Lecture 7: The Logarithmic correction (2^{nd} part)

17/01/2025

III.4.2) The lower bound

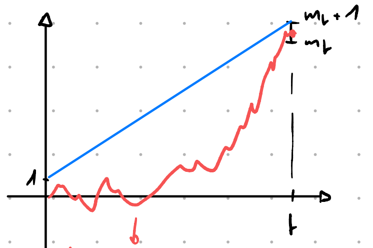
We prove here the following stronger result

Proposition LB: On the survival event, $M_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t + O_{\mathbb{P}}(1)$, which means that, for any $\varepsilon > 0$, there exists $y > 0$ such that for t large enough
$$\mathbb{P}\left(M_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t - y \,\middle|\, \text{survival}\right) \geq 1 - \varepsilon.$$

We define $m_t := \lambda_c t - \frac{3}{2\lambda_c}\log t$,

$\lambda_t := \frac{m_t}{t}$ the slope of our barrier,

$$K_t := \sum_{u \in \mathcal{N}_t} \mathbb{1}_{X_u(t) \geq m_t \text{ and } \max_{s\in[0,t]} X_u(s) - \lambda_t s \leq 1}.$$



a trajectory contributing to K_t

We apply the 1^{st} and 2^{nd} moment argument to K_t.

We first check in the two following lemmas that $\mathbb{E}[Z_t]$ and $\mathbb{E}[Z_t^2]$ are of order 1 as $t \to \infty$.

Lemma 1: $\frac{e^{-\lambda_c}}{\sqrt{2\pi}} \leq \liminf_{t\to\infty} \mathbb{E}[K_t] \leq \limsup_{t\to\infty} \mathbb{E}[K_t] \leq \frac{1}{\sqrt{2\pi}}$.

Proof: By the many-to-one lemma,
$$\mathbb{E}[K_t] = e^{mt}\,\mathbb{P}\Big(\underbrace{B_t \geq m_t}_{\iff B_t - \lambda_t t \geq 0},\ \max_{s\in[0,t]} B_s - \lambda_t s \leq 1\Big)$$

Girsanov $\Big\downarrow$

$$= e^{mt}\,\mathbb{E}\left[e^{-\lambda_t B_t - \frac{\lambda_t^2}{2}t}\, \mathbb{1}_{\underbrace{B_t \geq 0,\ \max_{s\in[0,t]} B_s \leq 1}_{\implies B_t \in [0,1]}}\right]$$

Note that $\frac{\lambda_t^2}{2}t = \frac{t}{2}\left(\underbrace{\lambda_c^2}_{=2m} - 3\frac{\log t}{t} + \left(\frac{3}{2\lambda_c}\frac{\log t}{t}\right)^2\right) = mt - \frac{3}{2}\log t + o(1)$

Moreover, on $\{B_t \in [0,1]\}$, we have $e^{-\lambda_t B_t} \in [e^{-\lambda_t}, 1] \subset [e^{-\lambda_c}, 1]$, so we get
$$e^{-\lambda_c} t^{3/2}\,\mathbb{P}\Big(B_t \geq 0,\ \max_{s\in[0,t]} B_s \leq 1\Big) \leq \mathbb{E}[K_t] \leq t^{3/2}\,\mathbb{P}\Big(B_t \geq 0,\ \max_{s\in[0,t]} B_s \leq 1\Big)$$

But $\mathbb{P}\Big(B_t \geq 0,\ \max_{s\in[0,t]} B_s \leq 1\Big) \sim \frac{1}{\sqrt{2\pi}\, t^{3/2}}$ by Corollary 2 of Lecture 6 (with $a = y = 1$)

so we get the result. ∎

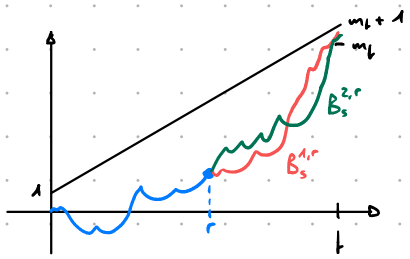
Lemma 2: There exists $C>0$ such that for t large enough $E[K_t^2] \le C$.

