Lecture 4: More on additive martingales

20/12/2024

II.2) Martingales

For $\lambda \in \mathbb{R}$, $t \geq 0$, we define $W_t^\lambda := \sum_{u \in \mathcal{N}(t)} e^{\lambda X_u(t) - (\frac{\lambda^2}{2} + m)t}$.

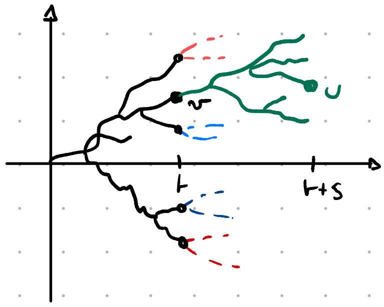
<u>Remark</u>: $W_t^0 = e^{-mt} N_t$.

<u>Proposition</u>: $(W_t^\lambda)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$-martingale, called <u>additive martingale</u>.

<u>Proof</u>: First note that, for $t \geq 0$,

$$E[W_t^\lambda] \underset{\text{many-to-one}}{=} e^{\cancel{mt}}\, E\left[e^{\lambda B_t - \frac{\lambda^2}{2}t - \cancel{mt}}\right] = 1 \quad \text{because } E[e^{\lambda B_t}] = e^{\frac{\lambda^2}{2}t}.$$

Now let $t, s \geq 0$.



$$W_{t+s}^\lambda = \sum_{v \in \mathcal{N}(t)} \sum_{\substack{u \in \mathcal{N}(t+s) \\ \text{s.t. } u \geq v}} e^{\lambda X_u(t+s) - (\frac{\lambda^2}{2}+m)(t+s)}$$

$$= \sum_{v \in \mathcal{N}(t)} e^{\lambda X_v(t) - (\frac{\lambda^2}{2}+m)t} \underbrace{\sum_{\substack{u \in \mathcal{N}(t+s) \\ \text{s.t. } u \geq v}} e^{\lambda (X_u(t+s) - X_v(t)) - (\frac{\lambda^2}{2}+m)s}}$$

by the branching property, conditionally on $\mathcal{F}_t$, these are independent random variables with the same law as W_s^λ

So $E[W_{t+s}^\lambda \mid \mathcal{F}_t] = \sum_{v \in \mathcal{N}(t)} e^{\lambda X_v(t) - (\frac{\lambda^2}{2}+m)t} \times \underbrace{E[W_s^\lambda]}_{=1} = W_t$. ∎

<u>Consequence</u>: $W_t^\lambda \xrightarrow{a.s.} W_\infty^\lambda$. (positive martingale!)

<u>Proposition</u>: Let $q = P(\forall t, N_t \geq 1)$ be the probability of survival.

For any $\lambda \in \mathbb{R}$, $P(W_\infty^\lambda > 0) \in \{0, q\}$.

$\Rightarrow$ either $W_\infty^\lambda > 0$ a.s. on survival, or $W_\infty = 0$ a.s.

<u>Proof</u> (similar to the case $\lambda = 0$!)

At the first branching time T_1, there are L children each one starts an independent BBM, write $W_\infty^{\lambda,1}, \dots, W_\infty^{\lambda,L}$ their associated limit of additive martingales. Then, $W_\infty^\lambda = 0 \iff \forall i \in \{1, \dots, L\}, W_\infty^{\lambda,i} = 0$

So $P(W_\infty^\lambda = 0) = E[P(W_\infty^\lambda = 0)^L]$ hence it is a fixed point of $f(s) = E[s^L]$

The two fixed points of f are $1-q$ and 1.

Last time we saw a theorem saying $\forall |\lambda| < \sqrt{2m}$, (W_t^λ) is uniformly integrable.
The proof was based on a 2^{nd} moment calculation for a truncated version of W_t^λ.
$\rightarrow$ we will see today a stronger result using a different proof:

Theorem (Neveu 1988): Assume $\mathbb{E}[L^2] < \infty$.

