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Polarisation des partialarpartial-variétés de Calabi-Yau SKT par des classes d'Aeppli et l'hyperbolicité sG

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Résumé

Cette thèse se concentre sur deux sujets principaux dans le contexte de la géométrie non-Kählerienne : l'étude des déformations locales polarisées par la classe de cohomologie d'Aeppli $[\omega]_A$ d'une métrique SKT, et l'exploration de l'hyperbolicité sur les variétés complexes compactes.

La première partie de la thèse approfondit les déformations des $\partial\bar{\partial}$-variétés X avec un fibré canonique trivial et équipées d'une métrique ω telle que $\partial\bar{\partial}\omega = 0$ (c'est-à-dire ω est SKT). Dans ce contexte, nous introduisons le concept de petites déformations de X polarisées par la classe de cohomologie d'Aeppli $[\omega]_A$ d'une métrique SKT ω. Il existe une correspondance entre les variétés polarisées par $[\omega]_A$ dans la famille Kuranishi de X et les classes de Bott-Chern qui sont primitives dans un sens que nous définissons. Nous étudions également l'existence d'un élément primitif dans une classe primitive de Bott-Chern arbitraire et comparons les métriques sur l'espace de base de la sous-famille de variétés polarisées par $[\omega]_A$ au sein de la famille Kuranishi.

La dernière partie de cette thèse propose l'hyperbolicité sG comme nouvel outil pour étudier l'hyperbolicité sur les variétés complexes. Elle démontre que cette notion conduit à une classe plus large de variétés hyperboliques par rapport à l'hyperbolicité équilibrée. Nous introduisons également des structures hyperboliques faiblement p-Kähler et des métriques hyperboliques plurifermées étoilées comme nouvelles pistes possibles d'exploration.

Mots clés: Géométrie non-Kählerienne, Familles holomorphes de variétés complexes compactes, $\partial\bar{\partial}$-variétés de Calabi-Yau , Métriques SKT, Hyperbolicité des variétés complexes.

Abstract

This thesis centers around two main topics within the context of non-Kähler geometry: the study of local deformations polarised by the Aeppli cohomology class $[\omega]_A$ of an SKT metric, and the exploration of hyperbolicity on compact complex manifolds.

The first part of the thesis delves into the deformations of $\partial\bar{\partial}$-manifolds X with trivial canonical bundle and equipped with a metric ω such that $\partial\bar{\partial}\omega = 0$ (i.e., ω is SKT). In this context, we introduce the concept of small deformations of X polarised by the Aeppli cohomology class $[\omega]_A$ of an SKT metric ω. There is a correspondence between the manifolds polarised by $[\omega]_A$ in the Kuranishi family of X and the Bott-Chern classes that are primitive in a sense that we define. We also investigate the existence of a primitive element in an arbitrary Bott-Chern primitive class and compare the metrics on the base space of the subfamily of manifolds polarised by $[\omega]_A$ within the Kuranishi family.

The last part of this thesis proposes sG-hyperbolicity as a new tool for studying hyperbolicity on complex manifolds. It demonstrates that this notion leads to a wider class of divisorially hyperbolic manifolds compared to balanced hyperbolicity. We also introduce weakly p-Kähler hyperbolic structures and pluriclosed star split hyperbolic metrics as possible new avenues for exploration.

Keywords: Non-Kähler Geometry, Holomorphic families of compact complex manifolds, Calabi-Yau $\partial\bar{\partial}$-Manifolds, SKT Metrics, Hyperbolicity of Complex Manifolds.

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Contents

Chapitre 1

Introduction

Dans cette thèse, nous explorons les deux sujets suivants :

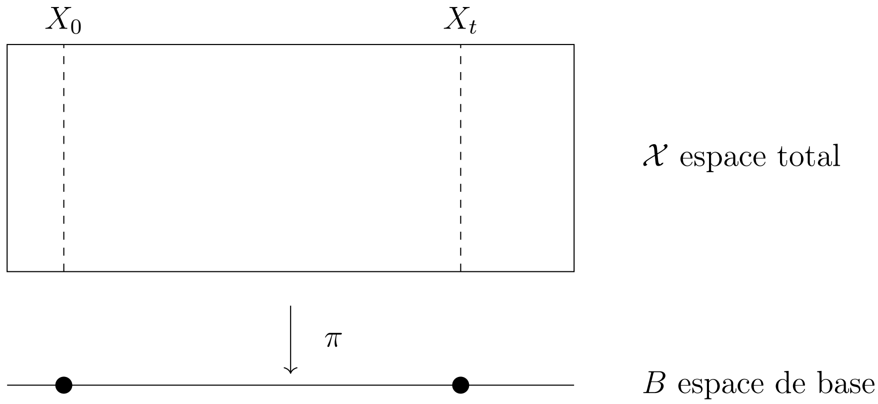
- les déformations locales polarisées par une classe SKT $[\omega]_A$,
- l'hyperbolicité sur les variétés complexes compactes.

En géométrie non-kählérienne, la déformation et la métrique sont deux sujets incroyablement importants et intrigants. La géométrie non-kählérienne étudie les variétés complexes qui ne satisfont pas la condition kählérienne, révélant des structures plus riches et plus diversifiées que celles généralement trouvées dans les variétés kählérienne.

La théorie de la déformation explore comment les objets dans des espaces complexes changent continuellement avec des paramètres variables, servant d'outil crucial pour étudier la diversité des structures complexes. À travers la déformation, nous pouvons comprendre comment les propriétés géométriques et topologiques globales d'une structure complexe répondent à des modifications mineures. La théorie de la déformation dans un cadre non-kählérien explore comment ces structures complexes évoluent avec les changements de paramètres, révélant des phénomènes géométriques et topologiques nouveaux spécifiques aux variétés non-kählérienne.

Dans le contexte des variétés non-kählérienne, l'étude des métriques telles que SKT (strong Kähler with torsion), équilibrée et Gauduchon prend une importance accrue en raison de l'absence de métriques kählérienne. Ces métriques sont intrigantes car elles permettent aux variétés complexes d'exhiber des comportements géométriques riches qui ne sont pas observés dans les environnements kählérien. Les métriques sur une variété complexe influencent profondément la structure géométrique et les caractéristiques topologiques globales de la variété.

Dans cette thèse, notre discussion sur la déformation se centre autour des familles holomorphes de variétés complexes compactes. Rappelons qu'une famille holomorphe de variétés complexes compactes est une submersion holomorphe propre $\pi : \mathcal{X} \to B$ entre deux variétés complexes $\mathcal{X}$ et B. Nous appelons $\mathcal{X}$ l'espace total de la famille, B l'espace de base de la famille, et $X_t = \pi^{-1}(t)$ la fibre au-dessus de t pour $t \in B$. Nous utilisons le diagramme ci-dessous pour illustrer ces concepts de manière plus intuitive.



L'espace total, l'espace de base et les fibres sont intimement liés et présentent des connexions intrinsèques profondes. Grâce à [Ehr47], les fibres d'une telle famille partagent la même structure différentielle, ainsi que des structures complexes différentes. Notre premier sujet est d'étudier le sous-ensemble de l'espace de base lorsque une fibre admet certaines conditions. Nous pourrions déduire plusieurs types de métriques intéressantes sur le sous-ensemble de l'espace de base grâce aux propriétés des fibres. Pour être plus précis, nous étudions l'espace de déformation polarisée universel de X lorsque X est une $\partial\bar{\partial}$-variété de Calabi-Yau SKT.

Rappelons qu'une métrique hermitienne sur une variété complexe X est une C^∞ famille de cartes sesquilinéaires définies positives sur $T_x^{1,0}X \times T_x^{1,0}X$ par rapport à x. Nous pouvons identifier canoniquement une métrique hermitienne avec une C^∞ $(1,1)$ forme ω. Il est bien connu que les variétés complexes avec une métrique d-fermée ; c'est-à-dire, les variétés kählérienne, possèdent une richesse de propriétés favorables. Cependant, dans le domaine de la géométrie non-kählérienne, différents types de métriques imposent certaines propriétés à la variété sous-jacente.

Le tableau suivant montre brièvement plusieurs types de métriques hermitiennes importantes et les relations entre elles.

$$\begin{array}{ccccc} \omega \text{ est kählérienne} & \Longrightarrow & \omega \text{ est Hermitien-symplectique} & \Longrightarrow & \omega \text{ est SKT} \\ \Downarrow & & & & \\ \omega \text{ est équilibré} & \Longrightarrow & \omega \text{ est fortement Gauduchon} & \Longrightarrow & \omega \text{ est Gauduchon} \end{array}$$

Les différentes métriques sur les variétés complexes impliquent diverses propriétés géométriques et topologiques des variétés. Explorer des variétés avec diverses métriques offre un domaine de recherche fascinant. Notre second sujet est d'explorer divers types d'hyperbolicité des métriques ou des variétés complexes, ainsi que les relations entre les différents types d'hyperbolicité. En étudiant cela, nous pourrions en apprendre davantage sur les propriétés des variétés hyperboliques.

Les déformations locales polarisées par une classe SKT $[\omega]_A$

En citant [Pop19], une variété complexe compacte X satisfaisant au $\partial\bar\partial$ lemme avec un fibré canonique trivial peut être déformée dans n'importe quelle direction dans $H^{0,1}(X, T^{1,0}X)$. En d'autres termes, la famille Kuranishi d'une variété de Calabi-Yau est non obstruée. Dans [Tia87], Tian a étudié l'espace de déformation polarisée universel de X lorsque X admet une métrique kählérienne avec la condition équilibrée. Popovici a exploré les déformations locales copolarisées par une classe équilibrée dans [Pop19], profitant de l'ouverture de la déformation des $\partial\bar\partial$ propriétés et équilibrées.

Inspirés par ces travaux, nous pouvons considérer la déformation polarisée d'une $\partial\bar\partial$-variété de Calabi-Yau SKT. Remarquons d'abord que les propriétés $\partial\bar\partial$ et SKT d'une variété complexe compacte sont ouvertes sous les déformations holomorphes. Comme la propriété SKT implique seulement la fermeture $\partial\bar\partial$ de ω, nous devons considérer la classe de cohomologie d'Aeppli de ω plutôt que la classe de cohomologie de De Rham ou de Dolbeault. Le groupe de cohomologie d'Aeppli de bidegré (p,q) est défini comme :

$$H_A^{p,q}(X,\mathbb{C}) = \frac{\ker(\partial\bar\partial : C^{p,q}(X) \to C^{p+1,q+1}(X))}{\mathrm{Im}(\partial : C^{p-1,q}(X) \to C^{p,q}(X)) + \mathrm{Im}(\bar\partial : C^{p,q-1}(X) \to C^{p,q}(X))}.$$

Grâce à l'hypothèse $\partial\bar\partial$ sur X, nous avons les décompositions

$$\begin{aligned} H^2_{DR}(X,\mathbb{C}) &\simeq H_A^{2,0}(X,\mathbb{C}) \oplus H_A^{1,1}(X,\mathbb{C}) \oplus H_A^{0,2}(X,\mathbb{C}) \\ &\simeq H_{\bar\partial}^{2,0}(X,\mathbb{C}) \oplus H_{\bar\partial}^{1,1}(X,\mathbb{C}) \oplus H_{\bar\partial}^{0,2}(X,\mathbb{C}) \end{aligned}$$

et la symétrie

$$H_A^{2,0}(X,\mathbb{C}) \cong H_A^{0,2}(X,\mathbb{C}).$$

Ainsi, sur une $\partial\bar\partial$-variété de Calabi-Yau SKT, nous pouvons considérer la polarisation par la classe d'Aeppli $[\omega]_A$ d'une métrique SKT. Laissons $\pi : \mathcal{X} \to B$ être la famille Kuranishi de X.

Definition 1.0.1 (voir Définition 3.3.1)**.** *Fixons la classe d'Aeppli* $[\omega]_A \in H_A^{1,1}(X,\mathbb{C})$ *d'une métrique SKT* ω *sur* $X_0 = X$*. Pour* $t \in B$*, nous disons que* X_t *est polarisée par* $[\omega]_A$ *si la projection* $[\omega]_{A,t}^{0,2}$ *de* $\{\omega\}_{DR}$ *sur* $H_A^{0,2}(X_t,\mathbb{C})$ *est* 0.

Cette définition est naturelle dans le sens suivant. Si ω est en outre kählérienne, nous avons effectivement que pour une déformation X_t de X, X_t est polarisée par $[\omega]_{\bar\partial}$ si et seulement si X_t est polarisée par $[\omega]_A$. Selon [Pop19], nous savons que pour une déformation X_t d'une variété kählérienne (X,ω), X_t est polarisée par $[\omega]_{\bar\partial}$ si et seulement si X_t est copolarisée par la classe équilibrée $[\omega^{n-1}]_{\bar\partial}$. Ainsi, pour une métrique kählérienne sur X, ces trois types de polarisation coïncident sans surprises.

Avec la classe d'Aeppli $[\omega]_A$ d'une métrique SKT, nous pouvons considérer la carte

$$\begin{aligned} L_{[\omega]} : H_{BC}^{p,q}(X,\mathbb{C}) &\longrightarrow H_A^{p+1,q+1}(X,\mathbb{C}) \\ [\gamma]_{BC} &\longmapsto [\omega\wedge\gamma]_A \end{aligned}$$

Bien que nous n'ayons pas de décomposition de Lefschetz ou le théorème de Lefschetz fort en général en raison de la 'torsion' dans la définition de 'strong Kähler with torsion', nous avons l'isomorphisme des groupes de cohomologie de Dolbeault aux groupes de cohomologie d'Aeppli et de Bott-Chern correspondants grâce à l'hypothèse $\partial\bar{\partial}$ de nouveau. Cela nous donne une autre version de l'isomorphisme de Lefschetz via des isomorphismes.

L'hypothèse Calabi-Yau, c'est-à-dire, la trivialité du fibré canonique K_X, nous fournit une forme holomorphe n connue sous le nom de forme Calabi-Yau. Avec cette forme holomorphe n, nous avons des isomorphismes

$$H^{0,1}(X, T^{1,0}X) \simeq H^{n-1,1}(X, \mathbb{C})$$

et

$$H^{0,q}(X, \mathbb{C}) \simeq H^{n,q}(X, \mathbb{C})$$

Avec tous ces isomorphismes mentionnés ci-dessus, nous pouvons maintenant décrire le sous-ensemble des déformations locales de X polarisées par la classe SKT $[\omega]_A$ par le théorème suivant :

Theorem 1.0.2 (voir Théorème 3.4.3)**.** *Si ω est une métrique SKT sur une $\partial\bar{\partial}$-variété de Calabi-Yau X, alors*

$$\widetilde{T_{[u]}} : H^{0,1}(X, T^{1,0}X)_{[\omega]} \longrightarrow H^{n-1,1}_{BC,prim}(X, \mathbb{C})$$

est un isomorphisme.

Cette description est à nouveau cohérente avec la version kählérienne et la version équilibrée si la métrique SKT ω est kählérienne. Soulignons la cruciale de l'hypothèse $\partial\bar{\partial}$ ici en utilisant l'exemple de la variété Calabi-Eckmann $X := S^3 \times S^3$ qui est une variété SKT. En calculant ses groupes de cohomologie de Bott-Chern, nous savons que les isomorphismes échouent. Ainsi, aussi le Théorème 1.0.2. En fait, il existe un t dans n'importe quel petit voisinage de 0 tel que la déformation correspondante X_t de X n'est pas SKT.

Nous considérons ensuite les propriétés géométriques de l'espace de base $B_{[\omega]_A}$ du sous-ensemble des déformations holomorphes de X polarisées par la classe SKT $[\omega]_A$. Nous avons l'ouverture de la déformation des propriétés combinées $\partial\bar{\partial}$ et Calabi-Yau sous les déformations holomorphes. Par conséquent, nous avons $\dim_{\mathbb{C}} H^{n,0}(X_t, \mathbb{C}) = 1$ pour t suffisamment proche de 0. Avec cette observation, nous remarquons que la carte périodique identifiant la ligne complexe $H^{n,0}(X_t, \mathbb{C})$ à un point dans $\mathbb{P}H^n(X, \mathbb{C})$ est une immersion holomorphe locale. En d'autres termes, le Théorème de Torelli Local est valable sous nos conditions.

Nous pourrions donc dériver une métrique kählérienne sur B à partir de la carte périodique. Analogiquement à [Pop19], nous proposons également deux versions des métriques de Weil-Petersson sur $B_{[\omega]_A}$: $\omega^{(1)}_{WP}$ et $\omega^{(2)}_{WP}$. Pour comparer ces trois métriques, nous remarquons que $\omega^{(1)}_{WP} = \omega^{(2)}_{WP}$ si $\mathrm{Ric}(\omega_t) = 0$ pour tous les t dans $B_{[\omega]_A}$. Sous condition kählérienne, nous pouvons supposer que ω est Kähler Einstein, ce qui est Ricci plat. Par conséquent, les métriques de

Weil-Petersson dérivées et celle dérivée par la carte périodique et la métrique de Fubini-Study sur $\mathbb{P}H^n(X, \mathbb{C})$ coïncident. Mais sous condition SKT, si ces métriques coïncident dépend de certaines conditions.

Hyperbolicité sur les variétés complexes compactes

La hyperbolicité de Kobayashi est introduite dans [Kob67], nécessitant que la pseudo-distance de Kobayashi soit une distance. Chaque application holomorphe de $\mathbb{C}$ vers une variété hyperbolique de Kobayashi est constante, ce qui est déduit du fait que chaque application holomorphe réduit la distance par rapport à la pseudo-distance de Kobayashi. Plus tard, Brody a prouvé qu'une variété complexe compacte X est hyperbolique de Kobayashi si chaque application holomorphe $f\colon \mathbb{C} \to X$ est constante [Bro78].

Une variété complexe X est dite hyperbolique de Brody si chaque application holomorphe $f\colon \mathbb{C} \to X$ est constante.

Gromov a introduit la notion d'hyperbolicité kählérienne pour les variétés complexes compactes dans [Gro91], nécessitant une métrique kählérienne ω sur une variété complexe compacte pour être $\tilde{d}$(bornée). Marouani et Popovici l'ont généralisée et ont proposé la notion d'hyperbolicité équilibrée dans [MP22a]. Une variété complexe compacte de dimension $n \geq 2$ est hyperbolique équilibrée si elle admet une métrique équilibrée ω telle que ω^{n-1} soit $\tilde{d}$(bornée) par rapport à ω.

Bien qu'une variété hyperbolique équilibrée n'ait pas besoin d'être hyperbolique de Brody, dans [MP22a] ils ont prouvé qu'une application holomorphe non dégénérée $f\colon \mathbb{C}^{n-1} \to X$ doit satisfaire certaines conditions de croissance non sous-exponentielle. Une variété complexe compacte X telle que chaque application holomorphe satisfait à ces conditions est dite être hyperbolique divisoriellement par [MP22a]. Pour résumer, nous avons le diagramme suivant :

$$\begin{array}{ccc} X \text{ est hyperbolique kählérienne} & \Longrightarrow & X \text{ est hyperbolique de Kobayashi/Brody} \\ \Big\Downarrow & & \Big\Downarrow \\ X \text{ est hyperbolique équilibrée} & \Longrightarrow & X \text{ est hyperbolique divisoriellement} \end{array}$$

Nous remarquons que la propriété équilibrée n'est pas ouverte sous les déformations holomorphes, alors que la propriété de fortement Gauduchon (sG) l'est. Dans cette perspective, il est naturel de proposer l'hyperbolicité sG. Nous observons que l'hyperbolicité sG est effectivement ouverte sous les déformations holomorphes. Ainsi, les petites déformations des variétés hyperboliques équilibrées sont sG hyperboliques. Cela nous donne une avenue pour obtenir des exemples de variétés hyperboliques sG qui ne sont pas nécessairement hyperboliques équilibrées. Notre théorème principal est que chaque variété hyperbolique sG est hyperbolique divisoriellement. Par conséquent, nous avons une classe plus large de variétés hyperboliques divisoriellement que la classe des variétés hyperboliques équilibrées.

La notion de propriété équilibrée dégénérée introduite dans [Pop15] est plus forte que la propriété hyperbolique équilibrée. Nous proposons la notion de métrique sG dégénérée. Nous observons que la propriété sG dégénérée implique l'hyperbolicité sG et que la propriété équilibrée dégénérée implique la propriété sG dégénérée. La propriété sG dégénérée est ouverte sous la déformation holomorphe. De nouveau, les petites déformations des variétés équilibrées dégénérées sont sG dégénérées. L'existence d'une métrique sG dégénérée sur une variété complexe compacte est équivalente à la dégénérescence du cône pseudo-effectif de la variété complexe compacte. Rappelons que le cône pseudo-effectif d'une variété complexe compacte est défini dans [Dem92] comme

$$\mathcal{E}_X := \left\{ [T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \middle| T \geq 0 \text{ d-fermé } (1,1)\text{-courant sur } X \right\}.$$

Dans la dernière partie, nous avons généralisé l'hyperbolicité équilibrée et sG à d'autres dimensions, et proposé l'hyperbolicité p-kählérienne faible. Nous avons également introduit l'hyperbolicité plurifermée étoilée, en s'appuyant sur la notion de métrique plurifermée étoilée introduite dans [Pop23]. Il est intéressant de constater que l'hyperbolicité plurifermée étoilée implique également l'hyperbolicité divisoriellement. Quant à l'hyperbolicité p-kählérienne faible, elle exige que chaque application holomorphe non dégénérée $f : \mathbb{C}^p \to X$ satisfasse à la condition de croissance non sous-exponentielle.