Proof: We decompose $E[K_t^2] = E[K_t] + E[K_t(K_t-1)]$ and we already know that $E[K_t]$ is bounded for t large. Then, we bound

$$E[K_t(K_t-1)] = E\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} \mathbb{1}_{X_u(t)\ge m_t,\ \max_{s\in[0,t]} X_u(s)-\lambda_t s\le 1}\ \mathbb{1}_{X_v(t)\ge m_t,\ \max_{s\in[0,t]} X_v(s)-\lambda_t s\le 1}\Big]$$

many-to-two

$$= E[L(L-1)]\int_0^t e^{2mt-mr}\ \underbrace{P\left(\begin{Bmatrix} B_t^{1,r}\ge m_t,\ \max_{s\in[0,t]} B_s^{1,r}-\lambda_t s\le 1 \\ B_t^{2,r}\ge m_t,\ \max_{s\in[0,t]} B_s^{2,r}-\lambda_t s\le 1\end{Bmatrix}\right)}\ dr$$



see more details at the end of this note $\underset{\bigcirc}{=}\ E\left[\mathbb{1}_{\max_{s\in[0,r]} B_s-\lambda_t s\le 1}\ \varphi(B_r)^2\right]$

with $\varphi(x) := P\left(\max_{s\in[0,t-r]} B_s + x - \lambda_t(s+r)\le 1,\ B_{t-r}+x\ge m_t\right)$

We first bound $\varphi(x)$ for $x\le \lambda_t r+1$ (note that $\varphi(x)=0$ for $x>\lambda_t r+1$)

$$\varphi(x) = P\Big(\max_{s\in[0,t-r]} B_s-\lambda_t s\le 1+\lambda_t r - x,\ B_{t-r}-\lambda_t(t-r)\ge \underbrace{m_t - x-\lambda_t(t-r)}_{=\lambda_t t - x - \lambda_t t+\lambda_t r = \lambda_t r - x}\Big)$$

Girsanov

$$= E\Big[e^{-\underbrace{\lambda_t B_{t-r}}_{\ge \lambda_t(\lambda_t r - x)}-\frac{\lambda_t^2}{2}(t-r)}\ \mathbb{1}_{\max_{s\in[0,t-r]} B_s\le 1+\lambda_t r - x,\ B_{t-r}\ge\lambda_t r - x}\Big]$$

$$\le e^{\lambda_t(x-\lambda_t r)-\frac{\lambda_t^2}{2}(t-r)}\ P\Big(\max_{s\in[0,t-r]} B_s\le \underbrace{1+\lambda_t r - x}_{=a},\ B_{t-r}\ge \underbrace{\lambda_t r - x}_{=a-y}\Big) \Rightarrow y=1$$

Corollary 2 in Lecture 6

$$\le e^{\lambda_t(x-\lambda_t r)-\frac{\lambda_t^2}{2}(t-r)}\left(\frac{1+\lambda_t r - x}{(t-r)^{3/2}}\wedge 1\right).$$

we can add this because $P(\cdots)\le 1$
(useful to make the function integrable at $r=t$)

$$E[K_t(K_t-1)] \le C\int_0^t e^{2mt-mr}\ E\left[\mathbb{1}_{\max_{s\in[0,r]} B_s-\lambda_t s\le 1}\left(e^{\lambda_t(B_r-\lambda_t r)-\frac{\lambda_t^2}{2}(t-r)}\left(\frac{1+\lambda_t r - B_r}{(t-r)^{3/2}}\wedge 1\right)\right)^2\right] dr$$

$$= C\int_0^t e^{2mt-mr-\lambda_t^2(t-r)}\ \underbrace{E\left[\mathbb{1}_{\max_{s\in[0,r]} B_s-\lambda_t s\le 1}\ e^{2\lambda_t(B_r-\lambda_t r)}\left(\frac{(1+\lambda_t r - B_r)^2}{(t-r)^3}\wedge 1\right)\right]}\ dr$$

$$E[\cdots] = E\left[e^{-\lambda_t B_r-\frac{\lambda_t^2}{2}r}\ \mathbb{1}_{\max_{s\in[0,r]} B_s\le 1}\ e^{2\lambda_t B_r}\left(\frac{(1-B_r)^2}{(t-r)^3}\wedge 1\right)\right] \quad \text{(by Girsanov)}$$

$$= e^{-\frac{\lambda_t^2}{2}r}\ E\left[e^{\lambda_t B_r}\ \mathbb{1}_{\max_{s\in[0,r]} B_s\le 1}\left(\frac{(1-B_r)^2}{(t-r)^3}\wedge 1\right)\right]$$

we distinguish according to values of B_r (note that $B_r\le 1$ on the considered event)

$$= e^{-\frac{\lambda_t^2}{2}r} \sum_{k\geq 0} \mathbb{E}\left[\mathbb{1}_{B_r\in[-k,-k+1]}\ \underbrace{e^{\lambda_t B_r}}_{\leq e^{\lambda_t(-k+1)}}\ \mathbb{1}_{\max_{s\in[0,r]} B_s\leq 1}\left(\underbrace{\frac{(1-B_r)^2}{(t-r)^3}\wedge 1}_{\leq \left(\frac{(1+k)^2}{(t-r)^3}\wedge 1\right) \leq (1+k)^2\left(\frac{1}{(t-r)^3}\wedge 1\right)}\right)\right]$$

