Lecture 6 : The Logarithmic correction

10/01/2025

III. 3) Preliminaries on Brownian motion

For the proof of the second order of $M_t = \max_{u \in \mathcal{N}_t} X_u(t)$, we need estimates for Brownian motion staying below a barrier that we prove in this section.

We first need to state the strong Markov property for Brownian motion.

Theorem (strong Markov property): Let T be a stopping time for the canonical filtration of a Brownian motion $(B_t)_{t\geq 0}$. Define, for $t \geq 0$,

$$B_t^{(T)} = \begin{cases} B_{T+t} - B_T & \text{if } T < \infty \\ 0 & \text{otherwise.} \end{cases}$$

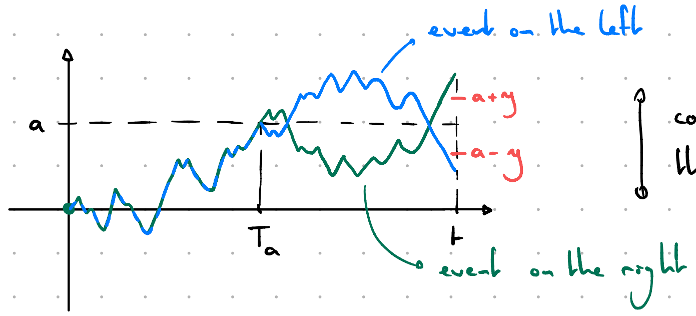
Assume $\mathbb{P}(T<\infty) > 0$. Then, under $\mathbb{P}(\cdot \mid T<\infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion started at 0 and independent of $\mathcal{F}_T$.

Proof: See Theorem 14.15 in Le Gall, "Measure Theory, Probability, and Stochastic Processes". It relies on a discretization of time by distinguishing according to the events $\left\{ \frac{k-1}{2^n} < T < \frac{k}{2^n} \right\}$ for $k \geq 1$, in order to then apply the simple Markov property. ∎

Proposition (reflection principle): For $t \geq 0$, $a > 0$ and $y \geq 0$, we have

$$\mathbb{P}\left(\max_{s\leq t} B_s \geq a,\ B_t \leq a-y \right) = \mathbb{P}(B_t \geq a+y).$$

A picture of the proof:



↕ correspondence by reflecting the trajectory after T_a with respect to level a.

Proof: Let $T_a = \inf\{t \geq 0 : B_t = a\}$. Note that $\left\{ \max_{s\leq t} B_s \geq a \right\} = \{T_a \leq t\}$.

So $$\mathbb{P}\left(\max_{s\leq t} B_s \geq a,\ B_t \leq \underbrace{a}_{= B_{T_a}} - y \right) = \mathbb{P}\left(T_a \leq t,\ B_t - B_{T_a} \leq -y\right)$$
$$= \mathbb{P}\left(T_a \leq t,\ B_{t-T_a}^{(T_a)} \leq -y\right).$$

↳ defined as in the theorem above

$$= \mathbb{P}\left(\left(T_a, B^{(T_a)}\right) \in H\right)$$

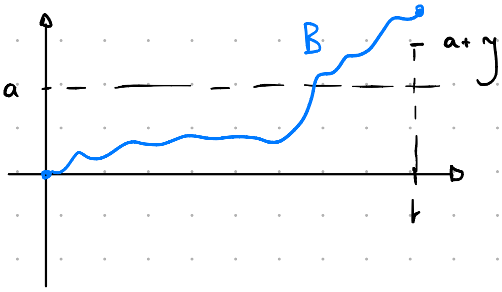
where $H = \{(s,w) \in \mathbb{R}_+ \times \underbrace{\mathcal{C}(\mathbb{R}_+)}_{\text{continuous functions from } \mathbb{R}_+ \to \mathbb{R}} : s \leq t,\ w(t-s) \leq b-a\}$ is measurable. (exercise!)