- If $|\lambda| < \sqrt{2m}$, then $W_t^\lambda \longrightarrow W_\infty^\lambda$ in L^p for any $p \in (1,2] \cap \left(1, \frac{2m}{\lambda^2}\right)$ and in particular $W_\infty^\lambda > 0$ a.s. on survival.
- If $|\lambda| \geq \sqrt{2m}$, then $W_\infty^\lambda = 0$ a.s.

Lemma (von Bahr-Esseen 1965): Let $n \geq 1$, $p \in [1,2]$. Let $X_1, \dots, X_n \in L^p$ be independent centered r.v. Then $\mathbb{E}\left[\left|\sum_{k=1}^n X_k\right|^p\right] \leq 2 \sum_{k=1}^n \mathbb{E}[|X_k|^p]$.

Proof: see Exercise 3 at the end of the file if you are curious.

Remark: This is clearly true for $p=1$ and $p=2$ (with constant 1 instead of 2).

Proof of the theorem:

- Case $|\lambda| < \sqrt{2m}$: Recall
$$W_{t+s}^\lambda = \sum_{v \in \mathcal{N}(t)} e^{\lambda X_v(t) - \left(\frac{\lambda^2}{2}+m\right)t} \underbrace{\sum_{\substack{u \in \mathcal{N}(t+s) \\ \text{s.t. } u \geq v}} e^{\lambda(X_u(t+s) - X_v(t)) - \left(\frac{\lambda^2}{2}+m\right)s}}_{=: W_s^\lambda(v,t)}$$
(the additive martingale for the BBM starting from v at time t)

and, by the branching property, conditionally on $\mathcal{F}_t$, the $W_s^\lambda(v,t)$ for $v \in \mathcal{N}(t)$ are independent and have the same law as W_s^λ.

Then
$$W_{t+s}^\lambda - W_t^\lambda = \sum_{v \in \mathcal{N}(t)} \underbrace{e^{\lambda X_v(t) - \left(\frac{\lambda^2}{2}+m\right)t}}_{\mathcal{F}_t\text{-measurable}} \underbrace{\left(W_s^\lambda(v,t) - 1\right)}_{\text{indep. and centered given } \mathcal{F}_t}$$

So by the lemma:
$$\mathbb{E}\left[|W_{t+s}^\lambda - W_t^\lambda|^p \mid \mathcal{F}_t\right] \leq \sum_{v \in \mathcal{N}(t)} \mathbb{E}\left[\left(e^{\lambda X_v(t) - \left(\frac{\lambda^2}{2}+m\right)t} |W_s^\lambda(v,t) - 1|\right)^p \,\middle|\, \mathcal{F}_t\right]$$
$$= \left(\sum_{v \in \mathcal{N}(t)} e^{\lambda p X_v(t) - \left(\frac{\lambda^2}{2}+m\right)pt}\right) \mathbb{E}\left[|W_s^\lambda - 1|^p\right]$$
$$= e^{\left(\frac{(\lambda p)^2}{2}+m\right)t - \left(\frac{\lambda^2}{2}+m\right)pt} \underbrace{W_t^{\lambda p}}_{\text{has expectation } 1} \mathbb{E}\left[|W_s^\lambda - 1|^p\right]$$

Taking the expectation: $\mathbb{E}\left[|W^\lambda_{t+s}-W^\lambda_t|^p\right] \leq e^{(p-1)\left(\frac{\lambda^2}{2}-m\right)t}\, \mathbb{E}\left[|W^\lambda_s-1|^p\right]$

$(p-1)\left(\frac{\lambda^2}{2}-m\right) < 0$ because $p < \frac{2m}{\lambda^2}$.

Now take $t\in\mathbb{N}$ and $s=1$.

Note that $\mathbb{E}\left[|W^\lambda_1-1|^p\right] \underset{\text{Jensen}}{\leq} \mathbb{E}\left[(W^\lambda_1-1)^2\right]^{p/2} \leq C(\lambda,p)$ (check it with the many-to-two).

so $\|W^\lambda_{t+1}-W^\lambda_t\|_p$ is summable and $(W^\lambda_t)_{t\in\mathbb{N}}$ is bounded in L^p.

