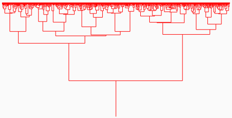

Theorem: *Given that $X_t = x_t$, the red tree of height t, with time and masses rescaled by t, converges locally in distribution to a time-inhomogeneous branching Markov process defined on $[0, 1)$ such that:*

- *It starts at time 0 with a single particle with mass x.*
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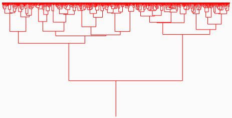

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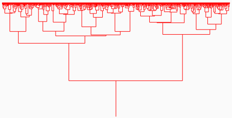

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Wide open question: universality among other hierarchical renormalization models?

Number and total mass of red leaves

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with $\eta_x := \int_0^1 r^2(s)\,ds$ and r a 4-dimensional Bessel bridge from 0 to $2\sqrt{x}$.

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Last open question: What is the law of the mass of a typical red leaf?

Thanks for your attention!