Chapter 2

Preliminaries

2.1 Deformation

2.1.1 Holomorphic family of compact complex manifold

Definition 2.1.1. *A holomorphic family of compact complex manifolds is a proper holomorphic submersion $\pi : \mathcal{X} \to B$ between two complex manifolds $\mathcal{X}$ and B.*

We call $\mathcal{X}$ the total space of the family, B the base space of the family, and $X_t = \pi^{-1}(t)$ the fibre above t for $t \in B$. We have the compactness of the fibres X_t from π being proper.

Fix a base point $0 \in B$, we could consider the fibre X_0 as a reference fibre. A fibre X_t is called a deformation of X.

The most fundamental fact about the holomorphic family of compact complex manifolds is the following

Theorem 2.1.2 ([Ehr47])**.** *(i) Every holomorphic family of compact complex manifolds is locally C^∞ trivial in the following sense.*

There exists a C^∞ manifold X such that every point $t_0 \in B$ has an open neighbourhood $U \subset B$ for which there exists a C^∞ diffeomorphism

$$T : \mathcal{X}_U \to X \times U$$

such that $pr_2 \circ T = \pi$, where $\mathcal{X}_U = \pi^{-1}(U) \subset X$ and $pr_2 : X \times U \to U$ is the projection on the second factor.

(ii) If the base space B is contractible, the family is even globally C^∞ trivial in the sense that there exists a C^∞ manifold X and a C^∞ diffeomorphism

$$T : \mathcal{X} \to X \times B$$

such that $pr_2 \circ T = \pi$, where $pr_2 : X \times B \to B$ is the projection on the second factor.

(iii) Suppose that the base space B of the family is an open ball about the origin in some $\mathbb{C}^m$. The local trivialisation $T = (T_0, \pi) : \mathcal{X} \to X_0 \times B$ of

(i), obtained after possibly replacing B by a neighbourhood U of $0 \in B$, can be chosen such that the fibres of the map $T_0 : \mathcal{X} \to X_0$ are complex submanifolds of $\mathcal{X}$.

From this theorem, we know that the fibres are locally C^∞-diffeomorphic to each other, i.e.,

$$X_t \overset{C^\infty}{\simeq} X_0 \quad \text{for } t \in U.$$

Another fact deduced from this theorem is that the differential structure remains unchanged under small deformations. For example, for t in a sufficiently small neighborhood of 0, we have

$$H^k_{DR}(X_t, \mathbb{C}) = H^k_{DR}(X, \mathbb{C}).$$

However, in general, the Dolbeault cohomology groups, Hodge numbers, the operators $\bar\partial, \bar\partial^*, \Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ vary with the complex structure J_t of X_t.

Due to the compactness of X_0, there is an atlas consisting of finite open cover $\{V_j\}_j$ of X_0, $\{V_j \times U\}_j$ is a finite open cover of $X \times U$. We have holomorphic transition maps $f_{jk}(z,t)$ with respect to this cover, which are holomorphic in both z and t.

Now it is clear that the complex structures of X_t depend holomorphically on $t \in B$.

2.1.2 Kodaira-Spencer map

Let $d\pi : T^{1,0}\mathcal{X} \to T^{1,0}B$ be the differential of π. We have

$$\ker(d\pi|_{X_0}) = T^{1,0}_{X_0}.$$

Hence we get the following exact sequence on X_0:

$$0 \to T^{1,0}X_0 \to T^{1,0}\mathcal{X}|_{X_0} \to \pi^*(T^{1,0}B)|_{X_0} \to 0$$

From this we deduce the connecting morphism

$$\rho : H^0(X_0, \pi^*(T^{1,0}B)|_{X_0}) \to H^1(X_0, \mathcal{O}_{X_0}(T^{1,0}X_0))$$

By Theorem 2.1.2, $\pi^*(T^{1,0}B)|_{X_0} = X_0 \times T^{1,0}_0 B$ is trivial. Therefore, we have

$$H^0(X_0, \pi^*(T^{1,0}B)|_{X_0}) \simeq T^{1,0}_0 B.$$

Besides, by the Dolbeault isomorphism

$$H^1(X_0, \mathcal{O}_{X_0}(T^{1,0}X_0)) \simeq H^{0,1}(X_0, T^{1,0}X_0)$$

we get the Kodaira-Spencer map at 0:

$$\rho : T^{1,0}_0 B \to H^{0,1}(X_0, T^{1,0}X_0)$$

For a holomorphic vector field $\frac{\partial}{\partial t}$ on a small neighbourhood of 0 in B, we now seek the cohomology class $\rho\left(\frac{\partial}{\partial t}\right)$.

Fix a finite open cover $\{V_j\}_j$ as in Section 2.1.1, for $1 \leq \alpha \leq n$, we have

$$f_{ik}^{\alpha}(z_k, t) = f_{ij}^{\alpha}(f_{jk}^{1}(z_k, t), \cdots, f_{jk}^{n}(z_k, t), t),$$

Putting $z_j^{\beta} = f_{jk}^{\beta}(z_k, t)$, we obtain

$$\frac{\partial f_{ik}^{\alpha}}{\partial t}(z_k, t) = \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}(z_k, t)}{\partial t}.$$

Let

$$\theta_{jk}(t) = \sum_{\beta=1}^{n} \frac{\partial f_{jk}^{\beta}(z_k, t)}{\partial t} \cdot \frac{\partial}{\partial z_j^{\beta}}$$

denote a vector field on $V_{jk} := V_j \cap V_k$.
We have

$$\frac{\partial f_{ik}^{\alpha}(z_k, t)}{\partial t} = \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial t} + \theta_{jk}(t)(f_{ij}^{\alpha}(z_k, t)).$$

By

$$\theta_{ik}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{ik}^{\alpha}(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^{\alpha}}$$

$$= \sum_{\alpha=1}^{n} \left(\frac{\partial f_{ij}^{\alpha}(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^{\alpha}} + \theta_{jk}(t)(f_{ij}^{\alpha}(z_k, t)) \frac{\partial}{\partial z_j^{\alpha}} \right),$$

$$= \theta_{ij}(t) + \theta_{jk}(t),$$

we see that $\{\theta_{ij}\}$ is a 1-cocycle. Consider ad charts compatible with the atlas, we can get the cohomology class in $H^1(X, \mathcal{O}(T^{1,0}X_0)) \simeq H^{0,1}(X, T^{1,0}X_0)$.

We can see $\theta(t)$ as the derivative of the complex structure of X_t with respect to t. The Kodaira-Spencer map can be seen as the differential at $t = 0$ in B.
We put

$$\frac{\partial X_t}{\partial t}|_{t=0} := \rho \left(\frac{\partial}{\partial t}|_{t=0} \right).$$

Definition 2.1.3. *([Kod86, §5.2., p. 228]) Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$ with $t \in B$, where B is an open ball about 0 in some $\mathbb{C}^m$.*

The family $\pi : \mathcal{X} \longrightarrow B$ is said to be **complete** *at $0 \in B$ if for every holomorphic family $\sigma : \mathcal{Y} \longrightarrow D$ of compact complex manifolds $Y_s := \sigma^{-1}(s)$ with $s \in D$, where D is an open ball about 0 in some $\mathbb{C}^l$, such that*

$$0 \in D \qquad \text{and} \qquad \sigma^{-1}(0) = \pi^{-1}(0) \ \left(\text{i.e., } Y_0 = X_0 \right),$$

there exist:

an open subset $\Delta \subset D$ *such that* $0 \in \Delta$

and

a holomorphic map $\Delta \ni s \xrightarrow{h} t = h(s) \in B$

such that $h(0) = 0$ *and* $Y_s = X_{h(s)}$ *for every* $s \in \Delta$.

The following key result by Kodaira and Spencer gives a sufficient condition for completeness of a family at a point.

Theorem 2.1.4 ([KS58], see also [Kod86, Theorem 6.1, p. 284]). *(Theorem of* **completeness***) Let* $\pi : \mathcal{X} \longrightarrow B$ *be a holomorphic family of compact complex manifolds* $X_t := \pi^{-1}(t)$ *with* $t \in B$*, where* B *is an open ball about* 0 *in some* $\mathbb{C}^m$.

If the **Kodaira-Spencer map**

$$\rho : T_0^{1,0}B \longrightarrow H^{0,1}(X_0, T^{1,0}X_0)$$

at 0 *is* **surjective***, the holomorphic family* $\pi : \mathcal{X} \longrightarrow B$ *is* **complete** *at* $0 \in B$.

As a consequence, all the possible directions of deformations of X are in $H^{0,1}(X, T^{1,0}X)$.

2.1.3 Deformation Openness and Closedness

It is a natural question to ask what properties of X_t can be deduced from the properties of X_0, and conversely, what properties of X_0 can be deduced from the properties of X_t. The notion of deformation openness and closedness was introduced in [Pop14].

Definition 2.1.5 ([Pop14]). *(i) A given property* (P) *of a compact complex manifold is said to be open under holomorphic deformations if for every holomorphic family of compact complex manifolds* $(X_t)_{t\in B}$ *and for every* $t_0 \in B$*, the following implication holds:*

X_{t_0} *has property* (P) $\implies$ X_t *has property* (P) *for all* $t \in B$ *sufficiently close to* t_0.

(ii) A given property (P) *of a compact complex manifold is said to be closed under holomorphic deformations if for every holomorphic family of compact complex manifolds* $(X_t)_{t\in B}$ *and for every* $t_0 \in B$*, the following implication holds:*

X_t *has property* (P) *for all* $t \in B \setminus \{t_0\}$ $\implies$ X_{t_0} *has property* (P).

We take the Kähler property for example, we have the following:

Theorem 2.1.6 ([KS60]). *The Kähler property of compact complex manifold is open under holomorphic deformations.*

Theorem 2.1.7 ([Hir62]). *The Kähler property of compact complex manifold of dimension at least 3 is not closed under holomorphic deformations.*

2.1.4 Existence Theorem

In this subsection, let us recall some results in the existence of deformations.

Theorem and Definition 2.1.8 ([Kur62], see also [Kod86, Theorem 6.5., p. 318]). *(Theorem of* **existence***) For every compact complex manifold X, there exist an open ball Δ_ε about 0 in $\mathbb{C}^m$, where $m := dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X)$, an* **analytic subset** *$B \subset \Delta_\varepsilon$ such that $0 \in B$ and a holomorphic family $(X_t)_{t\in B}$ of compact complex manifolds such that:*

(i) the family $(X_t)_{t\in B}$ is **complete** *at every point $t \in B$;*

(ii) $X_0 = X$.

This family $(X_t)_{t\in B}$ of compact complex manifolds is called the **Kuranishi family** *of X.*

In general, the base space B of the Kuranishi family is only an analytic subset of $H^{0,1}(X, T^{1,0}X)$. By the Kuranishi family being unobstructed, we mean that the base space B of the Kuranishi family is smooth. In other words, the unobstructedness of the Kuranishi family means X can be deformed in all the directions in $H^{0,1}(X, T^{1,0}X)$. When the Kuranishi family is unobstructed, its base B can be identified with a small ball about 0 in $H^{0,1}(X, T^{1,0}X)$.

Theorem 2.1.9 ([Kod86, Theorem 5.1.]). *Let X be a compact complex manifold and $\theta \in H^{0,1}(X, T^{1,0}X)$. If there exists a holomorphic family of compact complex manifolds $\pi : \mathcal{X} \to B$ such that $\pi^{-1}(0) = X$ and $\frac{\partial X_t}{\partial t}|_{t=0} = \theta$, it is necessary that $[\theta, \theta] = 0$.*

From this theorem, $[\theta, \theta] \in H^{0,2}(X, T^{1,0}X)$ is an obstruction given that there is no deformation in the direction θ if $[\theta, \theta] \neq 0$.

For a representative $\varphi \in C^\infty_{0,1}(X, T^{1,0}X)$ in the cohomology class θ, $[\varphi, \varphi] = 0$ is not necessary.

φ induces an almost complex structure J'.

J' is integrable if and only if $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$.

In this way, a deformation X_t of X is represented by a form $\varphi(t) \in C^\infty_{0,1}(X, T^{1,0}X)$.

Theorem 2.1.10 ([Kod86, Theorem 5.4.]). *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds, and $\varphi(t)$ be the family of forms given above.*

Then

$$\left.\frac{\partial \varphi(t)}{\partial t}\right|_{t=0}$$

is $\overline{\partial}_0$-closed, and

$$\rho\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\right) = -\left\{\left.\frac{\partial \varphi(t)}{\partial t}\right|_{t=0}\right\}_{\overline{\partial}_0}$$

We have the following existence theorem of Kodaira-Spencer.

Theorem 2.1.11 ([KNS58]). *Let X be a compact complex manifold such that $H^{0,2}(X, T^{1,0}X) = 0$. Then, there exists a holomorphic family $\pi : \mathcal{X} \to B \subset \mathbb{C}^m$ of compact complex manifolds, where $m := \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X)$ and B is a small open ball about the origin in $\mathbb{C}^m$, such that:*

(i) $\pi^{-1}(0) = X$;

(ii) the Kodaira-Spencer map at 0

$$\rho : T_0^{1,0}B \to H^{0,1}(X, T^{1,0}X)$$

is an isomorphism.

This theorem means that the space $H^{0,2}(X, T^{1,0}X)$ contains all the qualitative obstructions. Hence when $H^{0,2}(X, T^{1,0}X)$ vanishes, we can deform X in all directions in $H^{0,1}(X, T^{1,0}X)$. The next theorem discusses what requirements are needed to achieve the unobstructedness of the Kuranishi family without assuming the vanishing of $H^{0,2}(X, T^{1,0}X)$.

Before that, let us clarify the following definition given the diversity of definitions for the Calabi-Yau manifold.

Definition 2.1.12. *A compact complex manifold X is said to be Calabi-Yau if its canonical bundle $K_X := \wedge^{n,0}T^*X$ is trivial.*

With this definition, a compact complex manifold is Calabi-Yau if and only if there is a unique non-vanishing holomorphic n-form u up to a multiplicative constant. The form u is called the Calabi-Yau form.

Theorem 2.1.13 ([Tia87], [Tod89]). *Let X be a compact Kähler Calabi-Yau manifold. then the Kuranishi family of X is unobstructed.*

Later, Popovici relaxed the conditions Kählerness to $\partial\bar{\partial}$-lemma in [Pop19]. With the definition:

Definition 2.1.14. *A compact complex manifold X is said to be a $\partial\bar{\partial}$-manifold if for any d-closed pure-type form v on X, the following exactness properties are equivalent:*

$$v \text{ is } d\text{-exact} \Leftrightarrow v \text{ is } \partial\text{-exact} \Leftrightarrow v \text{ is } \bar{\partial}\text{-exact} \Leftrightarrow v \text{ is } \partial\bar{\partial}\text{-exact},$$

we have the following

Theorem 2.1.15 ([Pop19], [ACRT18]). *Let X be a Calabi-Yau $\partial\bar{\partial}$-manifold. Then the Kuranishi family of X is unobstructed.*

2.2 Cohomology Groups

2.2.1 Definitions

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. We denote the space of $\mathbb{C}$-valued smooth differential forms of bidegree (p, q) by $C^{\infty}_{p,q}(X, \mathbb{C})$, the space of $\mathbb{C}$-valued smooth differential forms of degree k by $C^{\infty}_{k}(X, \mathbb{C})$. Denote by d, ∂, $\bar{\partial}$ the exterior derivative on X, its $(1, 0)$-part and $(0, 1)$-part respectively.

First of all, here are some standard notions of cohomology groups:

Definition 2.2.1. *For $0 \leq p, q \leq n$ and $0 \leq k \leq n$, we have*

1. *De Rham cohomology group of degree k:*

$$H^k_{DR}(X, \mathbb{C}) = \frac{\ker d}{Im\, d};$$

2. *Dolbeault cohomology group of bidegree (p, q):*

$$H^{p,q}_{\bar{\partial}}(X, \mathbb{C}) = \frac{\ker \bar{\partial}}{Im\, \bar{\partial}};$$

3. *Conjugate Dolbeault cohomology group of bidegree (p, q):*

$$H^{p,q}_{\partial}(X, \mathbb{C}) = \frac{\ker \partial}{Im\, \partial};$$

4. *Bott-Chern cohomology group of bidegree (p, q):*

$$H^{p,q}_{BC}(X, \mathbb{C}) = \frac{\ker \partial \cap \ker \bar{\partial}}{Im \partial\bar{\partial}};$$

5. *Aeppli cohomology group of bidegree (p, q):*

$$H^{p,q}_{A}(X, \mathbb{C}) = \frac{\ker(\partial\bar{\partial})}{Im\, \partial + Im\, \bar{\partial}},$$

where the kernals and images are considered here as subspace of $C^{\infty}_{k}(X, \mathbb{C})$ or $C^{\infty}_{p,q}(X, \mathbb{C})$ accordingly.

2.2.2 Hodge Theory

Hodge theory delves into the intricate relationship between differential forms on complex manifolds and their topological and geometric properties. One of the fundamental facts is the Hodge isomorphism, a principle that allows for correspondence between harmonic forms to cohomology classes on a compact Hermitian manifold.

We need the definitions of Laplace-Beltrami operators firstly:

Definition 2.2.2. *On a compact Hermitian manifold* (X, ω) *with* $\dim_{\mathbb{C}} X = n$*, we can define:*

$$\Delta = \Delta_\omega := dd^* + d^*d : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}),$$
$$\Delta' = \Delta'_\omega := \partial\partial^* + \partial^*\partial : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C}),$$
$$\Delta'' = \Delta''_\omega := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C}),$$

for all $0 \leq k \leq 2n$ *and* $0 \leq p, q \leq n$*, where* d^* *(resp.* ∂^**,* $\bar\partial^*$*) is the formal adjoint of* d *(resp.* ∂*,* $\bar\partial$*) with respect to the metric* ω*.*

Now we can state the following theorem:

Theorem 2.2.3. *Let* (X, ω) *be a compact Hermitian manifold:*

(i) The operators $\Delta, \Delta', \Delta''$ *are self-adjoint elliptic operators of order two;*

(ii) The kernels of $\Delta, \Delta', \Delta''$ *are finite dimensional, and their images are closed and finite codimensional;*

(iii) We have the following orthogonal decompositions for $0 \leq k \leq 2n$*,* $0 \leq p, q \leq n$*:*

$$C^\infty_k(X, \mathbb{C}) = \ker \Delta \oplus Imd \oplus Imd^*,$$
$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta' \oplus Im\partial \oplus Im\partial^*,$$
$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta'' \oplus Im\bar\partial \oplus Im\bar\partial^*.$$

Furthermore, we have

$$Im\Delta = Imd \oplus Imd^*, \ \ker d = Imd \oplus \ker \Delta,$$
$$Im\Delta' = Im\partial \oplus Im\partial^*, \ \ker \partial = Im\partial \oplus \ker \Delta',$$
$$Im\Delta'' = Im\bar\partial \oplus Im\bar\partial^*, \ \ker \bar\partial = Im\bar\partial \oplus \ker \Delta''.$$

We adopt the following notations

$$\mathcal{H}^k_\Delta(X, \mathbb{C}) := \ker \Delta|_{C^\infty_k(X,\mathbb{C})},$$
$$\mathcal{H}^{p,q}_{\Delta'}(X, \mathbb{C}) := \ker \Delta'|_{C^\infty_{p,q}(X,\mathbb{C})},$$
$$\mathcal{H}^{p,q}_{\Delta''}(X, \mathbb{C}) := \ker \Delta''|_{C^\infty_{p,q}(X,\mathbb{C})}.$$

We get directly the following from Theorem 2.2.3

Corollary 2.2.4. *On a compact Hermitian manifold* (X, ω) *with* $\dim_{\mathbb{C}} X = n$*, we have Hodge isomorphisms:*

$$\mathcal{H}^k_\Delta(X, \mathbb{C}) \simeq H^k_{DR}(X, \mathbb{C}),$$
$$\mathcal{H}^{p,q}_{\Delta'}(X, \mathbb{C}) \simeq H^{p,q}_{\partial}(X, \mathbb{C}),$$
$$\mathcal{H}^{p,q}_{\Delta''}(X, \mathbb{C}) \simeq H^{p,q}_{\bar\partial}(X, \mathbb{C}),$$

for $0 \leq k \leq 2n$*,* $0 \leq p, q \leq n$*.*

We have similar results in the Bott-Chern and Aeppli cases. Firstly, we need the Laplacians:

Definition 2.2.5 ([KS60], [Sch07]). *We denote the Bott-Chern Laplacian by*

$$\Delta_{BC} := \partial^*\partial + \bar\partial^*\bar\partial + (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial^*\bar\partial)(\partial^*\bar\partial)^* + (\partial^*\bar\partial)^*(\partial^*\bar\partial),$$

and the Aeppli Laplacian by

$$\Delta_A := \partial\partial^* + \bar\partial\bar\partial^* + (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial^*)(\partial\bar\partial^*)^* + (\partial\bar\partial^*)^*(\partial\bar\partial^*).$$