$$\leq \left(\frac{1}{(t-r)^3}\wedge 1\right) e^{-\frac{\lambda_t^2}{2}r}\, e^{\lambda_t}\sum_{k\geq 0}(k+1)^2 e^{-\lambda_t k}\, P\Big(\max_{s\in[0,r]} B_s \leq \underbrace{1}_{=a},\ B_r\geq \underbrace{-k}_{=a-y}\Big) \Rightarrow y = k+1$$

Corollary 2 of Lecture 6 $\leq \frac{(k+1)^2}{r^{3/2}}\wedge 1 \leq \left(\frac{1}{r^{3/2}}\wedge 1\right)(k+1)^2$

$$\leq \left(\frac{1}{(t-r)^3}\wedge 1\right)\left(\frac{1}{r^{3/2}}\wedge 1\right) e^{-\frac{\lambda_t^2}{2}r} \times \underbrace{e^{\lambda_t}\sum_{k\geq 0}(k+1)^4 e^{-\lambda_t k}}_{\leq C'}$$

$$\leq C'\left(\frac{1}{(t-r)^3}\wedge 1\right)\left(\frac{1}{r^{3/2}}\wedge 1\right) e^{-\frac{\lambda_t^2}{2}r}$$

recall $\lambda_t = \lambda_c - \frac{3}{2\lambda_c}\frac{\log t}{t} \in \left[\frac{\lambda_c}{2},\lambda_c\right]$ for t large enough

$$\mathbb{E}[K_t(K_t-1)] \leq C''\int_0^t e^{2mt - mr - \lambda_t^2 t + \frac{\lambda_t^2}{2}r}\left(\frac{1}{(t-r)^3}\wedge 1\right)\left(\frac{1}{r^{3/2}}\wedge 1\right)dr$$

But $\lambda_t^2 = \lambda_c^2 - 3\frac{\log t}{t} + \left(\frac{3}{2\lambda_c}\frac{\log t}{t}\right)^2 \begin{cases} \geq 2m - 3\frac{\log t}{t} \\ \leq 2m - 3\frac{\log t}{t} + 1 \text{ for } t \text{ large enough}\end{cases}$

so $2mt - \lambda_t^2 t \leq 3\log t$ and $\frac{\lambda_t^2}{2}r - mr \leq -\frac{3}{2}\frac{r}{t}\log t + \frac{1}{2}$ which yields:

$$\mathbb{E}[K_t(K_t-1)] \leq C'''\int_0^t t^{3-\frac{3r}{2t}}\left(\frac{1}{(t-r)^3}\wedge 1\right)\left(\frac{1}{r^{3/2}}\wedge 1\right)dr$$

$$\leq C'''\left(\int_0^{t/2} t^3\frac{1}{(t/2)^3}\left(\frac{1}{r^{3/2}}\wedge 1\right)dr + \int_{t/2}^t t^{3-\frac{3r}{2t}}\left(\frac{1}{(t-r)^3}\wedge 1\right)\frac{1}{(t/2)^{3/2}}\,dr\right)$$

$s = t-r$

$$\leq C''''\Big(\underbrace{\int_0^{t/2}\left(\frac{1}{r^{3/2}}\wedge 1\right)dr}_{\leq \text{cste}} + \underbrace{\int_0^{t/2} t^{\frac{3s}{2t}}\left(\frac{1}{s^3}\wedge 1\right)ds}_{\leq \int_0^{\sqrt{t}} \underbrace{t^{\frac{3}{2\sqrt{t}}}}_{\xrightarrow{t\to\infty} 1}\left(\frac{1}{s^3}\wedge 1\right)ds + \underbrace{\int_{\sqrt{t}}^{t/2} t^{\frac{3}{4}}\frac{1}{s^3}\,ds}_{= t^{3/4}\left[\frac{-1}{2s^2}\right]_{\sqrt{t}}^{t/2} \leq \frac{t^{3/4}}{2t}} \leq \text{cste}}\Big)$$

$$\leq C'''''$$

$\blacksquare$

Proof of Proposition LB:

Step 1: Lower bounding the probability to have a large maximum.