But, by the strong Markov property, we have

- $B^{(T_a)}$ is independent of $\mathcal{F}_{T_a}$ so it is independent of T_a.
- $B^{(T_a)}$ has the same law as $-B^{(T_a)}$ (both the Brownian motion and the 0 process have a law invariant by a change of sign).

So $(T_a, B^{(T_a)}) \overset{(d)}{=} (T_a, -B^{(T_a)})$.

Therefore
$$\begin{aligned}
\mathbb{P}\left(\left(T_a, B^{(T_a)}\right) \in H\right) &= \mathbb{P}\left(\left(T_a, -B^{(T_a)}\right) \in H\right) \\
&= \mathbb{P}\big(T_a \leq t,\ -(B_t - \underbrace{B_{T_a}}_{=a}) \leq -y\big) \\
&= \mathbb{P}(T_a \leq t,\ B_t \geq a+y) \\
&= \mathbb{P}(B_t \geq a+y)
\end{aligned}$$
because $\{T_a \leq t\} \subset \{B_t \geq a+y\}$ by the intermediate value theorem ($B_0 = 0 \leq a$ and $B_t \geq a+y \geq a$ + continuity of B) ∎



Corollary 1: For $t \geq 0$, $\max_{s \leq t} B_s \overset{(d)}{=} |B_t|$.

In particular, for $a > 0$, $\mathbb{P}\left(\max_{s\leq t} B_s < a\right) = \int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$

and, as $t \to \infty$, with a possibly depending on t such that $a = o(\sqrt{t})$,
$$\mathbb{P}\left(\max_{s\leq t} B_s < a\right) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\frac{a}{\sqrt{t}}.$$

Proof: Let $a > 0$

$$\mathbb{P}\left(\sup_{s\leq t} B_s \geq a\right) = \mathbb{P}\big(\underbrace{\sup_{s\leq t} B_s \geq a}_{\in \{B_t > a\}},\ B_t > a\big) + \underbrace{\mathbb{P}\big(\sup_{s\leq t} B_s \geq a,\ B_t \leq a\big)}_{\text{Proposition above}}$$

$$= \mathbb{P}(B_t > a) + \mathbb{P}(B_t \geq a) = \mathbb{P}(B_t \leq -a) + \mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)$$

This characterizes the distribution (because the r.v. are positive).

Then $P\left(\max_{s\le t} B_s < a\right) = P(|B_t| < a) = \int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$

Note that $1 - \frac{x^2}{2t} \le e^{-x^2/2t} \le 1$ which gives $\int_{-a}^{a} e^{-x^2/2t}\,dx = 2a + O\left(\frac{a^3}{t}\right) \sim 2a$ as $t \to \infty$ because $a = o(\sqrt{t})$, and so $\int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\frac{a}{\sqrt{t}}$. ■

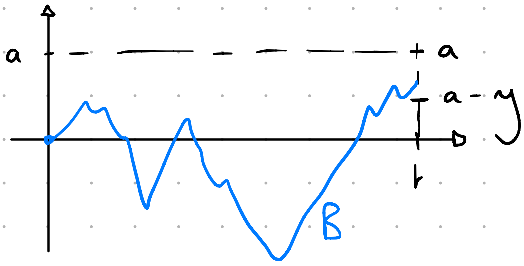
Corollary 2: Let $a>0$, $y \geq 0$ and $t \geq 0$.

$$P\left(\max_{s\le t} B_s < a,\ B_t \geq a-y\right) = \int_{-a}^{a} \frac{e^{-x^2/2t} - e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\,dx \le \frac{a y^2}{\sqrt{2\pi}\, t^{3/2}}.$$

Moreover, as $t \to \infty$, with a, y possibly dependent on t with $y = o(\sqrt{t})$ and $a = o(\sqrt{t})$,

$$P\left(\max_{s< t} B_s < a,\ B_t \geq a-y\right) \underset{t\to\infty}{\sim} \frac{a y^2}{\sqrt{2\pi}\, t^{3/2}}.$$

A picture of the event:



Exercise 1: Proof of the corollary

1.| Exact formulas

1.a| Prove that $P\left(\max_{s\le t} B_s < a,\ B_t \le a-y\right) = P(B_t \le a-y) - P(B_t \geq a+y)$

1.b.| Deduce that $P\left(\max_{s\le t} B_s < a,\ B_t \le a-y\right) = P(B_t \in [y-a, y+a])$

1.c.| Using Corollary 1, prove the equality in the statement.