Theorem 2.2.6 ([KS60], [Sch07]). *On a compact Hermitian manifold* (X, ω) *with* $\dim_{\mathbb{C}} X = n$*, we have*

1. *The operators* Δ_{BC}, Δ_A *are self-adjoint elliptic operators of order four.*

2. *The kernels of* Δ_{BC}, Δ_A *are finite dimensional, while their images are closed and finite codimensional.*

3. *We have the following orthogonal decompositions for* $0 \leq p, q \leq n$*:*

$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta_{BC} \oplus Im(\partial\bar\partial) \oplus (Im\partial^* + Im\bar\partial^*),$$

$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta_A \oplus (Im\partial + Im\bar\partial) \oplus Im(\partial\bar\partial)^*.$$

Furthermore, we have:

$$Im\Delta_{BC} = Im(\partial\bar\partial) \oplus (Im\partial^* + Im\bar\partial^*),$$

$$Im\Delta_A = (Im\partial + Im\bar\partial) \oplus Im(\partial\bar\partial)^*,$$

$$\ker\partial \cap \ker\bar\partial = \ker \Delta_{BC} \oplus Im(\partial\bar\partial),$$

$$\ker(\partial\bar\partial) = \ker \Delta_A \oplus (Im\partial + Im\bar\partial).$$

Therefore, we have the isomorphisms

$$H^{p,q}_{BC}(X, \mathbb{C}) \simeq \mathcal{H}^{p,q}_{\Delta_{BC}}(X, \mathbb{C}),$$

$$H^{p,q}_{A}(X, \mathbb{C}) \simeq \mathcal{H}^{p,q}_{\Delta_{A}}(X, \mathbb{C}),$$

where

$$\mathcal{H}^{p,q}_{\Delta_{BC}}(X, \mathbb{C}) := \ker \Delta_{BC}|_{C^\infty_{p,q}(X,\mathbb{C})},$$

$$\mathcal{H}^{p,q}_{\Delta_{A}}(X, \mathbb{C}) := \ker \Delta_{A}|_{C^\infty_{p,q}(X,\mathbb{C})}.$$

2.2.3 $\partial\bar{\partial}$-manifolds

We have mentioned the definition of $\partial\bar{\partial}$-manifold. The most essential fact about $\partial\bar{\partial}$-manifolds is that Hodge theory still holds on $\partial\bar{\partial}$-manifolds,

Theorem 2.2.7 ([Pop], Theorem 1.3.2). *A compact complex n-dimensional manifold X is a $\partial\bar{\partial}$-manifold if and only if the identity induces an isomorphism between $\bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$ and $H^k_{DR}(X,\mathbb{C})$, for every $k \in \{0,\dots,2n\}$, in the following sense:*

(i) for every bidegree (p,q) with $p+q=k$, every Dolbeault cohomology class $[\alpha^{p,q}] \in H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$ contains a d-closed representative $\alpha^{p,q}$;

(ii) the linear map

$$\bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) \ni \sum_{p+q=k} [\alpha^{p,q}] \mapsto \left\{ \sum_{p+q=k} \alpha^{p,q} \right\} \in H^k_{DR}(X,\mathbb{C})$$

is well-defined by means of d-closed representatives (in the sense that it does not depend on the choices of d-closed representatives $\alpha^{p,q}$ of the Dolbeault classes $[\alpha^{p,q}]$) and bijective.

Definition 2.2.8. *If the identity induces an isomorphism*

$$\bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) \cong H^k_{DR}(X,\mathbb{C})$$

in the sense of Theorem 2.2.7 for every $k \in \{0,\dots,2n\}$, we say that the manifold X has the Hodge Decomposition property.

Definition 2.2.9. *Fix $p,q \in \{0,\dots,n\}$. We say that the conjugation induces an isomorphism between $H^{p,q}_{\bar{\partial}}(X)$ and the conjugate of $H^{q,p}_{\bar{\partial}}(X)$ if the following two conditions are satisfied:*

(i) every class $[\alpha^{p,q}] \in H^{p,q}_{\bar{\partial}}(X)$ contains a d-closed representative $\alpha^{p,q}$;

(ii) the linear map

$$H^{p,q}_{\bar{\partial}}(X) \ni [\alpha^{p,q}] \mapsto \overline{[\overline{\alpha^{p,q}}]} \in \overline{H^{q,p}_{\bar{\partial}}(X)}$$

is well-defined and bijective.

Moreover, if the conjugation induces an isomorphism $H^{p,q}_{\bar{\partial}}(X) \cong H^{q,p}_{\bar{\partial}}(X)$ for every $p,q \in \{0,\dots,n\}$, we say that the manifold X has the Hodge Symmetry property.

With these definitions, we find that $\partial\bar{\partial}$-manifolds have the Hodge Decomposition property and the Hodge Symmetry property, as a direct consequence of Theorem 2.2.7.

For the relationship between Betti numbers and Hodge numbers, we have

$$\sum_{p+q=k} h^{p,q} \geq b_k, \quad 0 \leq k \leq 2n,$$

for any compact complex manifold.

By Theorem 2.2.7, we have

$$\sum_{p+q=k} h^{p,q} = b_k$$

for $\partial\bar{\partial}$-manifolds.

Besides, by Definition 2.2.9, we have

$$h^{p,q} = h^{q,p}$$

for $\partial\bar{\partial}$-manifolds.

Now let us see the results for the Bott-Chern and Aeppli cohomology groups. We denote the dimensions of the cohomology groups by

$$h^{p,q}_{BC} := \dim_{\mathbb{C}} H^{p,q}_{BC}(X, \mathbb{C}),$$
$$h^{p,q}_{A} := \dim_{\mathbb{C}} H^{p,q}_{A}(X, \mathbb{C}).$$

Theorem 2.2.10. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent.*

(i) X is a $\partial\bar{\partial}$-manifold;

(ii) For every bidegree (p, q), the canonical linear map $H^{p,q}_{BC}(X, \mathbb{C}) \to H^{p,q}_{A}(X, \mathbb{C})$ is injective;

(iii) For every bidegree (p, q), the canonical linear map $H^{p,q}_{BC}(X, \mathbb{C}) \to H^{p,q}_{A}(X, \mathbb{C})$ is surjective.

What's more, considering the canonical linear maps

$$H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^{p,q}_{\bar{\partial}}(X, \mathbb{C}) \longrightarrow H^{p,q}_{A}(X, \mathbb{C})$$
$$[\omega]_{BC} \longmapsto [\omega]_{\bar{\partial}} \longmapsto [\omega]_{A},$$

we actually have that these maps are isomorphisms, leading to

$$h^{p,q}_{BC} = h^{p,q}_{A} = h^{p,q}.$$

For an arbitrary compact complex manifold, we have

Theorem 2.2.11 ([AT13]). *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} (h^{p,q}_{BC} + h^{p,q}_{A}) \geq 2b_k.$$

Moreover, the equality holds for every $k \in \mathbb{N}$ if and only if X is a $\partial\bar{\partial}$-manifold.

However, for a $\partial\bar{\partial}$-manifold, we have

$$b_k = \sum_{p+q=k} h_{BC}^{p,q} = \sum_{p+q=k} h_A^{p,q} = \sum_{p+q=k} h^{p,q}, \quad 0 \leq k \leq 2n.$$

What's more, we have

$$H_{\mathrm{DR}}^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H_{BC}^{p,q}(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H_A^{p,q}(X,\mathbb{C}).$$

2.2.4 Deformation of cohomology groups

Though the De Rham cohomology remains unchanged under small deformations, the Dolbeault cohomology, the Bott-Chern cohomology, and the Aeppli cohomology vary with the complex structure in general.

Theorem 2.2.12. *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $\dim_{\mathbb{C}} X_t = n$ for all $t \in B$. Fix an arbitrary bidegree (p,q).*

(i) The functions:

$$B \ni t \mapsto h^{p,q}(t) := \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X_t,\mathbb{C}),$$
$$B \ni t \mapsto h_{BC}^{p,q}(t) := \dim_{\mathbb{C}} H_{BC}^{p,q}(X_t,\mathbb{C}),$$
$$B \ni t \mapsto h_A^{p,q}(t) := \dim_{\mathbb{C}} H_A^{p,q}(X_t,\mathbb{C}),$$

are upper-semicontinuous.

(ii) If the Hodge number $h^{p,q}(t)$ is independent of $t \in B$, then the map

$$B \ni t \mapsto H_{\bar{\partial}}^{p,q}(X_t,\mathbb{C})$$

defines a C^∞ vector bundle on B.

The analogous statement holds for $h_{BC}^{p,q}(t)$ and $h_A^{p,q}(t)$.

Proof. This result follows directly from Theorem 2.2.3, Theorem 2.2.6 and Theorems 7.3 and 7.4 in [Kod86]. □

As a result of Theorem 2.2.12, we have the deformation openness of $\partial\bar{\partial}$-manifold.

Theorem 2.2.13 ([AT13]). *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$. If the fibre X_0 is a $\partial\bar{\partial}$-manifold, then, for all $t \in B$ sufficiently close to 0, we have:*

(a) the fibre X_t is a $\partial\bar{\partial}$-manifold;

(b) $h_{BC}^{p,q}(t) = h_{BC}^{p,q}(0)$ and $h_A^{p,q}(t) = h_A^{p,q}(0)$ for every bidegree (p,q).

2.3 Hermitian Metrics

In this section, we will discuss several kinds of Hermitian metrics on compact complex manifolds. We can identify a Hermitian metric with a C^∞ positive definite $(1,1)$-form ω, which is variously called fundamental form or Kähler form.

The following chart shows the relations among several kinds of Hermitian metrics in this section:

$$\begin{array}{ccccc} \omega \text{ is Kähler} & \Longrightarrow & \omega \text{ is Hermitian-symplectic} & \Longrightarrow & \omega \text{ is SKT} \\ \Downarrow & & & & \\ \omega \text{ is balanced} & \Longrightarrow & \omega \text{ is strongly Gauduchon} & \Longrightarrow & \omega \text{ is Gauduchon} \end{array}$$

2.3.1 Gauduchon metrics

Definition 2.3.1 ([Gau77b]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A C^∞ positive definite $(1,1)$-form ω on X is said to be a Gauduchon metric if $\partial\bar{\partial}\omega^{n-1} = 0$.*

Theorem 2.3.2. *Every compact complex manifold admits Gauduchon metrics.*

On non-Kähler manifolds, many wonderful properties of Kähler metrics no longer apply, and the Gauduchon metric provides an alternative tool that can help us understand the geometric and topological properties of these compact complex manifolds.

The theorem below states that not only does a Gauduchon metric exist on every compact complex manifold, but also every Hermitian metric is conformally equivalent to a Gauduchon metric.

Theorem 2.3.3 ([Gau77b]). *Let X be a compact complex manifold. For an arbitrary Hermitian metric ω on X, there exists a unique positive-valued C^∞ function φ on X up to multiplications by positive constants, such that $\varphi\omega$ is a Gauduchon metric.*

Consider a holomorphic family of a compact complex manifolds $\pi : \mathcal{X} \to B$, a Gauduchon metric on a fibre X_0 can deform in a C^∞ way to Gauduchon metrics on X_t for t close to 0.

Proposition 2.3.4. *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds, where $B \subset \mathbb{C}^m$ is an open ball about 0 for some $m \in \mathbb{N}^*$. Put $X_t := \pi^{-1}(t)$ for $t \in B$.*

Let ω_0 be a Gauduchon metric on X_0. After possibly shrinking B about 0, there exists a C^∞ family $(\omega_t)_{t\in B}$ of 2-forms on the fibres $(X_t)_{t\in B}$ whose member for $t = 0$ is ω_0 and such that ω_t is a Gauduchon metric on X_t for every $t \in B$.

Sketch of proof. For a Hermitian metric ω on a compact complex manifold Y, denote by $\star_\omega$ the Hodge star operator and by $P_\omega := i\Lambda_\omega\bar{\partial}\partial$. Then the adjoint operator is given by

$$P^*_\omega : C^\infty(X,\mathbb{C}) \to C^\infty(X,\mathbb{C})$$
$$f \mapsto i \star_\omega \bar\partial\partial(f\frac{\omega^{n-1}}{(n-1)!}),$$

which is an elliptic operator of order 2.

Let $(\gamma_t)_{t\in B}$ be any family of Hermitian metrics varying in a C^∞ way with t on $(X_t)_{t\in B}$ such that $\gamma_0 = \omega_0$. By [Gau77b, Theorem 1, Lemma 1 and Lemma 2], we get a C^∞ function $f > 0$ on X_0, such that $P^*_{\gamma_0}(f) = 0$. By [Gau77b, Theorem 1], we have $\dim\ker P^*_{\gamma_t} = 1$. Then the kernels defines a C^∞ vector bundle on B. Hence, for a sufficiently small neighbourhood of $0 \in B$, we have a C^∞ section $t \mapsto f_t$ such that $f_0 = f$. We have the positivity of f_t by [Gau77b, Lemma 1 and Lemma 2]. Therefore, we have a C^∞ family of Gauduchon metrics $(f_t^{\frac{1}{n-1}}\gamma_t)_t$ for t in a small neighbourhood about 0. If ω_0 is Gauduchon, we can choose $f \equiv 1$ to get the required family of Gauduchon metrics. □

2.3.2 Strongly Gauduchon Metrics

Definition 2.3.5 ([Pop13a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

(i) (i) A C^∞ positive definite $(1,1)$-form ω on X is said to be a strongly Gauduchon (sG) metric if the $(n,n-1)$-form $\partial\omega^{n-1}$ is $\bar\partial$-exact.

(ii) If X carries such a metric, X is said to be a strongly Gauduchon (sG) manifold.

We note that every sG metric is a Gauduchon metric. The converse is true if the $\partial\bar\partial$-lemma holds.

Let us characterize sG manifolds by the following theorem.

Proposition 2.3.6 ([Pop13a]). *1. Let X be a compact complex manifold of complex dimension n. Then, X carries an sG metric if and only if there exists a C^∞ real d-closed $(2n-2)$-form Ω on X with the $(n-1,n-1)$-component positive definite.*

2. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then, X carries a strongly Gauduchon metric ω if and only if there is no non-zero current T of bidegree $(1,1)$ on X such that $T \geq 0$ and T is d-exact on X.

Similar to the case with Gauduchon metrics, considering a holomorphic family of an sG manifold (X,ω), the sG metric ω deforms in a C^∞ way to sG metrics on the nearby fibres. This also means that the sG property of the compact complex manifold is open under holomorphic deformations.

Theorem 2.3.7 ([Pop10]). *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds. Fix a point $0 \in B$ and suppose that the fibre $X_0 := \pi^{-1}(0)$ is a strongly Gauduchon manifold. Then, $X_t := \pi^{-1}(t)$ is a strongly Gauduchon manifold for every $t \in B$ sufficiently close to 0.*

The Kähler property of compact complex manifolds is not stable under modification. We have the following definition.

Definition 2.3.8 ([Fuj78]). *A compact complex manifold X is said to be a* (Fujiki) *class $\mathcal{C}$ manifold if there exists a proper holomorphic bimeromorphic map (also known as a modification)*

$$\mu : \tilde{X} \to X$$

from a compact Kähler manifold $\tilde{X}$.

Unlike the Kähler property, we have the stability under modification of strongly Gauduchon property.

Theorem 2.3.9 ([Pop13b]). *Let $\mu : \tilde{X} \to X$ be a modification of compact complex manifolds. Then, $\tilde{X}$ is a strongly Gauduchon manifold if and only if X is a strongly Gauduchon manifold.*

As an immediate consequence, we have the following

Corollary 2.3.10. *Every class $\mathcal{C}$ manifold is a strongly Gauduchon manifold.*

2.3.3 Balanced Metrics

The balanced metric is the third class of Hermitian metrics we will mention, which was introduced in [Gau77a] under the name of semi-Kähler and was renamed balanced by Michelsohn in [Mic82].

Definition 2.3.11 ([Gau77a],[Mic82]). *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be a balanced metric if $d\omega^{n-1} = 0$.

(ii) If X carries such a metric, X is said to be a balanced manifold.

For the characterization of balanced manifolds, we have the following theorem.

Proposition 2.3.12 ([Mic82]). *Let X be a compact complex manifold. Then, X carries a balanced metric ω if and only if there is no non-zero current T of bidegree $(1,1)$ on X such that $T \geq 0$ and T is the $(1,1)$-component of some d-exact current of degree 2 on X.*

The balanced property is not open under holomorphic deformations of complex structure. A counter-example is given by [Nak72] and [AB90]. The Iwasawa manifold is the compact complex manifold defined as the quotient of the Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} ; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$$

by its discrete subgroup.

$$\Gamma := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} ; z_1, z_2, z_3 \in \mathbb{Z}[i] \right\}$$

The Kuranishi family $(X_t)_{t\in B}$ of $X := G/\Gamma$ is unobstructed, whereas, for any small neighborhood of 0 in B, there is a t in the neighborhood such that X_t is not balanced.

However, we have the deformation openness of the combined $\partial\bar{\partial}$ and balanced properties.

Theorem 2.3.13 ([Wu06]). *Let $(X_t)_{t\in B}$ be a holomorphic family of compact complex manifolds over an open ball B containing the origin in some $\mathbb{C}^N$.*

If the fibre X_0 is a balanced $\partial\bar{\partial}$-manifold, the fibre X_t is again a balanced $\partial\bar{\partial}$-manifold for every $t \in B$ sufficiently close to 0.

Moreover, if X_0 is a $\partial\bar{\partial}$-manifold, any balanced metric ω_0 on X_0 deforms to a family of balanced metrics ω_t on X_t varying in a C^∞ way with t for t in a small enough neighborhood of 0.

The $\partial\bar{\partial}$-condition provides stability of dimensions of cohomology groups under deformation, which is crucial in the proof of Theorem 2.3.13.

For balanced manifolds, we also have the stability under modification.

Theorem 2.3.14 ([AB+95]). *Let $\mu : \tilde{X} \to X$ be a modification of compact complex manifolds. Then, $\tilde{X}$ is a balanced manifold if and only if X is a balanced manifold.*

As a consequence, we still have

Corollary 2.3.15. *Every class C manifold is a balanced manifold.*

2.3.4 SKT and Hermitian-symplectic metrics

Definition 2.3.16. *Let X be a complex manifold.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be an SKT (strong Kähler with torsion) metric if $\partial\bar{\partial}\omega = 0$.

(ii) If X carries a SKT metric, X is said to be a SKT manifold.

(iii) *([ST10]) A C^∞ positive definite $(1,1)$-form ω on X is said to be a Hermitian-symplectic (H-S) metric if ω is the component of bidegree $(1,1)$ of a real C^∞ d-closed 2-form $\tilde{\omega}$ on X.*

(iv) *If X carries a Hermitian-symplectic metric, X is said to be a Hermitian-symplectic manifold.*

We first notice the relation between these two notions.

Proposition 2.3.17. *Let ω be a Hermitian metric on a compact complex manifold X.*

(i) *If ω is Hermitian-symplectic, then ω is SKT.*

(ii) *If X is a $\partial\bar{\partial}$-manifold and ω is SKT, then ω is Hermitian-symplectic.*

Proof. (i) If ω is H-S, then we get $\partial\omega$ is $\bar{\partial}$-exact. Therefore, we have $\partial\bar{\partial}\omega = 0$;

(ii) If X is a $\partial\bar{\partial}$-manifold, and ω is an SKT metric on X, we have $\partial\bar{\partial}\omega = 0$. Hence $\bar{\partial}\omega$ is ∂-closed. Due to the $\partial\bar{\partial}$-condition, there exists a form $\rho^{0,2}$ such that $\bar{\partial}\omega = \partial\rho^{0,2}$. It is obvious $\bar{\partial}\rho^{0,2}$ is ∂-closed and of degree $(0,3)$. By the $\partial\bar{\partial}$-condition, $\bar{\partial}\rho^{0,2} = 0$.

Therefore, $-\overline{\partial\rho^{0,2}} + \omega - \rho^{0,2}$ is d-closed and ω is a H-S metric.

□

Let us characterize these two notions.

Proposition 2.3.18 ([Egi01],[Sul76])**.** *Let X be a complex manifold with* $\dim_{\mathbb{C}} X = n$.

(i) *X is SKT if and only if X carries no non-zero current T of bidegree $(n-1,n-1)$ such that $T \geq 0$ and T is $\partial\bar{\partial}$-exact.*

(ii) *X is Hermitian-symplectic if and only if X carries no non-zero current T of bidegree $(n-1,n-1)$ such that $T \geq 0$ and T is d-exact.*

The Hermitian-symplectic condition is open under holomorphic deformations.

Theorem 2.3.19 ([Yan15])**.** *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds. Fix a point $0 \in B$ and suppose that the fibre $X_0 := \pi^{-1}(0)$ is a Hermitian-symplectic manifold. Then, $X_t := \pi^{-1}(t)$ is a Hermitian-symplectic manifold for every $t \in B$ sufficiently close to 0.*

Proof. Suppose that ω is a Hermitian-symplectic metric on X_0. There exists a real C^∞ d-closed 2-form $\tilde{\omega}$ on X_0, such that ω is the component of bidegree $(1,1)$ of $\tilde{\omega}$ on X_0. Let ω_t be the component of bidegree $(1,1)$ of $\tilde{\omega}$ on X_t. For t close to 0, ω_t is positive definite. Hence ω_t is a Hermitian-symplectic metric on X_t. □

This also implies the openness of combined $\partial\bar{\partial}$ and SKT under holomorphic deformations.