We use the following version of Paley-Zygmund inequality for integer-valued r.v.:

$$P(K_t\geq 1) \geq \frac{\mathbb{E}[K_t]^2}{\mathbb{E}[K_t^2]}.$$

(Proof: $\mathbb{E}[K_t]^2 \underset{K_t\in\mathbb{N}}{=} \mathbb{E}[K_t\mathbb{1}_{K_t\geq 1}]^2 \underset{\text{Cauchy-Schwarz}}{\leq} \mathbb{E}[K_t^2]\,\mathbb{E}[\mathbb{1}_{K_t\geq 1}^2] = \mathbb{E}[K_t^2]\,P(K_t\geq 1)$)

Together with lemma 1 and 2, we get: there exists $c>0$ and $t_0 \geq 0$ such that for any $t \geq t_0$, $\mathbb{P}(K_t \geq 1) \geq c$.

In particular, $\forall t \geq t_0$ $\mathbb{P}(\Pi_t \geq m_t) \geq c$.

Step 2: Letting branch at the beginning to improve the lower bound.

Let $\varepsilon > 0$. We want to prove that there exists $y>0$ such that for t large enough $\mathbb{P}(\Pi_t < m_t - y \mid S) \leq \varepsilon$, where S is the survival event.

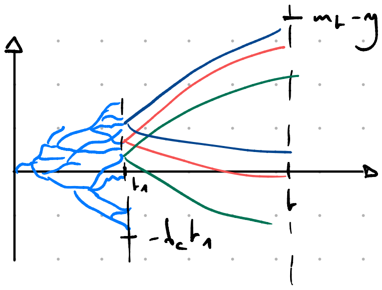
Let $n \geq 1$ to be chosen in terms of ε later.

On S, we have $N_t \xrightarrow[t\to\infty]{a.s.} \infty$, (more precisely, $e^{-mt} N_t = W_t^0 \xrightarrow{a.s.} W_\infty^0 > 0$ on S)

so $\mathbb{P}(N_t \geq n \mid S) \xrightarrow[t\to\infty]{} 1$.

On the other hand, $\mathbb{P}\left(\min_{u \in \mathcal{N}_t} X_u(t) \geq -\lambda_c t\right) \xrightarrow[t\to\infty]{} 1$, so there exists $t_1 \geq 0$ such that $\mathbb{P}(A_{t_1} \mid S) \geq 1 - \frac{\varepsilon}{2}$ where $A_{t_1} = \left\{ N_{t_1} \geq n, \min_{u \in \mathcal{N}_{t_1}} X_u(t_1) \geq -\lambda_c t_1 \right\}$

Now, let $t \geq t_0 + t_1$, we have



$$\mathbb{P}(\Pi_t < m_t - y \mid S) \leq \frac{\varepsilon}{2} + \mathbb{P}\left(A_{t_1} \cap \{\Pi_t < m_t - y\} \mid S\right)$$

$$\leq \frac{\varepsilon}{2} + \frac{1}{\mathbb{P}(S)} \mathbb{P}\left(\underbrace{A_{t_1}}_{\mathcal{F}_{t_1}\text{-measurable}} \cap \{\Pi_t < m_t - y\}\right)$$

Note that
$$\{\Pi_t < m_t - y\} = \left\{ \forall u \in \mathcal{N}_{t_1}, \max_{\substack{v \in \mathcal{N}_t \\ v \geq u}} X_v(t) < m_t - y \right\}$$

$$= \left\{ \forall u \in \mathcal{N}_{t_1}, \underbrace{\max_{\substack{v \in \mathcal{N}_t \\ v \geq u}} X_v(t) - X_u(t_1)}_{\substack{\text{by the branching property, independent of } \mathcal{F}_{t_1} \\ \text{and of each other, and } \overset{(d)}{=} \Pi_{t-t_1}}} < \underbrace{m_t - y - X_u(t_1)}_{\substack{\leq m_t - y + \lambda_c t_1 \\ \text{on the event } A_{t_1}}} \right\}$$

So on A_{t_1}, $\mathbb{P}(\Pi_t \leq m_t - y \mid \mathcal{F}_{t_1}) \leq \prod_{u \in \mathcal{N}_{t_1}} \mathbb{P}\left(\Pi_{t-t_1} < m_t - y + \lambda_c t_1\right)$

Choosing $y \geq 2\lambda_c t_1$, we have $m_t - y + \lambda_c t_1 \leq \lambda_c t - \frac{3}{2}\log t - \lambda_c t_1 \leq m_{t-t_1}$,

and so $\mathbb{P}\left(\Pi_{t-t_1} < m_t - y + \lambda_c t_1\right) \leq \mathbb{P}\left(\Pi_{t-t_1} < m_{t-t_1}\right) \underset{\text{by Step 1 } (t-t_1 > t_0)}{\leq} 1 - c$

Moreover, on A_{t_1}, we have $N_{t_1} \geq n$, so $\mathbb{P}(\Pi_t \leq m_t - y \mid \mathcal{F}_{t_1}) \leq (1-c)^n$.

