ON ALTERNATING QUASIPOSITIVE LINKS

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Abstract. An effectively verifiable condition for quasipositivity of links is given. In particular, it is proven that if a quasipositive link can be represented by an alternating diagram satisfying the condition that no pair of Seifert circles is connected by a single crossing, then the diagram is positive and the link is strongly quasipositive.

1. Introduction

An n-braid is called *quasipositive* if it is a product of conjugates of the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the braid group B_n. A braid is called *strongly quasipositive* if it is a product of braids of the form $\tau_{k,j}\sigma_j\tau_{k,j}^{-1}$ for $j \le k$ where $\tau_{k,j} = \sigma_k\sigma_{k-1}\dots\sigma_j$. All links in this paper are assumed to be oriented links in the 3-sphere S^3. A link is called *(strongly) quasipositive* if it is the braid closure of a (strongly) quasipositive braid. This class of links is impotrant for the study of plane algebraic curves. As shown in [3], a link is quasipositive if and only if it is cut on the standard embedded 3-sphere in $\mathbb{C}^2$ by a complex algebraic curve. It is also shown in [3] that a link is quasipositive if it is cut by an algebraic curve on any smoothly embedded 3-sphere bounding a strictly pseudoconvex domain in $\mathbb{C}^2$. Quasipositivity criterions play important role in the study of plane real algebraic curves (the first part of the 16th Hilbert problem), see e.g. [11].

A link diagram is called *positive* if all its crossings are positive. If one resolves all crossings according the orientations (i.e., replacing ⤫ or ⤬ with ≍), then the diagram transforms into a disjoint union of simple closed curves. They are called *Seifert circles* (see, e.g. [9, 14, 15]).

S. Baader [1, p. 268, Question (4)] asked: *Do quasipositive alternating links have positive diagrams?* Note that positive diagrams represent strongly quasipositive links (see [10], [13]) and alternating strongly quasipositive links have positive alternating diagrams by [2, Cor. 7.3]. Notice also that positive alternating diagrams are special (a diagram is called *special* [9] if its Seifert circles bound disjoint disks).

In this note we give an affirmative answer for a large class of alternating links: those which have an alternating diagram whose number of Seifert circles is equal to the braid index of the link. We call such diagrams *Diao–Hetyei–Liu* or *DHL diagrams* (and the corresponding links *DHL links*) because these authors gave in [5] the following very nice and simple characterization for them.

Theorem 1. ([5, Thm. 1.1]) *An alternating diagram is DHL if and only if there is no pair of Seifert circles connected by a single crossing.*

Our main result is the following.

Theorem 2. *Let D be a DHL diagram of a quasipositive link. Then D is positive.*

The proof is obtained by a combination of results from [6], [7], [9], [14], and [15] (see Section 2). Theorems 1 and 2 allow to produce a lot of examples of non-quasipositive links without any computations (see an example in Figure 1).

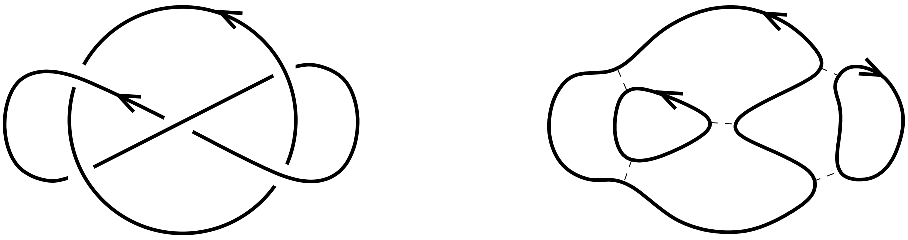

FIGURE 1. A link (on the left hand side) which is not quasipositive by Theorems 1 and 2. Its Seifert circles are shown on the right hand side.

Since any positive diagram represents a strongly quasipositive link (see [10], [13]), we obtain:

Corollary 1. *Let L be a DHL link. Then the following conditions are equivalent:*

- (i) *L is quasipositive;*
- (ii) *L is strongly quasipositive;*
- (iii) *L has a positive alternating diagram.*

In Section 3 we generalize Theorem 2 to all alternating links whose braid index is computed in [4]; see Theorem 4 and Remark 2.

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2. PROOF OF THE MAIN THEOREM

Let D be a connected link diagram. The *Seifert graph* of D is the graph G_D whose vertices correspond to Seifert circles and the edges correspond to the crossings. The sign of an edge is the sign of the corresponding crossing. A diagram D is called *reduced* if G_D does not have any edge whose removal disconnects G_D. Let $d(D)$ denote the sum of signs of all edges of a spanning tree of G_D, and let $w(D)$ be the *writhe* of D, i.e., the sum of signs of all crossings.