Proposition 2.3.20 ([YZZ23],[DP21]). *Every compact complex manifold X that admits a Hermitian-symplectic metric also admits a strongly Gauduchon (sG) metric.*

It is obvious that a Kähler metric is both SKT and balanced. For the converse, we have the following proposition.

Proposition 2.3.21 ([IP13]). *If a Hermitian metric ω on a compact complex manifold X is both SKT and balanced, then ω is Kähler.*

2.4 Hyperbolicity

2.4.1 Kobayashi hyperbolicity

We denote the unit disc in $\mathbb{C}$ by D and its Poincaré distance by ρ. We denote the disc of radius r in $\mathbb{C}$ by D_r.

Kobayashi introduced the Kobayashi pseudo-distance in [Kob67] to study holomorphic maps.

Definition 2.4.1 ([Kob67]). *Let X be a complex manifold. For two points $p, q \in X$, consider a chain of points $p = p_0, p_1, \ldots, p_k = q$ such that for $l = 0, 1, \ldots, k$, there is a holomorphic map $f_l : D \to X$ with $p_l, p_{l+1} \in Imf_l$. Denote such a chain by γ. Define the length of the chain by*

$$l(\gamma) = \sum_{i=0}^{k} \rho(f_i^{-1}(p_i), f_i^{-1}(p_{i+1}))$$

The Kobayashi pseudo-distance of p, q on X is defined as

$$d_X(p, q) = \inf_{\gamma} l(\gamma)$$

Proposition 2.4.2 ([Kob67]). *(i) Let X and X' be two complex manifolds. Every holomorphic map $f : X \to X'$ is distance decreasing, that is to say,*

$$d_X(p, q) \geq d_{X'}(f(p), f(q)).$$

(ii) For the unit disc D, we have

$$d_D = \rho.$$

(iii) Let X be a complex manifold. If there exists a complex Lie group G that acts transitively on X, then we have

$$d_X \equiv 0.$$

As a consequence of Proposition 2.4.2(3), we know that the Kobayashi pseudo-distance is actually 0 on $\mathbb{C}^n$ and all other complex Lie groups, as well as their quotients by lattices.

With the definition of Kobayashi distance, let us now recall the definition of Kobayashi hyperbolic manifold.

Definition 2.4.3 ([Kob67]). *Let X be a complex manifold. X is said to be Kobayashi hyperbolic if the Kobayashi distance d_X on X is actually a distance.*

From Proposition 2.4.2, we know that the unit disk D is Kobayashi hyperbolic, whereas all complex Lie groups as well as their quotients by lattices are not Kobayashi hyperbolic.

Proposition 2.4.4 ([Kob67]).

(i) The Cartesian product of Kobayashi hyperbolic manifolds is Kobayashi hyperbolic.

(ii) A complex submanifold of a Kobayashi hyperbolic manifold is Kobayashi hyperbolic.

(iii) Every bounded domain in $\mathbb{C}^n$ is Kobayashi hyperbolic.

(iv) Let X be a complex manifold and $\widetilde{X}$ be a covering manifold of X. Then X is Kobayashi hyperbolic if and only if $\widetilde{X}$ is Kobayashi hyperbolic.

The Kobayashi hyperbolicity is open under holomorphic deformations.

Theorem 2.4.5 ([Bro78]). *Let $\pi : X \to B$ be a holomorphic family of compact complex manifolds. If $X_0 = \pi^{-1}(0)$ for $0 \in B$ is Kobayashi hyperbolic, then $X_t = \pi^{-1}(t)$ is again Kobayashi hyperbolic for t sufficiently close to 0.*

2.4.2 Brody hyperbolicity

Definition 2.4.6 ([Bro78]). *A complex manifold X is said to be Brody hyperbolic if all holomorphic maps $f : \mathbb{C} \to X$ are constant.*

Proposition 2.4.7. *Let X be a complex manifold. If X is Kobayashi hyperbolic, then X is Brody hyperbolic.*

Proof. This is an immediate consequence of Proposition 2.4.2 (1) and (3). If there exists a holomorphic map $f : \mathbb{C} \to X$, we have $d_X \leq d_{\mathbb{C}} = 0$. If d_X is a distance, f is a constant map. □

Although the converse is not always true, for compact complex manifolds, we have

Theorem 2.4.8 ([Bro78]). *Let X be a compact complex manifold. If X is Brody hyperbolic, then X is Kobayashi hyperbolic.*

The Poincaré metric ds_R^2 of curvature -1 on the disc D_R of radius R is given by

$$ds_R^2 = \frac{4R^2 dz d\overline{z}}{(R^2 - |z|^2)^2}.$$

The proof of Theorem 2.4.8 is based on the following lemma:

Lemma 2.4.9 ([Bro78]). *Let X be a complex space with a length function F. Given $f \in Hol(D_R, X)$, define a function*

$$u = f^* F^2 / R^2 ds_R^2$$

on D_R. If $u(0) > c > 0$, then there is a map $g \in Hol(D_R, X)$ such that

(a) the function $g^* F^2 / R^2 ds_R^2$ is bounded by c on D_R and attains the maximum value c at the origin;

(b) $g = f \circ \mu_r \circ \varphi$, where φ is a holomorphic automorphism of D_R and μ_r is the multiplication by suitable r, $0 < r < 1$, (i.e., $\mu_r(z) = rz$ for $z \in D_R$).

For the Kobayashi pseudo-distance d_X, there is a corresponding infinitesimal form F_X, which is an intrinsic pseudo-metric. For $v \in T_x X$, $F_X(v)$ is defined as

$$F_X(v) := \inf_R \{ \frac{1}{R} | \exists f : D_R \to X \text{ holomorphic, } f(0) = x, (df)_0 (\frac{d}{dz} \bigg|_{z=0}) = v \}.$$

Sketch of proof of Theorem 2.4.8. Fix an arbitrary Hermitian metric on X. If X is not Kobayashi hyperbolic, there exists a sequence of vectors $v_n \in T_X$ such that $||v_n|| = 1$ and $\lim_{n\to\infty} F_X(v_n) = 0$. Assuming $F_X(v_n) < \frac{1}{n}$, by the definition of F_X, we have a sequence of holomorphic maps $f_n \in \text{Hol}(D_n, X)$. By applying Lemma 2.4.9 to $\{f_n\}$, we have an equicontinuous family of maps $\{g_n\} \subset \text{Hol}(D_r, X)$.

Using Arzelà-Ascoli's Lemma, we can extract a subsequence converging to $h_1 \in \text{Hol}(D_1, X)$. From this subsequence, we can extract again a subsequence converging to $h_2 \in \text{Hol}(D_2, X)$. Repeat this procedure, we have $h_k \in \text{Hol}(D_k, X)$ and $h_k|_{D_{k'}} = h_{k'}$ for $k \geq k'$. Hence we can get a map $h \in \text{Hol}(\mathbb{C}, X)$. □

2.4.3 Kähler hyperbolicity

Recall that a form α on a Hermitian manifold (X, ω) is said to be $\tilde{d}$(bounded) if the lift $\tilde{\alpha}$ of α to the universal cover $\tilde{X}$ od X is d-exact with a d-potential bounded with respect to the lift $\tilde{\omega}$ of ω.

Gromov proposed the notion of Kähler hyperbolic manifold in [Gro91]

Definition 2.4.10 ([Gro91]). *A compact complex manifold X is said to be Kähler hyperbolic if X admits a Kähler metric ω which is $\tilde{d}$(bounded).*

As for the relation between Kähler hyperbolicity and Kobayashi hyperbolicity, by [Gro91] we have :

(real hyperbolicity + Kähler) ⇒ Kähler hyperbolicity ⇒ Kobayashi hyperbolicity

where real hyperbolicity means $\pi_1(X)$ is hyperbolic and $\pi_2(X) = 0$.

Let us see some properties of Kähler hyperbolic manifolds.

Proposition 2.4.11. *(i) The Cartesian product of Kähler hyperbolic manifolds is Kähler hyperbolic.*

(ii) [CY18] The canonical bundle of a compact Kähler hyperbolic manifold is ample, whereas the cotangent bundle is not necessarily ample.

2.4.4 Balanced hyperbolicity

To generalize Kähler hyperbolicity, Marouani and Popovici proposed the notion of balanced hyperbolicity.

Definition 2.4.12 ([MP22a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$. X is said to be balanced hyperbolic if there is a balanced metric ω on X such that ω^{n-1} is $\tilde{d}$(bounded) with respect to ω.*

Proposition 2.4.13 ([MP22a]). *Let X be a compact complex manifold.*

(i) If X is Kähler hyperbolic, then X is balanced hyperbolic.

(ii) The Cartesian product of balanced hyperbolic manifolds is balanced hyperbolic.

As for the examples of balanced hyperbolic manifolds, quotients $X = G/\Gamma$ of a semi-simple complex Lie group by a lattice Γ are balanced hyperbolic. We first notice that there is no Kähler metric on this class of manifolds by [LM88]. We also know that such manifolds are not Kobayashi/Brody hyperbolic from Proposition 2.4.2(3), which means there is some non-constant holomorphic map $f : \mathbb{C} \to X$.

Instead of Brody hyperbolicity, Marouani and Popovici proposed divisorial hyperbolicity, which is implied by balanced hyperbolicity.

We denote the open ball (resp. sphere) of radius r centered at 0 in $\mathbb{C}^p$ by B_r (resp. S_r). Let $\star_\omega$ denote the Hodge star operator induced by a Hermitian metric ω, and let $\tau(z) := |z|^2$ be the squared Euclidean norm on $\mathbb{C}^p$.

Definition 2.4.14 ([MP22a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$. For $0 < p \leq n-1$, we say that a holomorphic map $f : \mathbb{C}^p \to X$ has* **subexponential growth** *if the following two conditions are satisfied:*

1. *There exist constants $C_1 > 0$ and $r_0 > 0$ such that*

$$\int_{S_t} |d\tau|_{f^\star\omega} d\sigma_{\omega,f,t} \leq C_1 t \operatorname{Vol}_{\omega,f}(B_t), \quad t > r_0,$$

where $d\sigma_{\omega,f,t} = \left(\star_{f^*\omega}\left(\frac{d\tau}{|d\tau|_{f^*\omega}}\right)\right)|_{S_t}$.

2. *For every constant $C > 0$, we have:*

$$\overline{\lim_{b\to+\infty}}\left(\frac{b}{C} - \log F(b)\right) = +\infty,$$

where $F(b) := \int_0^b \operatorname{Vol}_{\omega,f}(B_t)\, dt = \int_0^b \left(\int_{B_t} f^*\omega_{n-1}\right) dt, \quad b > 0.$

Definition 2.4.15 ([MP22a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$. X is said to be divisorially hyperbolic if there is no holomorphic map $f : \mathbb{C}^{n-1} \to X$ such that f is non-degenerate at some point and has subexponential growth.*

Theorem 2.4.16 ([MP22a]). *Every balanced hyperbolic compact complex manifold is divisorially hyperbolic.*

To summarize, here is a chart of relations among the various hyperbolicities on a compact complex manifold.

$$
\begin{array}{ccc}
X \text{ is Kähler hyperbolic} & \Longrightarrow & X \text{ is Kobayashi/Brody hyperbolic} \\
\Downarrow & & \Downarrow \\
X \text{ is balanced hyperbolic} & \Longrightarrow & X \text{ is divisorially hyperbolic}
\end{array}
$$

Chapter 3

Polarisation of SKT Calabi-Yau $\partial\bar{\partial}$-manifolds by Aeppli classes

3.1 Introduction

Let X be a compact complex manifold of dimension n. Recall that a manifold is said to be a *$\partial\bar{\partial}$-manifold* if for any pure-type d-closed form u, we have the following equivalences:

$$u \text{ is } d\text{-exact} \Leftrightarrow u \text{ is } \partial\text{-exact} \Leftrightarrow u \text{ is } \bar{\partial}\text{-exact} \Leftrightarrow u \text{ is } \partial\bar{\partial}\text{-exact}.$$

A complex manifold is called *Calabi-Yau* if its canonical bundle K_X is trivial.

A Hermitian metric ω, seen as a positive definite C^∞ $(1,1)$-form, on a complex manifold X is called strong Kähler with torsion (SKT for short) if $\partial\bar{\partial}\omega = 0$ and it is called Hermitian-symplectic [ST10] if ω is the component of bidegree $(1,1)$ of a real smooth d-closed 2-form on X. Obviously, on a $\partial\bar{\partial}$-manifold, a metric is SKT if and only if it is Hermitian-symplectic. The study of SKT metrics (also called pluriclosed, see [ST10]) has received a lot of attention over recent years. A necessary condition for the existence of a smooth family of SKT metrics on a differentiable family of complex manifolds is given in [PS21]. The existence of a left-invariant SKT structure on any even-dimensional compact Lie group G is obtained in [MS11].

The $\partial\bar{\partial}$-property is open under holomorphic deformations of the complex structure by [Wu06] and [AT13]. Namely, if $\pi : \mathcal{X} \to B$ is a proper holomorphic submersion between complex manifolds, and $X_0 := \pi^{-1}(0)$ is a $\partial\bar{\partial}$-manifold, then $X_t := \pi^{-1}(t)$ is a $\partial\bar{\partial}$-manifold for all t in a small neighbourhood of 0. Moreover, the Hodge numbers are independent of t in this case. If X_0 is a Calabi-Yau manifold and $h^{n,0}$ does not jump, which means that $h^{n,0}(t) = h^{n,0}(0)$ for t in a small neighbourhood of 0, then X_t is again a Calabi-Yau manifold for t close to 0. Though the SKT condition is not deformation open, if X_0 is an SKT $\partial\bar{\partial}$-manifold, then X_t is again an SKT $\partial\bar{\partial}$-manifold for t in a small neighbourhood of 0. Indeed, the Hermitian-symplectic condition is deformation open by [Yan15] (see also [Bel20]). Putting these things together, we get that if X_0 is an SKT Calabi-Yau $\partial\bar\partial$-manifold, then for all t close to 0, X_t is again an SKT Calabi-Yau $\partial\bar\partial$-manifold.

Now, fix an SKT metric ω on a Calabi-Yau SKT $\partial\bar\partial$-manifold X. Let $(X_t)_{t\in B}$ be the Kuranishi family of X. By [Bog78], [Tia87], [Tod89] (see also [Pop19]), the Kuranishi family is unobstructed. In particular, B can be seen as a ball about 0 in $H^{0,1}(X_0, T^{1,0}X_0)$, where $T^{1,0}X_0$ is the holomorphic tangent bundle of X_0. We can define the notion of X_t being polarised by the SKT Aeppli class $[\omega]_A$ by requiring the canonical image $\{\omega\}_{DR}$ of $[\omega]_A$ in $H^2_{DR}(X, \mathbb{C})$, where X is the C^∞ manifold underlying the fibres X_t, to be of type $(1,1)$ for the complex structure of X_t (see Definition 3.3.1). Somehow we can view the set $B_{[\omega]}$ of all the fibres X_t polarised by $[\omega]_A$ as the intersection of B and $H^{0,1}(X, T^{1,0}X)_{[\omega]}$, the latter being defined in Lemma 3.3.4 as a vector subspace of $H^{0,1}(X_0, T^{1,0}X_0)$.

In section 3.4, we define the primitivity of certain Bott-Chern classes w.r.t. $[\omega]_A$. Then we can identify the space of $H^{0,1}(X_0, T^{1,0}X_0)_{[\omega]}$ with the space of Bott-Chern primitive classes.

In section 3.6, we compare the Weil-Petersson metric and the metric induced by the period map on the base space $B_{[\omega]}$ of the family of $[\omega]$-polarised small deformations of X. In the case of Kähler polarised deformations, these two metrics coincide with each other by [Tia87, Theorem 2].

This work was inspired by [Pop19] where the notion of small deformations co-polarised by a balanced class was introduced and studied. Besides, deformations co-polarised by a Gauduchon class was studied in [Bel23].

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3.2 Preliminaries

Recall that a compact complex manifold X is a $\partial\bar\partial$-manifold if and only if there are canonical isomorphisms between the Bott-Chern, Aeppli and Dolbeaut cohomology groups, i.e., the canonical maps

$$\begin{array}{ccccc} H^{p,q}_{BC}(X,\mathbb{C}) = \dfrac{\ker\partial\cap\ker\bar\partial}{\mathrm{Im}\partial\bar\partial} & \longrightarrow H^{p,q}_{\bar\partial}(X,\mathbb{C}) = \dfrac{\ker\bar\partial}{\mathrm{Im}\bar\partial} & \longrightarrow H^{p,q}_{A}(X,\mathbb{C}) = \dfrac{\ker\partial\bar\partial}{\mathrm{Im}\partial+\mathrm{Im}\bar\partial} \\ [\alpha]_{BC} & \longmapsto \quad [\alpha]_{\bar\partial} & \longmapsto \quad [\alpha]_A \end{array}$$

are isomorphisms for all $0 \leq p, q \leq n$. For $\partial\bar\partial$-manifolds, we have a Hodge decomposition in the sense that the canonical map with

$$\bigoplus_{p+q=k} H^{p,q}_A(X,\mathbb{C}) \cong H^k_{DR}(X,\mathbb{C})$$
$$([\alpha^{p,q}]_A)_{p+q=k} \mapsto \{\sum_{p+q=k} \alpha^{p,q}\}$$

is an isomorphism for all $0 \leq k \leq 2n$, where all the forms $\alpha^{p,q}$ are d-closed. The $\partial\bar\partial$-assumption on X guarantees that every Aeppli cohomology class can be represented by a d-closed form. We could use any of the Bott-Chern, Aeppli or Dolbeaut cohomologies here. As in [KS60] and [Sch07], Bott-Chern and Aeppli Laplacians $\Delta_{BC}, \Delta_A : C^\infty_{p,q}(X,\mathbb{C}) \to C^\infty_{p,q}(X,\mathbb{C})$ are defined as:

$$\Delta_{BC} = \partial^*\partial + \bar\partial^*\bar\partial + (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial^*\bar\partial)(\partial^*\bar\partial)^* + (\partial^*\bar\partial)^*(\partial^*\bar\partial),$$
$$\Delta_A = \partial\partial^* + \bar\partial\bar\partial^* + (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial^*)(\partial\bar\partial^*)^* + (\partial\bar\partial^*)^*(\partial\bar\partial^*),$$

and then we have

$$\ker\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker(\partial\bar\partial)^*, \tag{3.1}$$
$$\ker\Delta_A = \ker(\partial\bar\partial) \cap \ker\partial^* \cap \ker\bar\partial^*. \tag{3.2}$$

Let ω be an SKT metric on a $\partial\bar\partial$-manifold X. By the $\partial\bar\partial$-property, there exists a form $\alpha \in C^\infty_{0,1}(X,\mathbb{C})$, such that

$$\bar\partial\omega = \partial\bar\partial\alpha. \tag{3.3}$$

Note that $d(\omega+\partial\alpha+\bar\partial\bar\alpha) = 0$, so $\omega+\partial\alpha+\bar\partial\bar\alpha$ is a d-closed representative of $[\omega]_A$. We define $\{\omega\}_{DR}$ (resp. $[\omega]_{\bar\partial}$) to be the image of $[\omega]_A$ under the canonical injection $H^{1,1}_A(X,\mathbb{C}) \hookrightarrow H^2_{DR}(X,\mathbb{C})$ (resp. the isomorphism $H^{1,1}_A(X,\mathbb{C}) \xrightarrow{\cong} H^{1,1}_{\bar\partial}(X,\mathbb{C})$), which means that $\{\omega\}_{DR} = \{\omega+\partial\alpha+\bar\partial\bar\alpha\}_{DR}$ (resp. $[\omega]_{\bar\partial} = [\omega+\partial\alpha]_{\bar\partial}$).

Given a Calabi-Yau manifold X, we fix a non-vanishing holomorphic n-form u on X. Note that u exists and is unique up to a multiplicative constant since K_X is trivial. It defines the *Calabi-Yau isomorphism*:

$$\begin{aligned} T_{[u]} : H^{0,1}(X, T^{1,0}X) &\longrightarrow H^{n-1,1}_{\bar\partial}(X,\mathbb{C}) \\ [\theta] &\longmapsto [\theta \lrcorner u], \end{aligned}$$

where the operator $\cdot\lrcorner\cdot$ combines the contraction of u by the vector field component of θ with the multiplication by the $(0,1)$-form component.

In local coordinates, we can write as the following

$$\theta = \sum_{\substack{|J|=q \\ 1\le j\le n}} \theta^j_{\bar J} d\bar z_J \otimes \frac{\partial}{\partial z_j} \in C^\infty_{0,q}(U, T^{1,0}X),$$
$$u = f\, dz_1 \wedge \cdots \wedge dz_n.$$

Therefore, we can write $T_u : C^\infty_{0,q}(X, T^{1,0}X) \xrightarrow{\cdot\lrcorner u} C^\infty_{n-1,q}(X,\mathbb{C})$ in local coordinates:

$$T_u(\theta) = \theta\lrcorner u = \sum_{\substack{|J|=q \\ 1\le j\le n}} (-1)^{j-1} f \theta^j_{\bar J} d\bar z_J \wedge dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n.$$

We can check directly that $T_u(\ker\bar\partial) = \ker\bar\partial$ and $T_u(\operatorname{Im}\bar\partial) = \operatorname{Im}\bar\partial$. What's more, we have that f does not vanish. Hence, $T_{[u]}$ is indeed an isomorphism.