1.d.| Prove that $P\left(\max_{s\le t} B_s < a,\ B_t \geq a-y\right) = \int_{a}^{a+y} \frac{e^{-(x-y)^2/2t} - e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$.

2.| Upper bound

2.a| Prove that for any $x, y \in \mathbb{R}$, $\left|e^{-(x-y)^2/2t} - e^{-x^2/2t}\right| \le \frac{y}{2t}|2x-y|$.

2.b| Prove the inequality in the statement.

Hint: Use formula in 1.d if $y \le 2a$ and formula in 1.c. if $y > 2a$.

3.| Asymptotic equivalent

3.a.| Prove that $e^{-(x-y)^2/2t} - e^{-x^2/2t} = \frac{y(2x-y)}{2t} + O\left(\frac{(x+y)x^2 y}{t^2}\right)$ uniformly in $x \in [-a-y, a+y]$.

3.b.| Prove the asymptotic equivalent.

Hint: same as in 2.b.

We conclude this section by recalling Girsanov's theorem for Brownian motion and by providing a proof in the following proof.
The proof presented below is based on an elementary approach, whereas the general Girsanov's theorem uses martingale theory.

Exercise 2: Girsanov's theorem

Let $t \geq 0$ and $\mathcal{C}([0,t])$ denote the set of continuous functions from $[0,t] \longrightarrow \mathbb{R}$.
We equip $\mathcal{C}([0,t])$ with the norm $\|-\|_\infty$ and the associated Borelian σ-field.

1. We prove here the following simple version (the most used in this class):
For any $\lambda \in \mathbb{R}$ and $F : \mathcal{C}([0,t]) \longrightarrow \mathbb{R}_+$ measurable, we have

$$\mathbb{E}\left[e^{\lambda B_t - \frac{\lambda^2}{2}t} F((B_s)_{s\in[0,t]})\right] = \mathbb{E}\left[F((B_s + \lambda s)_{s\in[0,t]})\right]$$

1.a. Prove it for F of the form $F((B_s)_{s\in[0,t]}) = f(B_t)$ with $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ measurable.

Hint: Use the density of B_t to write the expectations as integrals.

1.b. Conclude.

Hint: The Borelian σ-field on $\mathcal{C}([0,t])$ is generated by cylinders so it is enough to consider F of the form $F((B_s)_{s\in[0,t]}) = \prod_{k=1}^{n} f_k(B_{t_k} - B_{t_{k-1}})$ where $0 = t_0 < \dots < t_n = t$ and $f_1 \longrightarrow f_n : \mathbb{R} \longrightarrow \mathbb{R}_+$ are measurable.

2. Use the previous question to prove the more general form: for $h : [0,t] \to \mathbb{R}$ is continuous and piecewise $\mathcal{C}^1$ and F as before,

$$\mathbb{E}\left[\exp\left(\int_0^t h'(s)\, dB_s - \int_0^t \frac{h'(s)^2}{2} ds\right) F\left((B_s)_{s\in[0,t]}\right)\right] = \mathbb{E}\left[F\left((B_s + h(s))_{s\in[0,t]}\right)\right]$$

Hint: Use that h can be approximated by piecewise affine functions, together with the hint of question 1.b.

III.4) The logarithmic correction

Our goal in this section is to prove the following theorem:

Theorem (Bramson 1978) On the survival event $\dfrac{\Omega_t - \lambda_c t}{\log t} \xrightarrow[t\to\infty]{\mathbb{P}} -\dfrac{3}{2\lambda_c}$.