For a link L, let $\sigma(L)$ and $\mathbf{n}(L)$ be its signature and nullity (the latter is the nullity of a symmetrized Seifert form on a *connected* Seifert surface).

Theorem 3. (Traczyk [14]) *Let D be a connected reduced alternating diagram of a link L. Then $\sigma(L) = d(D) - w(D)$ and $\mathbf{n}(L) = 0$.*

This formula for $\sigma(L)$ is given in [14, Thm. 2(1)] (the factor 1/2 is erroneous there). The fact that $\mathbf{n}(L) = 0$ (equivalently, $\det(L) \neq 0$) is proven in [9, Lem. 5.1] and in the appendix to [14]. Otherwise it can be easily derived from [14, Thm. 1].

Proof of Theorem 2. Let D be a DHL diagram of a quasipositive link L. Then each connected component of D is evidently a DHL diagram and it represents a quasipositive link by [12]. So, it is enough to consider the case when D is connected.

Let n be the braid index of L. By definition of DHL diagrams, D has n Seifert circles. Hence, by [15, Thm. 1] (see the discussion of this theorem in the introduction to [15]), L can be represented by an n-braid β_1 with

$$w(\beta_1) = w(D). \tag{1}$$

By [7, Thm. 1.2] L can be represented by a quasipositive n-braid β_2. Then Murasugi–Tristram inequality [9] for quasipositive braids can be reformulated as follows (see [11, Cor. 3.2])

$$1 + \mathbf{n}(L) \geq |\sigma(L)| + n - w(\beta_2). \tag{2}$$

By Dynnikov–Prasolov Theorem [6] (Generalized Jones Conjecture) we have

$$w(\beta_1) = w(\beta_2) \tag{3}$$

Note that any DHL diagram is reduced. Hence, by combining (1) – (3) with Theorem 3, we obtain $|d(D)-w(D)| \leq 1-n+w(D)$ whence $w(D)-d(D) \leq 1-n+w(D)$, i.e., $d(D) \geq n-1$. Recall that $d(D)$ is the sum of signs of all edges of a spanning tree of G_D. Any spanning tree of G_D has $n-1$ edges, hence all its edges are positive. Since each edge of G_D belongs to some spanning tree, we conclude that all crossings of D are positive. Theorem 2 is proven. □

3. A generalization of the main theorem

Let D be an alternating diagram of a link L. Let $b = b(L)$ be the braid index of L and $s = s(D)$ be the number of Seifert circles of D. Define $d^{\pm} = d^{\pm}(D)$ as the number of edges of this sign in a spanning tree of G_D, thus $d = d(D) = d^{+} - d^{-}$.

Let β be a braid with b strands realizing L. Due to Dynnikov – Prasolov Theorem [6], $w(\beta)$ does not depend on the choice of β, which allows us to define the numbers $\mathbf{r}^{\pm} = \mathbf{r}^{\pm}(D)$ from the system of equations

$$\mathbf{r}^{+} + \mathbf{r}^{-} = s - b, \qquad \mathbf{r}^{+} - \mathbf{r}^{-} = w(D) - w(\beta).$$

Remark 1. The definition of the numbers $r^{\pm}$ in introduced in [4] is not quite clear but in all cases when they are computed in [4], they are equal to our $\mathbf{r}^{\pm}$; cf. [4, Rem. 3.1–3.3].

If D is a DHL diagram, then $\mathbf{r}^{+} = \mathbf{r}^{-} = 0$ (recall that in this case $w(D) = w(\beta)$ by [15, Thm. 1]), thus the following statement is a generalization of Theorem 2.

Theorem 4. *Let D be a reduced alternating diagram of a quasipositive link L, and*

$$2\mathbf{r}^{-}(D) \leq d^{-}(D). \tag{4}$$

Then D is positive (and hence L is strongly quasipositive by [10, 13]).

Proof. Since the arguments are almost the same as for Theorem 2, we just write down the final computation. So, we have $w(D) - d \leq |\sigma| \leq 1 - b + w(\beta)$, hence

$$d + 1 \geq w(D) - w(\beta) + b = (\mathbf{r}^{+} - \mathbf{r}^{-}) + s - (\mathbf{r}^{+} + \mathbf{r}^{-}) = s - 2\mathbf{r}^{-} \geq s - d^{-}$$

whence $d^+ \geq s - 1$ and the result follows. $\square$

Remark 2. In all cases when the braid index of a reduced alternating diagram is computed in [4], the inequality (4) holds, in particular it holds for minimal diagrams of two-bridge links and of alternating Montesinos links.

Question 1. Does (4) hold for any reduced alternating diagram?

Remark 3. Tetsuya Ito [8] generalized Theorem 2 to homogeneous diagrams satisfying the condition that the number of Seifert circles is equal to the braid index (however there is no known algorithm to check this condition). Some other related questions are also discussed in [8].

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