For a primitive form v of bidegree (p,q), we will often use the following formula (see [Voi02, Proposition 6.29])

$$\star v = (-1)^{\frac{(p+q)(p+q+1)}{2}} i^{p-q} \frac{\omega^{n-p-q} \wedge v}{(n-p-q)!} \tag{3.4}$$

for the Hodge star operator $\star$.

3.3 Polarisation by SKT classes

Let X be a compact SKT Calabi-Yau $\partial\bar{\partial}$-manifold and let ω be an SKT metric on it. Let $\pi : \mathcal{X} \to B$ be the Kuranishi family of X. In a small neighbourhood of 0, $X_t := \pi^{-1}(t)$ is again a Calabi-Yau SKT $\partial\bar{\partial}$-manifold, and we have the following Hodge decomposition by [Sch07]

$$H^2_{DR}(X, \mathbb{C}) \simeq H^{2,0}_A(X_t, \mathbb{C}) \oplus H^{1,1}_A(X_t, \mathbb{C}) \oplus H^{0,2}_A(X_t, \mathbb{C}), \quad t \sim 0, \tag{3.5}$$

where "$\cong$" stands for the canonical isomorphism whose inverse is defined by $([\alpha^{2,0}]_A, [\alpha^{1,1}]_A, [\alpha^{0,2}]_A) \mapsto \{\alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}\}_{DR}$, where $d\alpha^{p,2-p} = 0$ for $p = 0, 1, 2$.

Definition 3.3.1. *Fix the Aeppli class $[\omega]_A \in H^{1,1}_A(X, \mathbb{C})$ of an SKT metric ω on $X_0 = X$. For $t \in B$, we say that X_t is polarised by $[\omega]_A$ if the projection $[\omega]^{0,2}_{A,t}$ of $\{\omega\}_{DR}$ onto $H^{0,2}_A(X_t, \mathbb{C})$ w.r.t (3.5) is 0.*

Denote by $B_{[\omega]}$ the set of $t \in B$ such that X_t is polarised by $[\omega]_A$, namely

$$B_{[\omega]} = \{t \in B | [\omega]^{0,2}_{A,t} = 0 \in H^{0,2}_A(X_t, \mathbb{C})\}.$$

Thus, X_t being polarised by $[\omega]_A$ means that $\{\omega\}_{DR}$ is of J_t-pure-type $(1,1)$ for $t \in B_{[\omega]}$ since $[\omega]^{2,0}_{A,t} = 0$ if and only if $[\omega]^{0,2}_{A,t} = 0$. Indeed, $\{\omega\}_{DR}$ being real, $[\omega]^{2,0}_{A,t}$ is the conjugate of $[\omega]^{0,2}_{A,t}$.

To situate this definition in its context, we remind the reader of the following classical facts. For every bidegree (p, q), one considers the coherent analytic sheaf $\mathcal{H}^{p,q} := R^q\pi_\star\Omega^p_{\mathcal{X}/B}$ on B. Since, in our case, the Hodge numbers $h^{p,q}_{\bar{\partial}}(t) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X_t, \mathbb{C})$ are independent of t for t close enough to 0, the sheaf $\mathcal{H}^{p,q}$ is even locally free on an open neighbourhood of 0 in B (that we can assume to be B after possibly shrinking B). Thus, $\mathcal{H}^{p,q}$ defines a holomorphic vector bundle over B whose fibres are $B \ni t \mapsto H^{p,q}_{\bar{\partial}}(X_t, \mathbb{C})$. Moreover, thanks to the fibres X_t being $\partial\bar{\partial}$-manifolds, each vector space $H^{p,q}_{\bar{\partial}}(X_t, \mathbb{C})$ is canonically isomorphic to its Aeppli cohomology counterpart $H^{p,q}_A(X_t, \mathbb{C})$.

In our case, we conclude that the set $B_{[\omega]}$ is the (possibly singular) subvariety of B arising as the vanishing locus of the holomorphic section s of the holomorphic vector bundle $\mathcal{H}^{0,2} \longrightarrow B$ defined by

$$s(t) := [\omega]^{0,2}_{A,t} \in \mathcal{H}^{0,2}_t \simeq H^{0,2}_A(X_t, \mathbb{C}), \quad t \in B. \tag{3.6}$$

Theorem 3.3.2. *Let ω be an SKT metric on a compact $\partial\bar{\partial}$-manifold X and let $\pi : \mathcal{X} \to B$ be its Kuranishi family. Consider $\gamma^{1,1}_t \in H^{1,1}_A(X_t, \mathbb{C})$ the Aeppli component of J_t-type $(1,1)$ of $\{\omega\}_{DR}$ w.r.t. (3.5). Then there exists an SKT metric $\omega_t \in \gamma^{1,1}_t$ for all t in a small neighbourhood of 0.*

Proof. By the definition (3.3) of α, we construct two d-closed forms:

$$\tilde{\omega} = \omega + \partial\alpha + \bar{\partial}\bar{\alpha},$$
$$\hat{\omega} = -\partial\bar{\alpha} + \omega - \bar{\partial}\alpha.$$

We know that $\tilde{\omega}$ is of J_0-bidegree $(1,1)$ and ω is the J_0-$(1,1)$-component of the 2-form $\hat{\omega}$. Since $\tilde{\omega} = \hat{\omega} + d(\alpha + \bar{\alpha})$, we have that $\{\tilde{\omega}\}_{DR} = \{\hat{\omega}\}_{DR}$

Decompose $\hat{\omega}$ into components of pure J_t-type:

$$\hat{\omega} = \Omega_t^{2,0} + \omega_t^{1,1} + \Omega_t^{0,2}.$$

We know that $\omega_0^{1,1} = \omega$. Moreover, $\omega_t^{1,1}$ is real and $\partial_t\bar{\partial}_t$-closed because $\hat{\omega}$ is real and d-closed. By the continuity of $(\omega_t^{1,1})_t$ with respect to t and $\omega_0^{1,1} = \omega > 0$, we also have $\omega_t^{1,1} > 0$ for t near 0. This ensures that $\omega_t^{1,1}$ is an SKT metric on X_t. Moreover, we have $\omega_t^{1,1} \in \gamma_t^{1,1}$, $\forall t \sim 0$. We put $\omega_t := \omega_t^{1,1}$, $\forall t \sim 0$ and we are done. □

Lemma 3.3.3. *Let X be a compact complex manifold and $\theta \in C^\infty_{0,q}(X, T^{1,0}X)$, $\beta \in C^\infty_{1,q'}(X, \mathbb{C})$. We have*

$$\bar{\partial}(\theta \lrcorner \beta) = \bar{\partial}\theta \lrcorner \beta + (-1)^q \theta \lrcorner \bar{\partial}\beta.$$

Proof. In local coordinates, we can write as the following:

$$\theta = \sum_{\substack{1\leq j\leq n\\ |J|=q}} \theta^j_{\bar{J}}\, d\bar{z}_J \otimes \frac{\partial}{\partial z_j},$$

$$\beta = \sum_{\substack{|I|=q'\\ 1\leq j\leq n}} \beta_{j\bar{I}}\, dz_j \wedge d\bar{z}_I.$$

Therefore, we have the following in local coordinates:

$$\bar{\partial}(\theta \lrcorner \beta) = \sum_{\substack{|J|=q\\ |I|=q'\\ 1\leq j\leq n}} \frac{\partial(\theta^j_{\bar{J}}\beta_{j\bar{I}})}{\partial \bar{z}_k}\, d\bar{z}_k \wedge d\bar{z}_J \wedge d\bar{z}_I,$$

$$\bar{\partial}\theta \lrcorner \beta = \sum_{\substack{|J|=q\\ |I|=q'\\ 1\leq j\leq n}} \frac{\partial \theta^j_{\bar{J}}}{\partial \bar{z}_k}\, \beta_{j\bar{I}}\, d\bar{z}_k \wedge d\bar{z}_J \wedge d\bar{z}_I,$$

$$\theta \lrcorner \bar{\partial}\beta = \sum_{\substack{|J|=q\\ |I|=q'\\ 1\leq j\leq n}} \theta^j_{\bar{J}}\, \frac{\partial \beta_{j\bar{I}}}{\partial \bar{z}_k}\, d\bar{z}_J \wedge d\bar{z}_k \wedge d\bar{z}_I.$$

As a consequence, we get

$$\bar{\partial}(\theta \lrcorner \beta) = \bar{\partial}\theta \lrcorner \beta + (-1)^q \theta \lrcorner \bar{\partial}\beta.$$

□

Recall that $H^{0,1}(X_0, T^{1,0}X_0) := \dfrac{\ker \bar{\partial} : C^\infty_{0,1}(X_0, T^{1,0}X_0) \to C^\infty_{0,2}(X_0, T^{1,0}X_0)}{\mathrm{Im}\bar{\partial} : C^\infty_{0,0}(X_0, T^{1,0}X_0) \to C^\infty_{0,1}(X_0, T^{1,0}X_0)}$, where $\bar{\partial}$ is the holomorphic structure of $T^{1,0}X_0$. Because the Kodaira-Spencer map gives an isomorphism between T_0B and $H^{0,1}(X_0, T^{1,0}X_0)$. Then we have the following lemma:

Lemma 3.3.4. *Consider the following subspace of* $H^{0,1}(X, T^{1,0}X)$*:*

$$\begin{aligned} H^{0,1}(X, T^{1,0}X)_{[\omega]} :&= \{[\theta] \in H^{0,1}(X, T^{1,0}X) | [\theta \lrcorner \zeta]_A = 0 \in H^{0,2}_A(X, \mathbb{C})\} \\ &= \{[\theta] \in H^{0,1}(X, T^{1,0}X) | [\theta \lrcorner \zeta]_{\bar\partial} = 0 \in H^{0,2}_{\bar\partial}(X, \mathbb{C})\}, \end{aligned}$$

where ζ *is an arbitrary representative in* $[\omega]_{\bar\partial}$ *(the image of* $[\omega]_A$ *under the canonical isomorphism* $H^{1,1}_A(X, \mathbb{C}) \overset{\cong}{\to} H^{1,1}_{\bar\partial}(X, \mathbb{C})$*).*

This subspace is well defined and it is, when $B_{[\omega]}$ *is smooth at* 0*, the tangent space at* 0 *to* $B_{[\omega]}$*:*

$$T^{1,0}_0 B_{[\omega]} = H^{0,1}(X, T^{1,0}X)_{[\omega]}.$$

Proof. By Lemma 3.3.3, we have that for θ a representative in $[\theta]$, $\bar\partial(\theta \lrcorner \zeta) = 0$ and

$$\begin{aligned} \theta \lrcorner (\zeta + \bar\partial \zeta') &= \theta \lrcorner \zeta + \bar\partial(\theta \lrcorner \zeta'), \\ (\theta + \bar\partial \theta') \lrcorner \zeta &= \theta \lrcorner \zeta + \bar\partial(\theta' \lrcorner \zeta) \end{aligned}$$

for $\theta' \in C^\infty(X, T^{1,0}X), \zeta' \in C^\infty_{1,0}(X, \mathbb{C})$. Hence the classes $[\theta \lrcorner \zeta]_{\bar\partial}$ and $[\theta \lrcorner \zeta]_A$ are independent of the choices of representatives of $[\theta]$ and $[\zeta]_{\bar\partial}$. Therefore, the space $H^{0,1}(X, T^{1,0}X)_{[\omega]}$ is well defined. By the $\partial\bar\partial$-property, the classes $[\theta \lrcorner \zeta]_A$ and $[\theta \lrcorner \zeta]_{\bar\partial}$ correspond to each other under the isomorphism $H^{0,2}_A(X, \mathbb{C}) \overset{\cong}{\longrightarrow} H^{0,2}_{\bar\partial}(X, \mathbb{C})$, so we also have

$$\begin{aligned} &\{[\theta] \in H^{0,1}(X, T^{1,0}X) | [\theta \lrcorner \zeta]_A = 0 \in H^{0,2}_A(X, \mathbb{C})\} \\ =&\{[\theta] \in H^{0,1}(X, T^{1,0}X) | [\theta \lrcorner \zeta]_{\bar\partial} = 0 \in H^{0,2}_{\bar\partial}(X, \mathbb{C})\} \end{aligned}$$

For t near 0, X_t is a $\partial\bar\partial$-manifold, so we have the Hodge decompositions:

$$\begin{aligned} H^2_{DR}(X, \mathbb{C}) =& H^{2,0}_{\bar\partial}(X_t, \mathbb{C}) \oplus H^{1,1}_{\bar\partial}(X_t, \mathbb{C}) \oplus H^{0,2}_{\bar\partial}(X_t, \mathbb{C}) \\ \simeq& H^{2,0}_A(X_t, \mathbb{C}) \oplus H^{1,1}_A(X_t, \mathbb{C}) \oplus H^{0,2}_A(X_t, \mathbb{C}). \end{aligned}$$

Now, it is standard (see e.g. the proof of the Griffiths transversality property) that the differential map at $0 \in B$ of the section s defined in (3.6) under the Gauss-Manin connection ∇ of the vector bundle $\mathcal{H}^{0,\,2} \longrightarrow B$ is the linear map

$$H^{0,1}(X, T^{1,0}X) \ni [\theta] \longmapsto [\theta \lrcorner \zeta]_A \in H^{0,2}_A(X, \mathbb{C}). \tag{3.7}$$

In other words, if we take a tangent vector $\frac{\partial}{\partial t_i}|_{t=0}$ in $T^{1,0}_0 B_{[\omega]}$ and denote by $[\theta]$ its image under the Kodaira-Spencer map $\rho : T^{1,0}_0 B \to H^{0,1}(X_0, T^{1,0}X_0)$, we have

$$\nabla_{\frac{\partial}{\partial t_i}|_{t=0}} [\omega]^{0,2}_{A,t} = [\theta \lrcorner \zeta]_A,$$

where ∇ is the connection induced on $\mathcal{H}^{0,\,2} \longrightarrow B$, via the Hodge decomposition, by the Gauss-Manin connection of the constant bundle $\mathcal{H}^2$ over B of fibre the de Rham cohomology space $H^2_{dR}(X,\,\mathbb{C})$ (see [Voi02, Definition 9.13 and Proposition 9.14]).

Therefore, by the definition of $B_{[\omega]}$ as the vanishing locus of the section s defined in (3.6), if $B_{[\omega]}$ is smooth at 0 (in other words, if the linear map (3.7) is of maximal rank), we have

$$\begin{aligned} T_0^{1,0}B_{[\omega]} &= \{[\theta] \in H^{0,1}(X, T^{1,0}X) \mid [\theta \lrcorner \zeta]_A = 0 \in H_A^{0,2}(X, \mathbb{C})\} \\ &= H^{0,1}(X, T^{1,0}X)_{[\omega]}. \end{aligned}$$

□

Remark 3.3.5. *If ω is moreover Kähler, α (defined in (3.3)) can be taken as* 0*. Therefore all the results here coincide with the case of Kähler polarised deformation.*

3.4 Primitive classes

Lemma and Definition 3.4.1. *Let X be a compact complex manifold of dimension n, and ω be an SKT metric on X. Then the map*

$$\begin{aligned} L_{[\omega]} : H_{BC}^{p,q}(X, \mathbb{C}) &\longrightarrow H_A^{p+1,q+1}(X, \mathbb{C}) \\ [\gamma]_{BC} &\longmapsto [\omega \wedge \gamma]_A \end{aligned}$$

is well-defined and only depends on the Aeppli class of ω.

We say that a Bott-Chern class $[\gamma]_{BC}$ of bidegree $(p, n-p)$ is primitive (or $[\omega]_A$-primitive) if $L_{[\omega]}([\gamma]_{BC}) = 0$. We denote the space of primitive Bott-Chern classes of bidegree $(n-1, 1)$ by $H_{BC,prim}^{n-1,1}(X, \mathbb{C})$.

Proof. Since $\partial\gamma = \bar\partial\gamma = 0$, we get

$$\partial\bar\partial(\omega \wedge \gamma) = \partial\bar\partial\omega \wedge \gamma = 0,$$

so $\omega \wedge \gamma$ represents an Aeppli class. By

$$\begin{aligned} \omega \wedge (\gamma + \partial\bar\partial\beta) =&\omega \wedge \gamma + \partial(\omega \wedge \bar\partial\beta) + \bar\partial(\partial\omega \wedge \beta) + \partial\bar\partial\omega \wedge \beta \\ =&\omega \wedge \gamma + \partial(\omega \wedge \bar\partial\beta) + \bar\partial(\partial\omega \wedge \beta), \end{aligned}$$

we have $[\omega \wedge (\gamma + \partial\bar\partial\beta)]_A = [\omega \wedge \gamma]_A$. Hence the map $L_{[\omega]}$ is well-defined. From

$$(\omega + \partial\beta_1 + \bar\partial\beta_2) \wedge \gamma = \omega \wedge \gamma + \partial(\beta_1 \wedge \gamma) + \bar\partial(\beta_2 \wedge \gamma),$$

we see that the map $L_{[\omega]}$ only depends on the Aeppli class of ω. □

Let X be a $\partial\bar\partial$-manifold. We denote by

$$j : H_A^{p,q}(X, \mathbb{C}) \xrightarrow{\cong} H_{\bar\partial}^{p,q}(X, \mathbb{C})$$

the canonical isomorphism and by

$$\widetilde{T_{[u]}} : H^{0,1}(X, T^{1,0}X) \xrightarrow[\cong]{T_{[u]}} H_{\bar\partial}^{n-1,1}(X, \mathbb{C}) \xrightarrow[\cong]{i} H_{BC}^{n-1,1}(X, \mathbb{C})$$

the composition of canonical isomorphism i and Calabi-Yau isomorphism.

More precisely, for $[\theta] \in H^{0,1}(X, T^{1,0}X)$, we have $T_{[u]}([\theta]) = [\theta \lrcorner u] \in H^{n-1,1}_{\bar\partial}(X, \mathbb{C})$. Then the isomorphism i maps $[\theta \lrcorner u]$ to $[\theta \lrcorner u + \bar\partial\eta]_{BC}$, where η is a $(n-1,0)$-form such that $\partial(\theta \lrcorner u + \bar\partial\eta) = 0$. Such a form η exists because of the d-closedness, ∂-exactness of $\partial(\theta \lrcorner u)$ and the $\partial\bar\partial$-property of X. The class $[\theta \lrcorner u + \bar\partial\eta]_{BC}$ is independent of the choice of η such that $\partial(\theta \lrcorner u + \bar\partial\eta) = 0$, again by the $\partial\bar\partial$-property of X.

Lemma 3.4.2. *The following map*

$$f_{[u]} : H^{0,q}_{\bar\partial}(X, \mathbb{C}) \longrightarrow H^{n,q}_{\bar\partial}(X, \mathbb{C})$$
$$[\xi] \longmapsto [u \wedge \xi]$$

is well-defined and an isomorphism for all $q = 1, \cdots, n$. As a consequence, we have the equality between the Hodge numbers $h^{0,q} = h^{n,q}$.

Proof. To check that this is an isomorphism, we first check that

$$f_u : C^\infty_{0,q}(X, \mathbb{C}) \longrightarrow C^\infty_{n,q}(X, \mathbb{C})$$
$$\xi \longmapsto u \wedge \xi$$

is an isomorphism. In local coordinates, let

$$\xi = \sum_{|J|=q} \xi_{\bar J} d\bar z_J \quad \text{and} \quad u = g dz_1 \wedge \cdots \wedge dz_n$$

where g does not vanish.

$$u \wedge \xi = \sum_{|J|=q} g\xi_{\bar J} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar z_J.$$

Hence

$$C^\infty_{0,q}(X, \mathbb{C}) \longrightarrow C^\infty_{n,q}(X, \mathbb{C})$$
$$\xi \longmapsto u \wedge \xi$$

is an isomorphism because g does not vanish.

It is easy to check $f_u(\ker\bar\partial) = \ker\bar\partial$ and $f_u(\mathrm{Im}\,\bar\partial) \subset \mathrm{Im}\,\bar\partial$ since $\bar\partial u = 0$, which means that $f_{[u]}$ is well-defined and injective. Now we already have $h^{n,q} \geq h^{0,q}$ for all $q = 1, \cdots, n$ by injectivity. Take any $\bar\partial$-exact form $\bar\partial\eta \in C^\infty_{n,q}(X, \mathbb{C})$. In local coordinates, write

$$\eta = \sum_{|I|=q-1} \eta_{\bar I} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar z_I$$

Let $\zeta = \sum_{|I|=q-1} \frac{\eta_{\bar I}}{g} d\bar z_I$, it is easy to check that $u \wedge \bar\partial\zeta = \bar\partial\eta$ and this implies the surjectivity of $f_{[u]}$.