Remark: This has to be compared with the similar result in the iid case:

$$\frac{\tilde{\Omega}_t - \lambda_c t}{\log t} \xrightarrow[t\to\infty]{\mathbb{P}} -\frac{1}{2\lambda_c}.$$

III.4.1) The upper bound

We need to show that, for any $\varepsilon > 0$, $\mathbb{P}\left(\Omega_t \geq \lambda_c t - \left(\frac{3}{2\lambda_c} - \varepsilon\right)\log t\right) \xrightarrow[t\to\infty]{} 0$.

We prove the following stronger result.

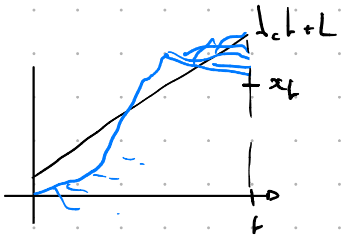
Proposition UB: $\mathbb{P}\left(\Omega_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t\right) \xrightarrow[t\to\infty]{} 0$.

Idea of the proof: Let $x_t = \lambda_c t - \frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t$.

Then $\mathbb{P}(\Omega_t \geq x_t) = \mathbb{P}(\exists u \in \mathcal{N}_t : X_u(t) \geq x_t) \leq \mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}\right]$,

but this cannot work: by the many-to-one this expectation is the same as in the iid case where there are particles above x_t with large probability (because $\tilde{\Omega}_t = \lambda_c t - \frac{1}{2\lambda_c}\log t + O(1)$).

So $\mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}\right]$ is large even if we expect $\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}$ to be small with high probability: this means that this expectation is dominated by an unlikely event on which $\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}$ is very large.



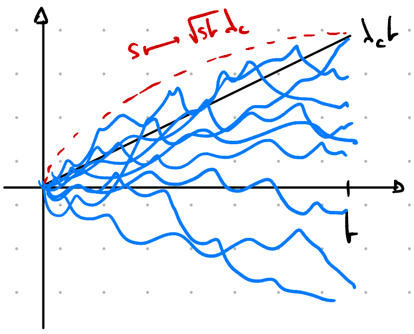
The issue is this type of event: one particle goes very high and then, by branching, drops many particles above x_t.

But this event is unlikely because Ω_s should be below $\lambda_c s$ for all s if L is large enough (already seen last time, see the lemma below).

So we first need to add the knowledge that particles stay below the line $s \mapsto \lambda_c s + L$ (we remove the bad event) before computing the first moment.

Such an argument could not be done in the iid case: at a time $s = bt$, $b \in (0,1)$, the maximum of the $\lfloor e^{mt} \rfloor$ particles is far above $\lambda_c s$:

$$\max_{1\le i\le \lfloor e^{mt}\rfloor} B_s^i \overset{(d)}{=} \sqrt{\frac{s}{t}} \max_{1\le i\le \lfloor e^{mt}\rfloor} B_t^i \underset{t\to\infty}{\sim} \sqrt{\frac{s}{t}} \times \lambda_c t = \frac{1}{\sqrt{b}}\lambda_c s \gg \lambda_c s \ !$$



This is because there are $\lfloor e^{mt} \rfloor$ particles from the beginning whereas in the BBM population grows progressively!

Lemma: For any $\varepsilon > 0$, there exists $L > 0$ such that $\mathbb{P}(E_L) \geq 1-\varepsilon$ where

$$E_L = \{\forall s \geq 0,\ M_s < \lambda_c s + L\}.$$

Proof: We have seen in the last lecture that $\limsup_{s\to\infty} M_s - \lambda_c s < \infty$ a.s.
But $s \mapsto M_s - \lambda_c s$ is continuous a.s. so $\sup_{s\geq 0} M_s - \lambda_c s < \infty$ a.s.,
so there exists $L > 0$ such that $\mathbb{P}\big(\underbrace{\sup_{s\geq 0} M_s - \lambda_c s \geq L}_{\supset E_L^c}\big) \leq \varepsilon$. ∎

Proof of Proposition UB: Let $\varepsilon > 0$ and $L > 0$ such that $\mathbb{P}(E_L) \geq 1-\varepsilon$.