Then, $(W^\lambda_t)_{t\geq 0}$ is bounded in L^p:

$$\mathbb{E}\left[|W^\lambda_t|^p\right] \underset{\text{martingale}}{=} \mathbb{E}\left[\,|\mathbb{E}[W^\lambda_{\lceil t\rceil}|\mathcal{F}_t]|^p\right] \underset{\text{Jensen}}{\leq} \mathbb{E}\left[\mathbb{E}\left[|W^\lambda_{\lceil t\rceil}|^p|\mathcal{F}_t\right]\right] = \mathbb{E}\left[|W^\lambda_{\lceil t\rceil}|^p\right] \text{ bounded.}$$

So $(W^\lambda_t)_{t\geq 0}$ converges in L^p.

- Case $|\lambda| > \sqrt{2m}$: Taking $s\to\infty$ in the decomposition of W^λ_{t+s} above, we get

$$W^\lambda_\infty = \sum_{v\in\mathcal{N}(t)} e^{\lambda X_v(t)-\left(\frac{\lambda^2}{2}+m\right)t}\, W^\lambda_\infty(v,t) \quad \text{where } W^\lambda_\infty(v,t) = \lim_{s\to\infty} W^\lambda_s(v,t) \text{ (a.s. limit).}$$

Let $p\in(0,1)$. Note that $\mathbb{E}\left[(W^\lambda_\infty)^p\right]<\infty$ because $W^\lambda_\infty\in L^1$ by Fatou ($\Rightarrow \mathbb{E}[W^\lambda_\infty]\leq 1$).

By subadditivity of $x\longmapsto x^p$,

$$(W^\lambda_\infty)^p \leq \sum_{v\in\mathcal{N}(t)} e^{\lambda p X_v(t)-\left(\frac{\lambda^2}{2}+m\right)pt}\left(W^\lambda_\infty(v,t)\right)^p$$

independent and same law as W^λ_∞

$$\mathbb{E}\left[(W^\lambda_\infty)^p\right] \leq \underbrace{\mathbb{E}\left[\sum_{v\in\mathcal{N}(t)} e^{\lambda p X_v(t)-\left(\frac{\lambda^2}{2}+m\right)pt}\right]}\times \mathbb{E}\left[(W^\lambda_\infty)^p\right]$$

$$= e^{\left(\frac{(\lambda p)^2}{2}+m\right)t-\left(\frac{\lambda^2}{2}+m\right)pt}\,\mathbb{E}\left[W^{\lambda p}_t\right] = e^{(p-1)\left(\frac{\lambda^2}{2}p-m\right)t} < 1 \quad \text{if } p\in\left(\tfrac{2m}{\lambda^2},1\right).$$

So $\mathbb{E}\left[(W^\lambda_\infty)^p\right]=0$ and $W^\lambda_\infty=0$ a.s.

- Case $|\lambda| = 2m$: see exercise at the end of the file. ▨

II.3) Back to the number of particles

For $x\in\mathbb{R}$ and $t\geq 0$, let $N_t(x) = \sum_{u\in\mathcal{N}(t)} \mathbb{1}_{X_u(t)\geq x}$.

Reminder: We have seen that $\mathbb{E}[N_t(at)] \sim \frac{1}{a\sqrt{2\pi t}}\, e^{\left(m-\frac{a^2}{2}\right)t}$ for $a>0$

and that $\forall a\in(0,\sqrt{2m})$, $\mathbb{P}(N_t(at)\geq 1) \xrightarrow[t\to\infty]{} 1$.

(or even in exercise $\mathbb{P}\left(N_t(at)\geq e^{\left(m-\frac{a^2}{2}-\varepsilon\right)t}\right) \xrightarrow[t\to\infty]{} 1$.)