□

Theorem 3.4.3. *If ω is an SKT metric on a Calabi-Yau $\partial\bar\partial$-manifold X, then*

$$\widetilde{T_{[u]}} : H^{0,1}(X, T^{1,0}X)_{[\omega]} \longrightarrow H^{n-1,1}_{BC,prim}(X, \mathbb{C})$$

is an isomorphism.

Proof. For $[\theta] \in H^{0,1}(X, T^{1,0}X)$, we have $\widetilde{T_{[u]}}([\theta]) = [\theta \lrcorner u + \bar\partial\eta]_{BC}$, where η is some $(n-1,0)$-form such that $\partial(\theta \lrcorner u + \bar\partial\eta) = 0$.

Let ζ be an arbitrary representative in $[\omega]_{\bar\partial}$. Then there exist $\beta_1 \in C^\infty_{0,1}(X, \mathbb{C})$ and $\beta_2 \in C^\infty_{1,0}(X, \mathbb{C})$, such that $\zeta - \omega = \partial\beta_1 + \bar\partial\beta_2$. By

$$0 = \theta \lrcorner (\omega \wedge u) = (\theta \lrcorner \omega) \wedge u + \omega \wedge (\theta \lrcorner u),$$

we get

$$\begin{aligned} L_{[\omega]}(\widetilde{T_{[u]}}([\theta])) =& [\omega \wedge (\theta \lrcorner u + \bar\partial\eta)]_A \\ =& [-(\theta \lrcorner \omega) \wedge u + \omega \wedge \bar\partial\eta]_A \\ =& [(\theta \lrcorner (\zeta - \omega)) \wedge u + \omega \wedge \bar\partial\eta]_A - [(\theta \lrcorner \zeta) \wedge u]_A. \end{aligned}$$

Moreover,

$$\begin{aligned} [(\theta \lrcorner (\zeta - \omega)) \wedge u + \omega \wedge \bar\partial\eta]_A =& [-(\zeta - \omega) \wedge (\theta \lrcorner u) + \omega \wedge \bar\partial\eta]_A \\ =& [-(\partial\beta_1 + \bar\partial\beta_2) \wedge (\theta \lrcorner u) + \omega \wedge \bar\partial\eta]_A \\ =& [\beta_1 \wedge \partial(\theta \lrcorner u) + \omega \wedge \bar\partial\eta]_A \\ =& [-\beta_1 \wedge \partial\bar\partial\eta + \omega \wedge \bar\partial\eta]_A \\ =& [(\omega + \partial\beta_1 + \bar\partial\beta_2) \wedge \bar\partial\eta]_A \\ =& [\zeta \wedge \bar\partial\eta]_A \\ =& 0. \end{aligned}$$

Hence we have

$$\begin{aligned} (j \circ L_{[\omega]})(\widetilde{T_{[u]}}([\theta])) &= j([\omega \wedge (\theta \lrcorner u + \bar\partial\eta)]_A) \\ &= [-(\theta \lrcorner \zeta) \wedge u]_{\bar\partial}. \end{aligned}$$

By Lemma 3.4.2, we have that $(j \circ L_{[\omega]})(\widetilde{T_{[u]}}([\theta])) = 0$ if and only if $[\theta] \in H^{0,1}(X, T^{1,0}X)_{[\omega]}$, i.e., $\widetilde{T_{[u]}}$ is an isomorphism. □

In the Kähler case, every primitive Dolbeaut class has one and only one d-closed primitive representative, which is the Δ''-harmonic element. In general, we have the following:

Lemma 3.4.4. *For a primitive form v of degree n, the following are equivalent:*

(a) *d-closed,*

(b) *d^*-closed,*

(c) *$\Delta v = 0$,*

(d) $\Delta_A v = 0$,

(e) $\Delta_{BC} v = 0$.

Proof. Recall that

$$\partial^* = - \star \bar{\partial} \star, \bar{\partial}^* = - \star \partial \star, d^* = - \star d \star .$$

Therefore by formula (3.4), v and $\star v$ are proportional. Hence, (a) and (b) are equivalent; $\partial v = 0$ if and only if $\bar{\partial}^* v = 0$; $\partial^* v = 0$ if and only if $\bar{\partial} v = 0$. Thus by equations (3.1) and (3.2), we have that (c), (d) and (e) are equivalent. The equivalence between (a) and (c) is trivial now. □

In the SKT case, there is at most one d-closed primitive form in a cohomology class of degree n, which is the harmonic form in the class according to Lemma 3.4.4. If we look for a d-closed primitive form in a primitive class on an SKT $\partial\bar{\partial}$-manifold, it does not matter whether we search in a Dolbeaut or Bott-Chern class.

We analyse an example of SKT manifold given in [TT17] from one point of view. Let $X = S^3 \times S^3$. Recall that the 3-sphere is diffeomorphic to the special unitary group $SU(2)$. We know that $\mathfrak{su}(2)$, the Lie algebra of $SU(2)$, has a basis $\{e_1, e_2, e_3\}$ with the following relations:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1.$$

Then by the Cartan formula, we have the following for the dual co-frame $\{e^1, e^2, e^3\}$:

$$de^1 = -2e^2 \wedge e^3, \quad de^2 = 2e^1 \wedge e^3, \quad de^3 = -2e^1 \wedge e^2.$$

On X, we take $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ to be two copies of this basis of $\mathfrak{su}(2)$, and the corresponding co-frames $\{e^1, e^2, e^3\}$ and $\{f^1, f^2, f^3\}$. Then we define a complex structure on X, namely the *Calabi-Eckmann complex structure* by:

$$Je_1 = e_2, \quad Jf_1 = f_2, \quad Je_3 = f_3.$$

We have

$$Je^1 = -e^2, \quad Jf^1 = -f^2, \quad Je^3 = -f^3.$$

For the complex co-frame of $(1,0)$-forms, we set

$$\varphi^1 = e^1 + ie^2, \quad \varphi^2 = f^1 + if^2, \quad \varphi^3 = e^3 + if^3.$$

Thus, we have

$$\begin{aligned} d\varphi^1 &= i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \bar{\varphi}^3, \\ d\varphi^2 &= \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \bar{\varphi}^3, \\ d\varphi^3 &= -i\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2. \end{aligned}$$

Equivalently,

$$\begin{aligned}
\partial\varphi^1 &= i\varphi^1 \wedge \varphi^3, \\
\partial\varphi^2 &= \varphi^2 \wedge \varphi^3, \\
\partial\varphi^3 &= 0, \\
\bar\partial\varphi^1 &= i\varphi^1 \wedge \bar\varphi^3, \\
\bar\partial\varphi^2 &= -\varphi^2 \wedge \bar\varphi^3, \\
\bar\partial\varphi^3 &= -i\varphi^1 \wedge \bar\varphi^1 + \varphi^2 \wedge \bar\varphi^2.
\end{aligned}$$

We define a Hermitian metric

$$\omega := \frac{i}{2}\sum_{j=1}^{3} \varphi^j \wedge \bar\varphi^j.$$

By direct calculation we know that $\partial\bar\partial\omega = 0$, which means that ω is an SKT metric on X. We calculate the Bott-Chern cohomology groups (see [TT17]):

$$\begin{aligned}
H^{0,0}_{BC}(X,\mathbb{C}) &= \langle[1]\rangle, \\
H^{1,1}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{1\bar1}], [\varphi^{2\bar2}]\rangle, \\
H^{2,1}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{23\bar2} + i\varphi^{13\bar1}]\rangle, \\
H^{1,2}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{2\bar2\bar3} - i\varphi^{1\bar1\bar3}]\rangle, \\
H^{2,2}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{12\bar1\bar2}]\rangle, \\
H^{3,2}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{123\bar1\bar2}]\rangle, \\
H^{2,3}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{12\bar1\bar2\bar4}]\rangle, \\
H^{3,3}_{BC}(X,\mathbb{C}) &= \langle[\varphi^{123\bar1\bar2\bar3}]\rangle,
\end{aligned}$$

where all the representatives above are the Δ_{BC}-harmonic ones. The other Bott-Chern cohomology groups are trivial.

Note that

$$[\omega \wedge (\varphi^{23\bar2} + i\varphi^{13\bar1})]_A = \frac{-1+i}{2}[\varphi^{123\bar1\bar2}]_A = \frac{-1+i}{2}[\partial\varphi^{13\bar1\bar3}]_A = 0.$$

Hence $[\varphi^{23\bar2} + i\varphi^{13\bar1}]_{BC} \in H^{2,1}_{BC}(X,\mathbb{C})$ is a primitive class but has no primitive representative. By conjugation, $[\varphi^{2\bar2\bar3} - i\varphi^{1\bar1\bar3}]_{BC} \in H^{1,2}_{BC}(X,\mathbb{C})$ is again a primitive class but has no primitive representative.

3.5 Period map

In this section, we recall the definition of the period map and the local Torelli theorem. Tis section closely follows [Pop19].

Fix a Hermitian metric ω on X. We have the Hodge star operator

$$\star : C^\infty_n(X,\mathbb{C}) \longrightarrow C^\infty_n(X,\mathbb{C})$$

and $\star^2 = (-1)^n id$, where $n = \dim_{\mathbb{C}} X$. When n is even , the eigenvalues of $\star$ is 1 and -1. When n is odd, the eigenvalues of $\star$ is i and $-i$. This induces a decomposition

$$C_n^\infty(X, \mathbb{C}) = \Lambda_+^n \oplus \Lambda_-^n,$$

where Λ_+^n (resp. Λ_-^n) is the eigenspace corresponding to 1 or i (resp. -1 or $-i$).

Since $\Delta = dd^* + d^*d$ commutes with $\star$, we have $\star(\mathcal{H}_\Delta^n(X, \mathbb{C})) = \mathcal{H}_\Delta^n(X, \mathbb{C})$. By $\mathcal{H}_\Delta^n(X, \mathbb{C}) = H_{DR}^n(X, \mathbb{C})$, we get a decomposition

$$H_{DR}^n(X, \mathbb{C}) = H_+^n(X, \mathbb{C}) \oplus H_-^n(X, \mathbb{C}).$$

The Hodge-Riemann bilinear form can be defined on $H_{DR}^n(X, \mathbb{C})$ without any assumption on ω:

$$\begin{aligned} Q : H_{DR}^n(X, \mathbb{C}) \times H_{DR}^n(X, \mathbb{C}) &\longrightarrow \mathbb{C} \\ (\{\alpha\}, \{\beta\}) &\longmapsto (-1)^{\frac{n(n-1)}{2}} \int_X \alpha \wedge \beta. \end{aligned}$$

It is non-degenerate. Indeed for every class $\{\alpha\}$, if α is the Δ-harmonic representative, then $\star\bar{\alpha}$ is also Δ-harmonic. We have

$$(-1)^{\frac{n(n-1)}{2}} Q(\{\alpha\}, \{\star\bar{\alpha}\}) = \int_X \alpha \wedge \star\bar{\alpha} = \|\alpha\|^2,$$

which is not 0 as long as α is not 0. Therefore we define a non-degenerate sesquilinear form

$$\begin{aligned} H : H_{DR}^n(X, \mathbb{C}) \times H_{DR}^n(X, \mathbb{C}) &\longrightarrow \mathbb{C} \\ (\{\alpha\}, \{\beta\}) &\longmapsto (-1)^{\frac{n(n+1)}{2}} i^n \int_X \alpha \wedge \bar{\beta}. \end{aligned}$$

Then one can define the period domain as follows.

Definition 3.5.1. *The period domain is defined as a subset of $\mathbb{P}H^n(X, \mathbb{C})$:*

- *If n is even,*

  $$D = \{\textit{complex line } l \in \mathbb{P}H^n(X, \mathbb{C}) | \, \forall \varphi \in l \backslash \{0\}, Q(\varphi, \varphi) = 0 \textit{ and } H(\varphi, \varphi) > 0\};$$

- *If n is odd,*

  $$D = \{\textit{complex line } l \in \mathbb{P}H^n(X, \mathbb{C}) | \, \forall \varphi \in l \backslash \{0\}, Q(\varphi, \varphi) = 0 \textit{ and } H(\varphi, \varphi) < 0\}.$$

We prove that $H^{n,0}(X, \mathbb{C})$ is a subset of the period domain by the following two lemmas:

Lemma 3.5.2. *Let X be a compact complex $\partial\bar{\partial}$-manifold, then*

- *if n is even, $H^{n,0}(X, \mathbb{C}) \subset H_+^n(X, \mathbb{C})$;*

- *if n is odd,* $H^{n,0}(X,\mathbb{C}) \subset H^n_-(X,\mathbb{C})$.

Proof. Because every element $\alpha \in C^\infty_{n,0}(X,\mathbb{C})$ is primitive for bidegree reasons, we have $\star\alpha = i^{n(n+2)}\alpha$ by (3.4). Hence α is $\bar\partial$-closed if and only if $\star\alpha$ is $\bar\partial$-closed. We have $i^{n(n+2)} = 1$ if n is even, and $i^{n(n+2)} = -i$ if n is odd. This proves the lemma. □

Lemma 3.5.3. *We have the following properties:*
$H(\{\alpha\},\{\alpha\}) > 0$ *for every class* $\{\alpha\} \in H^n_+(X,\mathbb{C}) \setminus \{0\}$,
$H(\{\alpha\},\{\alpha\}) < 0$ *for every class* $\{\alpha\} \in H^n_-(X,\mathbb{C}) \setminus \{0\}$.

Proof. If n is even, for every class $\{\alpha\} \in H^n_+(X,\mathbb{C}) \setminus \{0\}$, we have $\star\alpha = \alpha$. Then

$$H(\{\alpha\},\{\alpha\}) = i^{n(n+2)}\int_X \alpha\wedge\bar\alpha = \int_X \alpha\wedge\star\bar\alpha = \|\alpha\|^2 > 0.$$

If n is odd, for every class $\{\alpha\} \in H^n_+(X,\mathbb{C}) \setminus \{0\}$, we have $\star\alpha = i\alpha$. We still have

$$H(\{\alpha\},\{\alpha\}) = i^{n(n+2)}\int_X \alpha\wedge\bar\alpha = \int_X \alpha\wedge\star\bar\alpha = \|\alpha\|^2 > 0.$$

One can prove the second statement similarly. □

Theorem 3.5.4 (Local Torelli Theorem [Pop19, Thm 5.4])**.** *Let X be a compact Calabi-Yau $\partial\bar\partial$-manifold of dimension n, and $\pi : \mathcal{X} \longrightarrow B$ be its Kuranishi family. Then the associated period map*

$$\begin{aligned} \mathcal{P} : B &\to D \subset \mathbb{P}H^n(X,\mathbb{C}) \\ t &\mapsto H^{n,0}(X_t,\mathbb{C}) \end{aligned}$$

is a local holomorphic immersion.

Proof. Since $X = X_0$ is a $\partial\bar\partial$-manifold, the Hodge numbers $h^{p,\,q}(t)$ are independent of t varying in a small enough neighbourhood of 0. Since $X = X_0$ is also a Calabi-Yau manifold, $h^{n,\,0}(0) = 1$, hence $h^{n,\,0}(t) = 1$ and X_t is a Calabi-Yau $\partial\bar\partial$-manifold for all t in a neighbourhood of 0. Then $H^{n,0}(X_t,\mathbb{C})$ is a point in $\mathbb{P}H^n(X,\mathbb{C})$ for every $t \in B$ close enough to 0.

In particular, $\mathcal{H}^{n,\,0}$ (see notation in the explanations after Definition 3.3.1) is a holomorphic line bundle over (a sufficiently shrunk) B. Let u_0 be a nowhere vanishing holomorphic $(n,\,0)$-form on X_0. (Such a form exists thanks to the canonical line bundle K_{X_0} being trivial.) Equivalently, u_0 is a non-zero element in the fibre $\mathcal{H}^{n,\,0}_0 = H^{n,\,0}_{\bar\partial}(X_0,\,\mathbb{C}) \simeq H^0(X_0,\,K_{X_0})$ of $\mathcal{H}^{n,\,0}$ at $0 \in B$. If necessary, shrink B sufficiently such that the holomorphic vector bundle $\mathcal{H}^{n,\,0}$ is trivial over B. Take any holomorphic section u of $\mathcal{H}^{n,\,0}$ over B such that $u(0) = u_0$. Setting $u_t := u(t)$ for every $t \in B$, we get a holomorphic family $(u_t)_t \in B$ of nowhere vanishing n-forms on X such that u_t is a J_t-holomorphic $(n,0)$-form for every t. Since $h^{n,\,0}(t) = 1$, we conclude that $H^{n,0}(X_t,\mathbb{C}) = \mathbb{C}u_t$ for every $t \in B$. From this and the holomorphic dependence of u_t on $t \in B$, we get that the period map $\mathcal{P}$ is holomorphic.

If $\mathcal{P}$ is not a local immersion, then we can choose a point in B, say 0, and a tangent vector $\partial/\partial t \in T_0B$, such that $(d\mathcal{P})_0(\partial/\partial t) = 0$. Because X is $\partial\bar\partial$-manifold, we can choose a representative θ in $\rho(\partial/\partial t) \in H^{0,1}(X, T^{1,0}X)$ such that $d(\theta\lrcorner u_0) = 0$, where ρ is the Kodaira-Spencer map.

By Ehresmann's lemma, we have a smooth family of diffeomorphisms $\Phi_t^{-1}: X_0 \to X_t$. Taking a set of J_t-holomorphic coordinates $z_1(t), \cdots, z_n(t)$, we write $u_t = f_t dz_1(t) \wedge \cdots dz_n(t)$. Therefore, we have

$$\frac{\partial(\Phi_t^{-1})^* u_t}{\partial t}|_{t=0} = \theta\lrcorner u_0 + \frac{\partial f_t}{\partial t}|_{t=0} dz_1(t) \wedge \cdots dz_n(t), \tag{3.8}$$

where $v := \frac{\partial f_t}{\partial t}|_{t=0} dz_1(t) \wedge \cdots dz_n(t)$ is a $(n,0)$-form and $\theta\lrcorner u_0$ is a $(n-1,1)$-form. Then we know that $\theta = 0$. Hence the period map $\mathcal{P}$ is a local immersion.

□

3.6 Metrics on B

In this section we compare two versions of Weil-Petersson metric and the metric induced by the period map given in the previous section.

We use the definition of ω-minimal d-closed representative given in [Pop19]. The ω-minimal d-closed representative of a Dolbeaut cohomology class $[\beta]$ is $\beta_{min} = \beta + \bar\partial v_{min}$, where β is the Δ''-harmonic representative in $[\beta]$ and v_{min} is the solution of minimal L^2-norm of $\partial\beta = -\partial\bar\partial v$.

Definition 3.6.1. *Let $(u_t)_{t\in B}$ be a fixed holomorphic family of non-vanishing holomorphic n-forms on the fibres $(X_t)_{t\in B}$ and let $(\omega_t)_{t\in B_{[\omega]}}$ be a smooth family of SKT metrics on the fibres $(X_t)_{t\in B_{[\omega]}}$ such that $\omega_t \in \{\omega\}$ for any t and $\omega_0 = \omega$. The Weil-Petersson metrics $G_{WP}^{(1)}$ and $G_{WP}^{(2)}$ are defined on $B_{[\omega]}$ by*

$$G_{WP}^{(1)}([\theta_t], [\eta_t]) := \frac{\langle\langle \theta_t, \eta_t \rangle\rangle_{\omega_t}}{\int_{X_t} dV_{\omega_t}}$$

$$G_{WP}^{(2)}([\theta_t], [\eta_t]) := \frac{\langle\langle \theta_t\lrcorner u_t, \eta_t\lrcorner u_t \rangle\rangle_{\omega_t}}{i^{n^2}\int_{X_t} u_t \wedge \overline{u}_t}$$

for any $t \in B_{[\omega]}$, $[\theta_t], [\eta_t] \in H^{0,1}(X_t, T^{1,0}X_t)_{[\omega]}$. Here θ_t (resp. η_t) is chosen such that $\theta_t\lrcorner u_t$ (resp. $\eta_t\lrcorner u_t$) is the ω_t-minimal d-closed representative of the class $[\theta_t\lrcorner u_t] \in H^{n-1,1}(X_t, \mathbb{C})$ (resp. $[\eta_t\lrcorner u_t] \in H^{n-1,1}(X_t, \mathbb{C})$).

Remark 3.6.2. *Denote the $(1,1)$-forms associated with $G_{WP}^{(1)}$, $G_{WP}^{(2)}$ by $\omega_{WP}^{(1)}$, $\omega_{WP}^{(2)}$. If $Ric(\omega_t) = 0$ for all $t \in B_{[\omega]}$, we have $\omega_{WP}^{(1)} = \omega_{WP}^{(2)}$.*

Let $L = \mathcal{O}_{\mathbb{P}H^n(X,\mathbb{C})}(-1)$ be the tautological line bundle on $\mathbb{P}H^n(X, \mathbb{C})$.