It follows that $\mathbb{P}(\Pi_t < m_t - y \mid S) \leq \frac{\varepsilon}{2} + \frac{(1-c)^n}{\mathbb{P}(S)} \leq \varepsilon$ if n is chosen large enough. ▨

Additional details in the proof of Lemma 2 for the equality

$$\mathbb{P}\left(\begin{Bmatrix} B_t^{1,r} \geq m_t, \max_{s\in[0,t]} B_s^{1,r} - d_t s \leq 1 \\ B_t^{2,r} \geq m_t, \max_{s\in[0,t]} B_s^{2,r} - d_t s \leq 1 \end{Bmatrix}\right) = \mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s - d_t s \leq 1} \, \varphi(B_r)^2 \right]$$

where $\varphi(x) := \mathbb{P}\left(\max_{s\in[0,t-r]} B_s + x - d_t(s+r) \leq 1, B_{t-r} + x \geq m_t \right)$

First recall that $B^{1,r}$ and $B^{2,r}$ can be defined using 3 independent Brownian motions B^0, B^1, B^2 by setting, for $i \in \{1,2\}$, $B_s^{i,r} = \begin{cases} B_s^0 & \text{if } s \leq r, \\ B_{s-r}^i + B_r^0 & \text{if } s > r. \end{cases}$

Then we can rewrite

$$\mathbb{P}\left(\begin{Bmatrix} B_t^{1,r} \geq m_t, \max_{s\in[0,t]} B_s^{1,r} - d_t s \leq 1 \\ B_t^{2,r} \geq m_t, \max_{s\in[0,t]} B_s^{2,r} - d_t s \leq 1 \end{Bmatrix}\right)$$

$$= \mathbb{P}\left(\max_{s\in[0,r]} B_s^0 - d_t s \leq 1, \begin{cases} B_{t-r}^1 + B_r^0 \geq m_t, & \overbrace{\max_{s\in[r,t]} B_{s-r}^1 + B_r^0 - d_t s}^{= \max_{p\in[0,t-r]} B_p^1 + B_r^0 - d_t(p+r) \text{ by setting } p = s-r} \leq 1 \\ B_{t-r}^2 + B_r^0 \geq m_t, & \max_{s\in[r,t]} B_{s-r}^2 + B_r^0 - d_t s \leq 1 \end{cases} \right)$$

↓ write $\mathbb{P}(\cdots) = \mathbb{E}[\mathbb{P}(\cdots \mid B^0)]$

$$= \mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s^0 - d_t s \leq 1} \, \mathbb{P}\left(\begin{matrix} B_{t-r}^1 + B_r^0 \geq m_t, \max_{p\in[0,t-r]} B_p^1 + B_r^0 - d_t(p+r) \leq 1 \\ B_{t-r}^2 + B_r^0 \geq m_t, \max_{p\in[0,t-r]} B_p^2 + B_r^0 - d_t(p+r) \leq 1 \end{matrix} \,\middle|\, B^0 \right) \right]$$

↓ B^1 and B^2 are independent and have the same law

$$= \mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s^0 - d_t s \leq 1} \, \underbrace{\mathbb{P}\left(B_{t-r}^1 + B_r^0 \geq m_t, \max_{p\in[0,t-r]} B_p^1 + B_r^0 - d_t(p+r) \leq 1 \,\middle|\, B^0 \right)}_{= \varphi(B_r^0)}{}^2 \right]$$

We use here that if X and Y are independent r.v. then for any f measurable positive,
$\mathbb{E}[f(X,Y) \mid X] = g(X)$ where $g(x) := \mathbb{E}[f(x,Y)]$.

Proof: $g(X)$ is $\sigma(X)$-measurable and, for any h measurable positive

$$\mathbb{E}[g(X)\, h(X)] = \int g(x)\, h(x)\, \underbrace{dP_X(x)}_{\text{law of } X} \overset{\text{def of } g}{=} \int \left(\int f(x,y)\, dP_Y(y) \right) h(x)\, dP_X(x) \overset{\text{Fubini}}{=} \iint f(x,y)\, h(x)\, d(P_X \otimes P_Y)(x,y)$$

$$\overset{\text{independence}}{=} \iint f(x,y)\, h(x)\, dP_{(X,Y)}(x,y) = \mathbb{E}[f(X,Y)] \longrightarrow \text{this characterizes } \mathbb{E}[f(X,Y) \mid X].$$