Theorem: For any $a\in(0,\sqrt{2m})$, $a\sqrt{2\pi t}\, e^{-\left(m-\frac{a^2}{2}\right)t} N_t(at) \xrightarrow[t\to\infty]{\mathbb{P}} W^a_\infty$.

Remark: The convergence actually holds a.s. but this is harder to prove. See Louis Chataignier's master thesis for details.

Lemma: Let $(x_t)_{t\geq 0}$ be such that $\frac{x_t}{\sqrt{t}} \longrightarrow \infty$ as $t \longrightarrow \infty$.

Then $\mathbb{E}[N_t(x)] \sim \sqrt{\frac{t}{2\pi}} \frac{1}{x} e^{mt - \frac{x^2}{2t}}$ as $t \to \infty$, uniformly in $x \geq x_t$.

Proof: By the many-to-one $\mathbb{E}[N_t(x)] = e^{mt} \mathbb{P}(B_t \geq x) = e^{mt} \mathbb{P}(B_1 \geq \frac{x}{\sqrt{t}})$
and $\mathbb{P}(B_1 \geq y) \sim \frac{e^{-y^2/2}}{\sqrt{2\pi}\, y}$ as $y \longrightarrow \infty$. ∎

Proof of the theorem: The strategy is the following: consider some time $s = s(t) \leq t$ such that $s \longrightarrow \infty$ as $t \longrightarrow \infty$. Then we will prove

① $a\sqrt{2\pi t}\, e^{-(m - \frac{a^2}{2})t}\, \mathbb{E}[N_t(at) \mid \mathcal{F}_s] \xrightarrow{\mathbb{P}} W_\infty^a$ if $s = o(\sqrt{t})$.

② $\sqrt{t}\, e^{-(m-\frac{a^2}{2})t} \left(N_t(at) - \mathbb{E}[N_t(at) \mid \mathcal{F}_s]\right) \xrightarrow{L^p} 0$ for $p \in (1, \frac{2m}{a^2})$ if $\frac{s}{\log t} \longrightarrow \infty$.

This implies the result by choosing for example $s = t^{1/4}$.

Step ①: $N_t(at) = \sum_{v \in \mathcal{N}(s)} \sum_{\substack{u \in \mathcal{N}(t) \\ u \geq v}} \mathbb{1}_{X_u(t) - X_v(s) \geq at - X_v(s)}$.

Writing $\varphi(t,x) = \mathbb{E}[N_t(x)]$ and using the branching property

$$\mathbb{E}[N_t(at) \mid \mathcal{F}_s] = \sum_{v \in \mathcal{N}(s)} \underbrace{\varphi(t-s, at - X_v(s))}$$

to estimate this we restrict ourselves to an event where $|X_v(s)|$ is bounded deterministically

Let $E_s = \{\forall v \in \mathcal{N}(s), |X_v(s)| \leq \sqrt{2m}\, s\}$. Recall that $\mathbb{P}(E_s) \longrightarrow 1$ as $s \longrightarrow \infty$.

On E_s, uniformly in $v \in \mathcal{N}(s)$, (here the error term can be bounded deterministically)

$$\varphi(t-s, at - X_v(s)) = \sqrt{\frac{t-s}{2\pi}} \times \frac{(1+o(1))}{(at - X_v(s))} \times \exp\left(m(t-s) - \frac{(at - X_v(s))^2}{2(t-s)}\right)$$ by the lemma above

$$= \sqrt{\frac{t}{2\pi}} \frac{(1+o(1))}{at} \exp\Big(m(t-s) - \frac{a^2}{2} \frac{t^2}{t-s} + \underbrace{a\frac{t}{t-s}}X_v(s) - \underbrace{\frac{X_v(s)^2}{2(t-s)}}_{=o(1)}\Big)$$