We set:

$$\begin{aligned}
C_+ &:= \{\{\alpha\} \in H^n(X,\mathbb{C}) | H(\{\alpha\},\{\alpha\}) > 0\};\\
C_- &:= \{\{\alpha\} \in H^n(X,\mathbb{C}) | H(\{\alpha\},\{\alpha\}) < 0\};\\
U_+^n &:= \{[l] \in \mathbb{P}H^n(X,\mathbb{C}) | l \text{ is a complex line such that } l \subset C_+\};\\
U_-^n &:= \{[l] \in \mathbb{P}H^n(X,\mathbb{C}) | l \text{ is a complex line such that } l \subset C_-\}.
\end{aligned}$$

Then $\mathrm{Im}\mathcal{P}$ is a subset of U_+^n when n is even, $\mathrm{Im}\mathcal{P}$ is a subset of U_-^n when n is odd. We can get a Hermitian fibre metric h_L^+ on $L|_{U_+^n}$ from H. Then the associated Fubini-Study metric on U_+^n is

$$\omega_{FS}^+ = -i\Theta_{h_L^+}(L|_{U_+^n}).$$

Similarly, we get the associated Fubini-Study metric on U_-^n:

$$\omega_{FS}^- = -i\Theta_{h_L^-}(L|_{U_-^n}).$$

Then we have a Hermitian metric γ on B:

$$\gamma := \begin{cases} \mathcal{P}^*\omega_{FS}^+ & \text{if } n \text{ is even,}\\ \mathcal{P}^*\omega_{FS}^- & \text{if } n \text{ is odd.}\end{cases}$$

Lemma 3.6.3. *The Kähler metric γ defined on B is independent of the choice of metrics on $(X_t)_{t\in B}$ and is explicitly given by the formula:*

$$\gamma_t([\theta_t],[\theta_t]) = \frac{-\int_X(\theta_t\lrcorner u_t)\wedge\overline{(\theta_t\lrcorner u_t)}}{i^{n^2}\int_X u_t\wedge\overline{u_t}} = \frac{-H(\theta_t\lrcorner u_t,\theta_t\lrcorner u_t)}{i^{n^2}\int_X u_t\wedge\bar{u}_t}, \textit{ if } n \textit{ is even,}$$
$$\gamma_t([\theta_t],[\theta_t]) = \frac{-i\int_X(\theta_t\lrcorner u_t)\wedge\overline{(\theta_t\lrcorner u_t)}}{i^{n^2}\int_X u_t\wedge\bar{u}_t} = \frac{H(\theta_t\lrcorner u_t,\theta_t\lrcorner u_t)}{i^{n^2}\int_X u_t\wedge\bar{u}_t}, \textit{ if } n \textit{ is odd,}$$

for every $t \in B$ and every $[\theta_t] \in H^{0,1}(X_t, T^{1,0}X_t)$.

Proof. Because X is a Calabi-Yau manifold, there is a family of nowhere vanishing J_t-$(n,0)$-forms $(u_t)_{t\in B}$, such that u_t is J_t-holomorphic.

When n is even, we have $|u_t|^2_{h_L^+} = H(u_t,u_t)$. Now we know that

$$\omega_{FS}^+ = -i\partial_t\bar{\partial}_t \log(H(u_t,u_t)).$$

Taking a class $[\theta]$ in $H^{0,1}(X,T^{1,0}X)$, assume $[\theta]$ is the image of $\frac{\partial}{\partial t}|_{t=0}$ under the Kodaira-Spencer map. Then

$$\gamma_0([\theta],[\theta]) = -\frac{\partial^2\log(H(u_t,u_t))}{\partial t\partial\bar{t}}|_{t=0} = -\frac{\partial}{\partial t}\Big(\frac{H(u_t,\frac{\partial u_t}{\partial t})}{H(u_t,u_t)}\Big)|_{t=0}.$$

Because the left hand side of (3.8) is d-closed, v is a J_0-holomorphic $(n,0)$-form. Thus, v is Cu_0 for some constant C. So we have

$$\begin{aligned}
&-\frac{\partial}{\partial t}\Big(\frac{H(u_t, \frac{\partial u_t}{\partial t})}{H(u_t,u_t)}\Big)|_{t=0} \\
=&-\frac{H(\frac{\partial u_t}{\partial t}|_{t=0}, \frac{\partial u_t}{\partial t}|_{t=0})H(u_0,u_0) - H(\frac{\partial u_t}{\partial t}|_{t=0}, u_0)H(u_0, \frac{\partial u_t}{\partial t}|_{t=0})}{H^2(u_0,u_0)} \\
=&-\frac{H(\theta \lrcorner u_0, \theta \lrcorner u_0)H(u_0,u_0) + C\bar{C}H^2(u_0,u_0) - C\bar{C}H^2(u_0,u_0)}{H^2(u_0,u_0)} \\
=&-\frac{H(\theta \lrcorner u_0, \theta \lrcorner u_0)}{H(u_0,u_0)}.
\end{aligned}$$

The calculation of the case that n is odd differs with only a (-1) factor. □

Denote the space of global smooth forms of bidegree $(n-1,1)$ by $\Lambda^{n-1,1}$. Then we have two decompositions. The first one is Lefschetz decompositon:

$$\Lambda^{n-1,1} = \Lambda^{n-1,1}_{prim} \oplus (\omega \wedge \Lambda^{n-2,0}).$$

The second one is the decomposition into the eigenspaces of Hodge star operator:

$$\Lambda^{n-1,1} = \Lambda^{n-1,1}_{+} \oplus \Lambda^{n-1,1}_{-}.$$

Lemma 3.6.4. *These two decompositions coincide up to order. Specifically, we have*

- $\Lambda^{n-1,1}_{prim} = \Lambda^{n-1,1}_{-}$ *and* $\omega \wedge \Lambda^{n-2,0} = \Lambda^{n-1,1}_{+}$ *if n is even,*
- $\Lambda^{n-1,1}_{prim} = \Lambda^{n-1,1}_{+}$ *and* $\omega \wedge \Lambda^{n-2,0} = \Lambda^{n-1,1}_{-}$ *if n is odd.*

Proof. For a primitive form u of bidegree $(n-1,1)$, we have $\star u = (-1)^{n(n+1)/2} i^{n-2} u$. Therefore, we get $\Lambda^{n-1,1}_{prim} \subset \Lambda^{n-1,1}_{-}$ if n is even, and $\Lambda^{n-1,1}_{prim} \subset \Lambda^{n-1,1}_{+}$ if n is odd.

Now, it suffices to prove that for a form v of bidegree $(n-2,0)$, $\star(\omega\wedge v) = \omega \wedge v$ when n is even, and $\star(\omega \wedge v) = -i\omega \wedge v$ when n is odd. Firstly, for every form u of bidegree $(n-1,1)$ we have a decomposition $u = u_{prim} + \omega \wedge u_1$. Besides, v is primitive because it is of bidegree $(n-2,0)$. Hence, we get $\star v = \frac{i^{n(n-2)}}{2} v \wedge \omega^2$. Then

$$\begin{aligned}
\int_X u \wedge \star(\omega \wedge v) =& \langle\langle u, \omega \wedge \bar{v}\rangle\rangle \\
=& \langle\langle \omega \wedge u_1, \omega \wedge \bar{v}\rangle\rangle \\
=& 2\langle\langle u_1, \bar{v}\rangle\rangle \\
=& 2\int_X u_1 \wedge \star v \\
=& i^{n(n-2)} \int_X u_1 \wedge \omega^2 \wedge v \\
=& i^{n(n-2)} \int_X u \wedge \omega \wedge v.
\end{aligned}$$

Therefore we have $\star(\omega\wedge v)=i^{n(n-2)}\omega\wedge v$. $\square$

By Lemma 3.6.4, for any $\theta\in C^\infty_{0,1}(X,T^{1,0}X)$, we have the decomposition:

$$\theta\lrcorner u=\theta'\lrcorner u+\omega\wedge\zeta.$$

By orthogonality, we have

$$G^{(2)}_{WP}([\theta_t],[\theta_t]) = \frac{\langle\langle\theta_t\lrcorner u_t,\theta_t\lrcorner u_t\rangle\rangle}{i^{n^2}\int_{X_t}u_t\wedge\overline{u}_t}=\frac{\|\theta_t'\lrcorner u_t\|^2+2\|\zeta_t\|^2}{i^{n^2}\int_{X_t}u_t\wedge\overline{u}_t}.$$

If n is even, by Lemma 3.6.4, we have $\star(\theta'\lrcorner u)=-\theta'\lrcorner u$ and $\star(\omega\wedge\zeta)=\omega\wedge\zeta$. As a consequence, we have

$$\begin{aligned}\int_X(\theta_t\lrcorner u_t)\wedge\overline{(\theta_t\lrcorner u_t)}&=\int_X(\theta_t'\lrcorner u_t+\omega_t\wedge\zeta_t)\wedge\star(\overline{-\theta_t'\lrcorner u_t+\omega_t\wedge\zeta_t})\\&=-\|\theta_t'\lrcorner u_t\|^2+2\|\zeta_t\|^2.\end{aligned}$$

Then we have

$$\gamma_t([\theta_t],[\theta_t])=\frac{\|\theta_t'\lrcorner u_t\|^2-2\|\zeta_t\|^2}{i^{n^2}\int_{X_t}u_t\wedge\overline{u}_t}.$$

Similarly, we get the same expression for n odd.

Remark 3.6.5. *For all $[\theta_t]\in H^{0,1}(X_t,T^{1,0}X_t)_{[\omega]}\setminus\{0\}$, we have*

$$(G^{(2)}_{WP}-\gamma)_t([\theta_t],[\theta_t])=\frac{4\|\zeta_t\|^2}{i^{n^2}\int_{X_t}u_t\wedge\overline{u}_t}\geq 0.$$

Hence if every class in $H^{n-1,1}_{prim}(X_t,\mathbb{C})$ has a d-closed and primitive representative, or equivalently, $H^{n-1,1}_{BC,prim}(X_t,\mathbb{C})$ has a primitive representative, we would get $G^{(2)}_{WP}=\gamma$. The polarised deformation of a Kähler manifold satisfies this condition in [Tia87].

Chapter 4

Strongly Gauduchon Hyperbolicity and two other Types of Hyperbolicity

4.1 Introduction

Hyperbolicity is an important concept in the theory of complex manifolds, characterizing their geometric and topological properties. In recent years, the study of hyperbolicity has attracted widespread attention in the fields of complex analytic, algebraic and differential geometries, and has achieved a series of important results. Classical notions such as Kähler hyperbolicity, Kobayashi hyperbolicity, and Brody hyperbolicity have been intensively studied. Meanwhile, new notions such as balanced hyperbolicity and divisorial hyperbolicity have been introduced and studied, providing new perspectives and tools for the study of complex manifolds.

Let us first recall some notions of hyperbolicity.

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$.

1. A form α on (X, ω) is said to be $\tilde{d}$(bounded) if the lift $\tilde{\alpha}$ of α to the universal cover $\tilde{X}$ od X is d-exact with a d-potential bounded with respect to the lift $\tilde{\omega}$ of ω.

2. ([Gro91]) X is said to be Kähler hyperbolic if X admits a Kähler metric whose fundamental form ω is $\tilde{d}$(bounded).

3. ([Kob67]) X is said to be Kobayashi hyperbolic if the Kobayashi pseudo-distance on X is a distance.

4. ([Bro78]) X is said to be Brody hyperbolic if all holomorphic maps $f : \mathbb{C} \to X$ are constant.

5. ([MP22a]) X is said to be balanced hyperbolic if there is a balanced metric ω on X such that ω^{n-1} is $\tilde{d}$(bounded) with respect to ω.

6. ([MP22a]) X is said to be divisorially hyperbolic if there is no holomorphic map $f : \mathbb{C}^{n-1} \to X$ such that f is non-degenerate at some point and has subexponential growth in the sense of Definition 4.2.5.

As for the relations among these hyperbolicities, it is known that a compact complex manifold is Kobayashi hyperbolic if and only if it is Brody hyperbolic. Besides, for a compact complex manifold, we have the following implications:

$$\begin{array}{ccc} X \text{ is Kähler hyperbolic} & \Longrightarrow & X \text{ is Kobayashi/Brody hyperbolic} \\ \Downarrow & & \Downarrow \\ X \text{ is balanced hyperbolic} & \Longrightarrow & X \text{ is divisorially hyperbolic} \end{array}$$

This paper aims to further extend the research on hyperbolicity concepts. After reviewing existing notions of hyperbolicity and their mutual relationships, we introduce the notion of **sG-hyperbolicity** and investigate its connection to divisorial hyperbolicity. A key property of sG-hyperbolicity is its deformation openness — see Theorem 4.2.4. This property is known to hold for Kobayashi hyperbolic compact complex manifolds [Bro78], but it is still an open question whether it holds for Kähler hyperbolic, balanced hyperbolic and divisorially hyperbolic compact complex manifolds. We then construct examples of sG-hyperbolic manifolds that are not necessarily balanced hyperbolic. Finally, we propose new hyperbolicity notions, namely **weakly p-Kähler hyperbolicity**, **pluriclosed star split hyperbolicity**, and the relationship with divisorial hyperbolicity, laying the groundwork for further research.

4.2 sG-Hyperbolic Manifolds

4.2.1 Definition and Properties

Recall the definition of sG manifolds:

Definition 4.2.1 ([Pop13a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

1. *A C^∞ positive definite $(1,1)$-form ω on X is said to be a strongly Gauduchon (sG) metric if ω^{n-1} is the $(n-1, n-1)$-component of a real d-closed C^∞ $(2n-2)$-form Ω.*

2. *If X carries such a metric, X is said to be a strongly Gauduchon (sG) manifold.*

The first notion we introduce in this paper is contained in the

Definition 4.2.2. *Let X be a compact complex manifold with* $\dim_{\mathbb{C}} X = n$. *A Hermitian metric ω on X is said to be* **sG-hyperbolic** *(***strongly Gauduchon hyperbolic***) if there exists a real d-closed* $(2n-2)$*-form Ω on X such* *that the $(n-1, n-1)$-component of Ω is $\omega_{n-1} := \frac{\omega^{n-1}}{(n-1)!}$ and Ω is $\tilde{d}$(bounded) with respect to ω.*

The manifold X is said to be **sG-hyperbolic** *if it carries an sG-hyperbolic metric.*

The first property we observe for these manifolds is given in

Proposition 4.2.3. *The Cartesian product of sG-hyperbolic manifolds is sG-hyperbolic.*

Proof. Let (X_1, ω_1) and (X_2, ω_2) be sG-hyperbolic manifolds of respective dimensions m and n, and let $\pi_1 : \widetilde{X_1} \to X_1$ and $\pi_1 : \widetilde{X_1} \to X_1$ be their universal covers. ω_1^{m-1}(resp. ω_2^{n-1}) is the $(m-1, m-1)$ (resp.$(n-1, n-1)$) component of d-closed real form Γ_1 (resp. Γ_2).

We denote by $\omega = \sigma_1^*\omega_1 + \sigma_2^*\omega_2$ the induced product metric on X. We have that $\omega^{n+m-1} = \binom{n+m-1}{m-1}\sigma_1^*\omega_1^{m-1} \wedge \sigma_2^*\omega_2^n + \binom{n+m-1}{n-1}\sigma_1^*\omega_1^m \wedge \sigma_2^*\omega_2^{n-1}$ is the $(n+m-1, n+m-1)$-component of

$$\Gamma = \binom{n+m-1}{m-1}\sigma_1^*\Gamma_1 \wedge \sigma_2^*\omega_2^n + \binom{n+m-1}{n-1}\sigma_1^*\omega_1^m \wedge \sigma_2^*\Gamma_2,$$

which is a d-closed real $(n+m-1)$-form. Therefore ω is a strongly Gauduchon metric.

Besides, we know that

$$\begin{aligned}
\pi^*\Gamma &= \binom{n+m-1}{m-1}\pi^*\sigma_1^*\Gamma_1 \wedge \pi^*\sigma_2^*\omega_2^n + \binom{n+m-1}{n-1}\pi^*\sigma_1^*\omega_1^m \wedge \pi^*\sigma_2^*\Gamma_2 \\
&= \binom{n+m-1}{m-1}\tilde{\sigma}_1^*(\pi_1^*\Gamma_1) \wedge \sigma_2^*\omega_2^n + \binom{n+m-1}{n-1}\sigma_1^*\omega_1^m \wedge \tilde{\sigma}_2^*(\pi_2^*\Gamma_2) \\
&= d[\binom{n+m-1}{m-1}\tilde{\sigma}_1^*\Theta_1 \wedge \sigma_2^*\omega_2^n + \binom{n+m-1}{n-1}\sigma_1^*\omega_1^m \wedge \tilde{\sigma}_2^*\Theta_2].
\end{aligned}$$

Hence Γ is $\tilde{d}$(bounded) on $X_1 \times X_2$, i.e., $X_1 \times X_2$ is sG-hyperbolic. □

We have the deformation openness of the sG-hyperbolicity.

Theorem 4.2.4. *Let $\pi : X \to B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$. If the fibre X_0 is an sG-hyperbolic manifold, then, for all $t \in B$ sufficiently close to 0, the fibre X_t is again an sG-hyperbolic manifold.*

Proof. Let ω_0 be an sG-hyperbolic metric on X_0. By the definition of sG-hyperbolic metric, there exists a d-closed $\tilde{d}$(bounded) $(2n-2)$-form Ω such that ω_0^{n-1} is the $(n-1, n-1)$-component of Ω on X_0.

The $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ of Ω with respect to the complex structure of X_t is positive definite for $t \in B$ sufficiently close to 0. By Lemma ([Mic82], (4.8)), there exists a metric ω_t such that $\omega_t^{n-1} = \Omega_t^{n-1,n-1}$. Because of the compactness of the C^∞ manofold X underlying the fibres X_t and the continuity of ω_t with respect to t, Ω is $\tilde{d}$(bounded) with respect to ω_t for $t \in B$ sufficiently close to 0. Hence X_t is again an sG-hyperbolic manifold. □

Let us now finish recalling the definition of a divisorially hyperbolic manifold by recalling the definition of subexponential growth for entire holomorphic maps $f : \mathbb{C}^p \to (X, \omega)$. We denote the open ball (resp. sphere) of radius r centered at 0 in $\mathbb{C}^p$ by B_r (resp. S_r). Let $\star_\omega$ denote the Hodge star operator induced by a Hermitian metric ω, and let $\tau(z) := |z|^2$ be the squared Euclidean norm on $\mathbb{C}^p$.

Definition 4.2.5 ([MP22a]). *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$. For $0 < p \leq n-1$, we say that a holomorphic map $f : \mathbb{C}^p \to X$ has* **subexponential growth** *if the following two conditions are satisfied:*

1. *There exist constants $C_1 > 0$ and $r_0 > 0$ such that*

$$\int_{S_t} |d\tau|_{f^\star\omega} d\sigma_{\omega,f,t} \leq C_1 t \operatorname{Vol}_{\omega,f}(B_t), \quad t > r_0,$$

   where $d\sigma_{\omega,f,t} = \left(\star_{f^*\omega}\left(\frac{d\tau}{|d\tau|_{f^*\omega}}\right)\right)|_{s_t}$, $\operatorname{Vol}_{\omega,f}(B_t) := \int_{B_t} f^*\omega_p$.

2. *For every constant $C > 0$, we have:*

$$\overline{\lim_{b\to+\infty}}\left(\frac{b}{C} - \log F(b)\right) = +\infty,$$

   where

$$F(b) := \int_0^b \operatorname{Vol}_{\omega,f}(B_t)\, dt = \int_0^b \left(\int_{B_t} f^*\omega_p\right) dt, \quad b > 0.$$

Theorem 4.2.6. *Every sG-hyperbolic manifold is divisorially hyperbolic.*

Proof. Let X be a compact complex manifold of dimension n and let ω be an sG-hyperbolic metric on X. Suppose there exists a holomorphic map $f : \mathbb{C}^{n-1} \to X$ non-degenerate at some point that has subexponential growth.

Let $\pi : \tilde{X} \longrightarrow X$ be the universal cover of X. There exists a $\pi^*\omega$-bounded $(2n-3)$-form Γ on $\tilde{X}$ such that $d\Gamma = \pi^*\Omega = \pi^*\left(\Omega^{n,n-2} + \omega_{n-1} + \Omega^{n-2,n}\right)$.

Then $\tilde{f}^*\Gamma$ is $f^*\omega$-bounded because:

$$\begin{aligned}\left|\tilde{f}^*\Gamma\left(v_1, \cdots v_{2n-3}\right)\right| &= \left|\Gamma\left(f_*v_1, \cdots, f_*v_{2n-3}\right)\right| \\ &\leqslant C\left|\tilde{f}_*v_1\right|_{\pi^*\omega} \cdots \left|\tilde{f}_*v_{2n-3}\right|_{\pi^*\omega} \\ &= C\left|v_1\right|_{f^*\omega} \cdots \left|v_{2n-3}\right|_{f^*\omega}\end{aligned}$$

for any tangent vectors $v_1, \cdots, v_{2n-3}$ in $\mathbb{C}^{n-1}$ and some constant C.