Then $\mathbb{P}(M_t \geq x_t) \leq \underbrace{\mathbb{P}(E_L^c)}_{\leq \varepsilon} + \mathbb{P}(\{M_t \geq x_t\} \cap E_L)$

By inclusion of events,

$$\mathbb{P}(\{M_t \geq x_t\}\cap E_L) \leq \mathbb{P}\big(\exists u \in \mathcal{N}_t : X_u(t) \geq x_t \text{ and } \forall s\in[0,t],\ X_u(s) \leq \lambda_c s + L\big)$$

$$\leq \mathbb{E}\Big[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t \text{ and } \max_{s\in[0,t]}(X_u(s)-\lambda_c s)\leq L}\Big]$$

many-to-one

$$= e^{mt}\,\mathbb{P}\Big(B_t \geq x_t,\ \max_{s\in[0,t]}(B_s - \lambda_c s) \leq L\Big)$$

$$= e^{mt}\,\mathbb{P}\Big(B_t - \lambda_c t \geq x_t - \lambda_c t,\ \max_{s\in[0,t]}(B_s - \lambda_c s) \leq L\Big)$$

Girsanov

$$= e^{mt}\,\mathbb{E}\Big[\underbrace{e^{-\lambda_c B_t}}_{\leq e^{-\lambda_c(x_t-\lambda_c t)} = \frac{t^{3/2}}{(\log t)^3}} \cdot e^{-\overbrace{\frac{\lambda_c^2}{2}t}^{=mt}}\ \mathbb{1}_{B_t \geq \underbrace{x_t - \lambda_c t}_{= -\frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t},\ \max_{s\in[0,t]} B_s \leq L}\Big]$$

Corollary 2 in Section III.3

$$\leq \frac{t^{3/2}}{(\log t)^3} \, \mathbb{P}\left(\max_{s\in[0,t]} B_s \leq L \, , \, B_t \geq L - \left(\frac{3}{2\lambda_c}\log t - \frac{2}{\lambda_c}\log\log t + L\right)\right)$$

$$\sim \frac{t^{3/2}}{(\log t)^3} \, \frac{L \times \left(\frac{3}{2\lambda_c}\log t\right)^2}{\sqrt{2\pi}\, t^{3/2}} = O\left(\frac{1}{\log t}\right) \xrightarrow[t\to\infty]{} 0 . \quad \blacksquare$$

III.4.2) The lower bound

We prove here the following stronger result

<u>Proposition LB</u>: On the survival event, $M_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t + O_P(1)$, which means that, for any $\varepsilon > 0$, there exists $y > 0$ such that for t large enough

$$\mathbb{P}\left(M_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t - y \,\middle|\, \text{survival}\right) \geq 1-\varepsilon .$$

<u>Remark</u>: $O_P(1)$ is a notation for a random term depending on t which is tight for large t when seen as a process in t.

<u>Idea of the proof</u>:

Let K_t be the number of particles above $m_t = \lambda_c t - \frac{3}{2\lambda_c}\log t$ at time t and satisfying some trajectory condition to be chosen.

Then, use many-to-one and many-to-two to compute $\mathbb{E}[K_t]$ and $\mathbb{E}[K_t^2]$ and show that $\mathbb{E}[K_t^2] \leq C\,\mathbb{E}[K_t]^2$ (this requires to choose the condition in the definition of K_t such that these moments are not dominated by unlikely events.

By Paley-Zygmund / Cauchy-Schwarz inequality, we deduce $\mathbb{P}(Z_t \geq 1) \geq \frac{\mathbb{E}[Z_t]^2}{\mathbb{E}[Z_t^2]} \geq \frac{1}{C}$.

To conclude, let particles branch at the beginning until there is a large number of particles: each of them has a probability at least $\frac{1}{C}$ to have a very high descendant at a time t in the future, so it is very likely that at least one of them does.