$= aX_v(s) + O(\frac{s^2}{t})$ $=$ $aX_v(s) + o(1)$

$t \times (1 - \frac{s}{t})^{-1} = t\left(1 + \frac{s}{t} + O\left(\frac{s^2}{t^2}\right)\right) = t + s + o(1)$

using $s = o(\sqrt{t})$

$$= \frac{1+o(1)}{a\sqrt{2\pi t}} \exp\left(m(t-s) - \frac{a^2}{2}(t+s) + aX_v(s)\right)$$

Therefore, on E_s, $\mathbb{E}[N_t(at) \mid \mathcal{F}_s] = \frac{1+o(1)}{a\sqrt{2\pi t}} e^{(m - \frac{a^2}{2})t} \underbrace{\sum_{v \in \mathcal{N}(s)} e^{aX_v(s) - (m + \frac{a^2}{2})s}}$.

$= W_s^a \xrightarrow[t \to \infty]{} W_\infty^a$

Since $\mathbb{P}(E_s) \to 1$, this shows ①.

Step ② : Recall $N_t(at) = \sum_{v \in \mathcal{N}(s)} \Pi_v$ with $\Pi_v = \sum_{\substack{u \in \mathcal{N}(t) \\ u \geq v}} \mathbb{1}_{X_u(t) \geq at}$

So $N_t(at) - \mathbb{E}[N_t(at) | \mathcal{F}_s] = \sum_{v \in \mathcal{N}(s)} \big(\underbrace{\Pi_v - \mathbb{E}[\Pi_v | \mathcal{F}_s]}_{\text{given } \mathcal{F}_s \text{, indep. and centered r.v.}} \big)$

By von Bahr-Esseen inequality (applied given $\mathcal{F}_s$) : for any $p \in [1,2]$,

$$\mathbb{E}\Big[\big| N_t(at) - \mathbb{E}[N_t(at) | \mathcal{F}_s] \big|^p \Big| \mathcal{F}_s \Big] \leq \sum_{v \in \mathcal{N}(s)} \mathbb{E}\Big[\big| \Pi_v - \mathbb{E}[\Pi_v | \mathcal{F}_s] \big|^p \Big| \mathcal{F}_s \Big]$$

$|X - \mathbb{E}[X]|^p \leq 2(|X|^p + \mathbb{E}[|X|]^p)$

so $\mathbb{E}[|X - \mathbb{E}[X]|^p] \leq 2\, \mathbb{E}[|X|^p]$

$$\leq 2 \sum_{v \in \mathcal{N}(s)} \mathbb{E}\big[|\Pi_v|^p \big| \mathcal{F}_s \big]$$

But $\forall x \in \mathbb{R}$, $\mathbb{1}_{x \geq at} \leq e^{a(x - at)}$

so $\Pi_v \leq \sum_{\substack{u \in \mathcal{N}(t) \\ u \geq v}} e^{aX_u(t) - a^2 t} = e^{aX_v(s) + (m - \frac{a^2}{2})t - (m + \frac{a^2}{2})s} \underbrace{\sum_{\substack{u \in \mathcal{N}(t) \\ u \geq v}} e^{a(X_u(t) - X_v(s)) - (m + \frac{a^2}{2})(t-s)}}_{= W_s^a(v,t) \overset{(d)}{=} W_s^a}$

Choose $p < \frac{2m}{a^2}$, then $(W_s^a)_{s \geq 0}$ is bounded in L^p,

so $\mathbb{E}\big[|\Pi_v|^p \big| \mathcal{F}_s \big] \leq C\, e^{apX_v(s) + (m - \frac{a^2}{2})pt - (m + \frac{a^2}{2})ps}$.

$$\mathbb{E}\Big[\big| N_t(at) - \mathbb{E}[N_t(at) | \mathcal{F}_s] \big|^p \Big| \mathcal{F}_s \Big] \leq 2C e^{(m - \frac{a^2}{2})pt} e^{(m + \frac{(ap)^2}{2})s - (m + \frac{a^2}{2})ps} W_s^{ap}$$

Taking the expectation:

$$\mathbb{E}\Big[\Big| \sqrt{t}\, e^{-(m - \frac{a^2}{2})t} \big(N_t(at) - \mathbb{E}[N_t(at) | \mathcal{F}_s] \big) \Big|^p \Big] \leq 2C\, t^{p/2}\, e^{\overbrace{(p-1)(\frac{a^2}{2}p - m)}^{<0} s}$$

$\xrightarrow[t \to \infty]{} 0$ because $\frac{s}{\log t} \to \infty$. ■

Exercise 1 : Back to Neveu's theorem

We prove here the case $|\lambda| = 2m$ (and give a new proof of the case $|\lambda| > 2m$).

1. Prove that $W_\infty^\lambda = e^{\lambda X_\phi(\tau_\phi) - (\frac{\lambda^2}{2}+m)\tau_\phi} \left(W_\infty^{\lambda,1} + \dots + W_\infty^{\lambda, L_\phi} \right)$, where $(W_\infty^{\lambda,i})_{i\geq 1}$ are iid with the same law as W_∞^λ and independent of $(\tau_\phi, X_\phi(\tau_\phi), L_\phi)$.

2. For $p \in (0,1)$, show that $\mathbb{E}\left[(W_\infty^{\lambda,1} + \dots + W_\infty^{\lambda,L_\phi})^p\right] = \left(1 + p\left(\frac{\lambda^2}{2}+m\right) - \frac{(\lambda p)^2}{2}\right) \mathbb{E}\left[(W_\infty^\lambda)^p\right]$.

3. For $|\lambda| > \sqrt{2m}$, deduce that $W_\infty^\lambda = 0$ a.s.

4. Show that $\mathbb{E}\left[(W_\infty^{\lambda,1} + \dots + W_\infty^{\lambda,L_\phi})^p\right] = (m+1)\, \mathbb{E}\left[W_\infty^{\lambda,1} (W_\infty^{\lambda,1} + \dots + W_\infty^{\lambda,L_\phi})^{p-1}\right]$

5. Combining 2. and 4. and looking at the derivative at $p = 1^-$, prove that

$$(m+1)\, \mathbb{E}\left[W_\infty^{\lambda,1} \log(W_\infty^{\lambda,1} + \dots + W_\infty^{\lambda,L_\phi})\right] = \left(m - \frac{\lambda^2}{2}\right) \mathbb{E}\left[W_\infty^\lambda\right]$$

6. For $|\lambda| = \sqrt{2m}$, deduce that $W_\infty^\lambda = 0$ a.s.

Exercise 2 : Optimality of the p in Neveu's theorem for $|\lambda| < \sqrt{2m}$

Let $|\lambda| < \sqrt{2m}$. Using question 2 of exercise 1, prove that, for $p > \frac{\lambda^2}{2m}$, $\mathbb{E}\left[(W_\infty^\lambda)^p\right] = +\infty$.

Exercise 3 : von Bahr - Esseen inequality

Let $p \in [1,2]$.

1.a. For $x \geq 0$, prove that $2(|x|^p + 1) - |x-1|^p - |x+1|^p \geq 0$.

1.b. For $x, y \in \mathbb{R}$, prove that $|x+y|^p + |x-y|^p \leq 2(|x|^p + |y|^p)$

1.c. For $X, Y \in L^p$ independent with Y symmetric, prove that $\mathbb{E}[|X+Y|^p] \leq \mathbb{E}[|X|^p] + \mathbb{E}[|Y|^p]$.

2. For $X, Y \in L^p$ such that $\mathbb{E}[Y|X] = 0$, prove that $\mathbb{E}[|X|^p] \leq \mathbb{E}[|X+Y|^p]$.

3. Prove von Bahr - Esseen inequality.

Hint: Use 3.a. to get $\mathbb{E}\left[\left|\sum_{k=1}^n X_k\right|^p\right] \leq \mathbb{E}\left[\left|\sum_{k=1}^{n-1} X_k + X_n - X_n'\right|^p\right]$ where X_n' is independent of $(X_1, \dots, X_n)$ and has the same law as X_n.