Now, we have

$$\begin{aligned}\operatorname{Vol}_{\omega,f}(B_r) &= \int_{Br} f^*\omega_{n-1} = \int_{Br} f^*\Omega \\ &= \int_{Br} d\left(\tilde{f}^*\Gamma\right) \leqslant C\int_{S_r} d\sigma_{\omega,f,r}.\end{aligned}$$

We finish the proof by the following lemma:

Lemma 4.2.7. *There does not exist a holomorphic map $f : \mathbb{C}^p \to X$ for $0 < p \leq n-1$ that simultaneously satisfies the following conditions:*

- *f is non-degenerate at some point and has subexponential growth.*
- *$Vol_{\omega,f}(B_r) \leq C \int_{S_r} d\sigma_{\omega,f,r}$. for some constant C.*

Proof. Suppose there exists such a holomorphic map $f : \mathbb{C}^p \to X$. By the Hölder inequality, we have

$$\int_{s_r} \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,r} \cdot \int_{S_r} |d\tau|_{f^*w} d\sigma_{\omega,f,r} \geqslant \left(\int_{S_r} d\sigma_{\omega,f,r} \right)^2 .$$

We have $d\tau = 2rdr$. Let $d\mu_{\omega,f,r}$ be the measure on S_r such that

$$d\mu_{\omega,f,r} \wedge \frac{(d\tau)|_{S_r}}{2r} = (f^*\omega_{n-1})|_{S_r} .$$

Hence, we have

$$\frac{1}{2r} d\mu_{\omega,f,r} = \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,r}.$$

Then we get:

$$\begin{aligned}
&\mathrm{Vol}_{\omega,f}\left(B_r\right) = \textstyle\int_0^r \left(\int_{S_t} \frac{1}{|d\tau|_{f^*\omega}} d\sigma_{\omega,f,t} \right) d\tau \\
&\geqslant \textstyle\int_0^r \frac{\left(\int_{S_r} d\sigma_{\omega,f,r}\right)^2}{\int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,f,t}} 2tdt \\
&\geqslant \textstyle\frac{2}{C^2} \int_0^r \frac{\left(\mathrm{Vol}_{\omega,f}(B_t)\right)^2}{\int_{S_t} |d\tau|_{f^*\omega} d\sigma_{\omega,t}} tdt \\
&\geqslant \textstyle\frac{2}{C_1 C^2} \int_0^r \mathrm{Vol}_{\omega,f}\left(B_t\right) dt
\end{aligned}$$

for r big enough.

This is to say

$$F'(r) \geqslant \frac{2}{C_1 C^2} F(r) ,$$

for r big enough.

Hence, we get

$$(\log F(r))' \geqslant \frac{2}{C_1 C^2}.$$

Finally, we get

$$\log F\left(r_1\right) - \frac{2}{C_1 C^2} r_1 + \frac{2}{C_1 C^2} r_2 \geqslant \log F\left(r_2\right).$$

By (ii) of Definition 4.2.5, we have $F(r_2) = 0$ for r_2 big enough, which contradicts our assumption. □

Theorem 4.2.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \tilde{X} \to X$ be the universal cover of X. If X is sG-hyperbolic, then there exists no non-zero d-closed positive (1,1)-current $\tilde{T} \geq 0$ on $\tilde{X}$ such that $\tilde{T}$ is of $L^1_{\pi^*\omega}$.*

Proof. By the definition of sG-hyperbolic manifold, there exists a d-closed $(2n-2)$-form Ω on X where the $(n-1,n-1)$ component is ω^{n-1}, and there exists a $L^\infty_{\pi^*\omega}$-form Γ of degree $(2n-3)$ on $\tilde{X}$ such that $\pi^*\Omega = d\Gamma$.

For a d-closed $(1,1)$-current $\tilde{T}$ of $L^1_{\pi^*\omega}$ on $\tilde{X}$, $\tilde{T}\wedge\Gamma$ is again of $L^1_{\pi^*\omega}$. Hence also $d(\tilde{T}\wedge\Gamma)$. Now we have:

$$\int_{\tilde{X}} \tilde{T}\wedge\pi^*\omega^{n-1} = \int_{\tilde{X}} \tilde{T}\wedge\pi^*\Omega = \int_{\tilde{X}} d(\tilde{T}\wedge\Gamma) = 0.$$

Therefore, there exists no non-zero d-closed positive (1,1)-current $\tilde{T}\geq 0$ on $\tilde{X}$ such that $\tilde{T}$ is of $L^1_{\pi^*\omega}$.

□

4.2.2 Example

To further explore the properties of sG-hyperbolic manifolds, we will now look for some examples of sG-hyperbolic manifolds that do not necessarily belong to the category of balanced hyperbolic manifolds.

(a)For convenience, let us name a class of Hermitian metrics as follows:

Definition 4.2.9. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X \geq 2$. A C^∞ positive definite $(1,1)$-form ω on X is said to be a* **degenerate sG metric** *if $\omega^{n-1} = \partial\alpha + \bar{\partial}\beta$ for some $\alpha\in C^\infty_{n-2,n-1}(X,\mathbb{C})$, $\beta\in C^\infty_{n-1,n-2}(X,\mathbb{C})$. If X carries such a metric, X is said to be a* **degenerate sG manifold**.

In other words, we require ω^{n-1} to define the zero Aeppli cohomology class (i.e., to be Aeppli-exact). Recall the definitions of Bott-Chern and Aeppli cohomology groups of bidegree (p,q):

$$H^{p,q}_{BC}(X,\mathbb{C}) = \frac{\ker(\partial : C^{p,q}(X)\to C^{p+1,q}(X))\cap\ker(\bar{\partial} : C^{p,q}(X)\to C^{p,q+1}(X))}{\operatorname{Im}(\partial\bar{\partial} : C^{p-1,q-1}(X)\to C^{p,q}(X))},$$

$$H^{p,q}_{A}(X,\mathbb{C}) = \frac{\ker(\partial\bar{\partial} : C^{p,q}(X)\to C^{p+1,q+1}(X))}{\operatorname{Im}(\partial : C^{p-1,q}(X)\to C^{p,q}(X)) + \operatorname{Im}(\bar{\partial} : C^{p,q-1}(X)\to C^{p,q}(X))}.$$

It is obvious that degenerate sG metrics are strongly Gauduchon metrics.

Let us characterize degenerate sG manifolds, with contributions from [MP22b] and [Ale18].

Theorem 4.2.10. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

1. *Let ω be a Hermitian metric on X, ω is degenerate sG if and only if there exists a C^∞ d-exact $(2n-2)$-form Ω on X whose $(n-1,n-1)$-component is ω^{n-1}.*

2. *Let ω be a Gauduchon metric on X, ω is degenerate sG if and only if $H^{1,1}_{BC}(X,\mathbb{C})\wedge[\omega^{n-1}]_A = 0$, where $H^{1,1}_{BC}(X,\mathbb{C})$ is the Bott-Chern cohomology group of bidegree $(1,1)$ and $[\omega^{n-1}]_A$ is the Aeppli cohomology class determined by ω^{n-1}.*

3. *X is a degenerate sG manifold if and only if there exists no non-zero d-closed bidegree $(1,1)$-current $T \geq 0$ on X.*

Proof. (1) Let Ω be the d-exact $(2n-2)$-form mentioned in (1), then the $(n-1,n-1)$-component of Ω is obviously in $\mathrm{Im}\partial + \mathrm{Im}\bar\partial$. Conversely, if we have $\omega^{n-1} = \partial\alpha + \bar\partial\beta$ for some $\alpha \in C^\infty_{n-2,n-1}(X,\mathbb{C})$ and some $\beta \in C^\infty_{n-1,n-2}(X,\mathbb{C})$, then $d(\alpha+\beta)$ is a d-exact $(2n-2)$-form Ω on X whose $(n-1,n-1)$-component is ω^{n-1}.

(2) Let us check that $[\omega^{n-1}]_A \wedge \cdot : H^{1,1}_{BC}(X,\mathbb{C}) \to H^{n,n}_A(X,\mathbb{C})$ is well-defined first.

Because the Hermitian metric ω is Gauduchon, i.e., $\partial\bar\partial\omega^{n-1} = 0$, the Aeppli class $[\omega^{n-1}]_A$ is well-defined.

For a d-closed $(1,1)$-form α, we have

$$\partial\bar\partial(\omega^{n-1} \wedge \alpha) = \partial\bar\partial\omega^{n-1} \wedge \alpha = 0.$$

If $\alpha = \partial\bar\partial\varphi$ for some $\varphi \in C^\infty(X,\mathbb{C})$, we have

$$\omega^{n-1} \wedge \partial\bar\partial\varphi = \partial(\omega^{n-1} \wedge \bar\partial\varphi) + \bar\partial(\varphi\partial\omega^{n-1}) \in \mathrm{Im}\partial + \mathrm{Im}\bar\partial.$$

($\Rightarrow$) If $\omega^{n-1} = \partial\beta + \bar\partial\gamma$ for some $\beta \in C^\infty_{n-2,n-1}(X,\mathbb{C})$, $\gamma \in C^\infty_{n-1,n-2}(X,\mathbb{C})$, we have

$$\omega^{n-1} \wedge \alpha = \partial(\beta \wedge \alpha) - \bar\partial(\gamma \wedge \alpha) \in \mathrm{Im}\partial + \mathrm{Im}\bar\partial$$

for all d-closed $(1,1)$-forms α.

Hence, for all $[\alpha]_{BC} \in H^{1,1}_{BC}(X,\mathbb{C})$, we have

$$[\alpha]_{BC} \wedge [\omega^{n-1}]_A = [\omega^{n-1} \wedge \alpha]_A = 0.$$

($\Leftarrow$)Denote by Δ_A and Δ_{BC} the Aeppli Laplacian and Bott-Chern Laplacian induced by ω.

Because we have the orthogonal 3-space decomposition [Sch07]:

$$C^\infty_{n-1,n-1}(X,\mathbb{C}) = \ker\Delta_A \oplus (\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial) \oplus \mathrm{Im}(\bar\partial\partial)^*$$

and $\partial\bar\partial\omega^{n-1} = 0$, there is a decomposition

$$\omega^{n-1} = (\omega^{n-1})_h + \partial\Gamma + \bar\partial\Gamma'.$$

Due to another orthogonal 3-space decomposition ([KS60], see also[Sch07]):

$$C^\infty_{1,1}(X,\mathbb{C}) = \ker\Delta_{BC} \oplus \mathrm{Im}(\partial\bar\partial) \oplus (\mathrm{Im}\,\partial^* + \mathrm{Im}\,\bar\partial^*),$$

for all d-closed $(1,1)$-forms α, we have

$$\alpha = \alpha_h + \partial\bar\partial\varphi$$

for some $\varphi \in C^\infty(X,\mathbb{C})$.

$H^{1,1}_{BC}(X,\mathbb{C}) \wedge [\omega^{n-1}]_A = 0$ means for every $\alpha_h \in \ker\Delta_{BC}$, we have

$$\omega^{n-1} \wedge \alpha_h = (\omega^{n-1})_h \wedge \alpha_h + \partial(\alpha_h \wedge \varphi') + \bar\partial(\alpha_h \wedge \psi') \in \mathrm{Im}\partial + \mathrm{Im}\bar\partial.$$

The duality of the two decompositions mentioned above implies

$$\star(\omega^{n-1})_h \in \ker\Delta_{BC}.$$

Therefore, we have

$$(\omega^{n-1})_h \wedge \star(\omega^{n-1})_h = |(\omega^{n-1})_h|^2 dV \in \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}.$$

Hence we have

$$\int_X |(\omega^{n-1})_h|^2 dV = 0.$$

Thus, we deduce $(\omega^{n-1})_h = 0$ and $\omega^{n-1} \in \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$.

(3)($\Rightarrow$) Let Ω be a form as in (1). Suppose there is a non-zero d-closed $(1,1)$-current $T \geq 0$. Because ω^{n-1} is positive definite, we have

$$\int_X T \wedge \Omega = \int_X T \wedge \omega^{n-1} > 0.$$

On the other hand, by the d-closedness of T and the d-exactness of Ω, we know that $T \wedge \Omega$ is d-exact. Therefore, we have

$$\int_X T \wedge \Omega = 0.$$

This would be a contradiction.

($\Leftarrow$)Let $\mathcal{E}'_2(X)$ (resp. $\mathcal{E}'_{1,1}(X)$) be the space of currents of dimension 2 (resp. bidimension $(1,1)$), and let $\mathcal{A}$ be the convex closed subspace of $\mathcal{E}'_2(X)$ of d-closed currents of dimension 2.

Fix a Hermitian metric ω on X. We denote $\mathcal{B} = \{T \in \mathcal{E}'_{1,1}(X) \mid \int_X T \wedge \omega^{n-1} = 1\}$. Then $\mathcal{B}$ is a convex compact subset of $\mathcal{E}'_2(X)$ by [Sul76].

Suppose if there exists no non-zero $(1,1)$-current $T \geq 0$, i.e., $\mathcal{A} \cap \mathcal{B} = \varnothing$. By the Hahn-Banach separation theorem, there exists a linear functional that vanishes identically on $\mathcal{A}$ and is positive on $\mathcal{B}$. That is to say, there exists a d-exact form Ω of degree 2 whose $(1,1)$-component is a Hermitian metric. □

Due to the compactness of X, the d-exact $(2n-2)$-form Ω in Theorem 4.2.10 (1) is $\tilde{d}$(bounded). Hence, it is clear that every degenerate sG metric is sG-hyperbolic.

Corollary 4.2.11. *If a compact complex manifold X is degenerate sG, X is divisorially hyperbolic.*

We have the deformation openness of the degenerate sG condition.

Theorem 4.2.12. *Let $\pi : X \to B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$. If the fibre X_0 is a degenerate sG manifold, then, for all $t \in B$ sufficiently close to 0, the fibre X_t is again a degenerate sG manifold.*

Proof. This is quite obvious because of Theorem 4.2.10(1). Deformation does not change the d-exactness of Ω because it does not change the differentiable structure of the fibre X_0. The $(n-1,n-1)$-component $\Omega_t^{n-1,n-1}$ of Ω with respect to the complex structure of X_t is positive definite for $t \in B$ sufficiently close to 0 due to the continuity of $\Omega_t^{n-1,n-1}$ with respect to t. By Lemma ([Mic82], (4.8)), there exists a metric ω_t such that $\omega_t^{n-1} = \Omega_t^{n-1,n-1}$. □

(b)We now present a more concrete example.

Let G be a semi-simple complex Lie group, and Γ be a co-compact lattice of G. By [MP22a], G/Γ is balanced hyperbolic. An even stronger statement holds: it is actually degenerate balanced (see [Pop15]) by [Yac98].

We have the deformation openness of sG-hyperbolicity, but not of the balanced condition by [AB90]. Now we take $G = SL_2(\mathbb{C})$, Γ a co-compact lattice of G. The deformations of $X := G/\Gamma$ are sG-hyperbolic but not necessarily balanced hyperbolic. We basically follow the process of [Raj94].

We choose a co-compact lattice Γ of non-zero first Betti number. Let K be a maximal compact subgroup of G with an invariant Hermitian metric on G. Fix a maximal torus S of K and a system of positive roots of G with respect to S. Because the first Betti number of the lattice Γ is not zero, $H^{0,1}(X, T^{1,0}X)$ is not zero. Let λ be a highest weight of K on $H^{0,1}(X, T^{1,0}X)$, and V be the corresponding highest weight subspace. By Theorem 3 of [Raj94], G/Γ can be deformed in all directions in V.

(c)By Proposition 4.2.3, we have the following

Corollary 4.2.13. *If X_1 is a degenerate sG manifold and X_2 a balanced hyperbolic manifold, $X_1 \times X_2$ is an sG-hyperbolic manifold.*

But $X_1 \times X_2$ is not necessarily a degenerate sG or balanced hyperbolic manifold.

4.3 p-HS Hyperbolicity and Pluriclosed Star Split Hyperbolicity

Building on our exploration of sG-hyperbolic manifolds and their exemplifications, we now expand our horizon to encompass two other hyperbolicity variants: weakly p-Kähler hyperbolicity and pluriclosed star split hyperbolicity.

Let us recall the following definitions first.

Definition 4.3.1 ([Dem], Chapter III)**.** *1. Let V be a complex vector space of dimension n and let $V^\star$ be its dual. For an integer $1 \leq p \leq n-1$, a (p,p)-form $\alpha \in \Lambda^{p,p}V^\star$ is said to be strictly weakly positive if for all linearly independent $\tau_1, \ldots, \tau_q \in V^\star$, with $q = n-p$, the (n,n)-form*

$$\alpha \wedge i\tau_1 \wedge \overline{\tau_1} \wedge \cdots \wedge i\tau_q \wedge \overline{\tau_q} > 0.$$

2. *Let X be a complex manifold with* $\dim_{\mathbb{C}} = n$. *A C^∞ (p,p)-form α on X is said to be strictly weakly positive if, for every point $x \in X$, $\alpha(x) \in \Lambda^{p,p} T_x^\star X$ is a strictly weakly positive (p,p)-form.*

Definition 4.3.2. *[AA87] A complex manifold X is said to be a p-Kähler manifold if it admits a d-closed strictly weakly positive (p,p)-form, called the p-Kähler form.*

Definition 4.3.3. *[Bel20] Let X be a compact complex manifold of complex dimension n and let Ω be a C^∞ strictly weakly positive (p,p)-form on X. Ω is said to be a p-Hermitian-symplectic form if it is the (p,p)-component of a real d-closed $2p$-form $\hat{\Omega}$.*

Now, we define a type of hyperbolicity based on these definitions.

Definition 4.3.4. *Let X be a compact complex manifold with* $\dim_{\mathbb{C}} X = n \geqslant 2$. *$X$ is said to be* p**-HS hyperbolic** *if there exists a d-closed $2p$-form $\hat{\Omega}$, such that the (p,p)-component Ω of $\hat{\Omega}$ is strictly weakly positive, and $\hat{\Omega}$ is $\tilde{d}$(bounded) with respect to an arbitrary metric ω on X.*

For two metrics ω_1 and ω_2 on X , we have $\frac{1}{C}\omega_1 \leq \omega_2 \leq C\omega_1$ for some constant $C > 0$ because of the compactness of X. Thus, we deduce that the property of subexponential growth of a function is independent of the choice of metric, and that Definition 4.3.4 is well-posed.

Theorem 4.3.5. *Let X be a compact complex manifold of dimension n. If X is p-HS hyperbolic, then there is no holomorphic map $f : \mathbb{C}^p \to X$ such that f is non-degenerate at some point and has subexponential growth (with* $\mathrm{Vol}_{\Omega,f}(B_t) := \int_{B_t} f^*\Omega$*).*

Proof. Suppose there exists a holomorphic map $f : \mathbb{C}^p \to X$ non-degenerate at some point and has subexponential growth.

Let $\pi : \tilde{X} \longrightarrow X$ be the universal cover of X. Fix a metric ω on X. There exists a $\pi^*\omega$-bounded $(2p-1)$-form Γ on $\tilde{X}$, such that $d\Gamma = \pi^*\hat{\Omega}$.

Then $\tilde{f}^*\Gamma$ is $f^*\omega$-bounded:

$$\begin{aligned} \left|\tilde{f}^*\Gamma\left(v_1, \cdots v_{2p-1}\right)\right| &= \left|\Gamma\left(f_*v_1, \cdots, f_*v_{2p-1}\right)\right| \\ &\leqslant C\left|\tilde{f}_*v_1\right|_{\pi^*\omega} \cdots \left|\tilde{f}_*v_{2p-1}\right|_{\pi^*\omega} \\ &= C\left|v_1\right|_{f^*\omega} \cdots \left|v_{2p-1}\right|_{f^*\omega} \end{aligned}$$

for any tangent vectors $v_1, \cdots, v_{2p-1}$ in $\mathbb{C}^p$.

Now, we have

$$\begin{aligned} 0 < \langle f^*\Omega, [B_r]\rangle = \ \mathrm{Vol}_{\ \omega,f}(B_r) &= \int_{Br} f^*\hat{\Omega} \\ &= \int_{Br} d\left(\tilde{f}^*\Gamma\right) \leqslant C\int_{S_r} d\sigma_{\omega,f,r}, \end{aligned}$$

where $d\sigma_{\omega,f,r} = \left(\star_{f^*\omega}\left(\frac{d\tau}{|d\tau|_{f^*\omega}}\right)\right)|_{S_r}$.

By Lemma 4.2.7, the proof is complete. □

Theorem 4.3.6. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \tilde{X} \to X$ be the universal cover of X. If X is p-HS hyperbolic, then there exists no non-zero d-closed positive (n-p,n-p)-current $\tilde{T} \geq 0$ on $\tilde{X}$ such that $\tilde{T}$ is of $L^1_{\pi^*\omega}$.*

Proof. Let $\hat{\Omega}$ be a d-closed $2p$-form as in the definition of weakly p-Kähler hyperbolic manifold. There exists a $L^\infty_{\pi^*\omega}$-form Γ of degree $(2p-1)$ on $\tilde{X}$ such that $\pi^*\hat{\Omega} = d\Gamma$.

For a d-closed $(n-p, n-p)$-current $\tilde{T}$ of $L^1_{\pi^*\omega}$ on $\tilde{X}$, $\tilde{T} \wedge \Gamma$ is again of $L^1_{\pi^*\omega}$. Hence also $d(\tilde{T} \wedge \Gamma)$. Now we have:

$$\int_{\tilde{X}} \tilde{T} \wedge \pi^*\hat{\Omega} = \int_{\tilde{X}} d(\tilde{T} \wedge \Gamma) = 0.$$

However, (p,p)-component of $\pi^*\hat{\Omega}$ is weakly strictly positive. Therefore, there exists no non-zero d-closed positive $(n-p, n-p)$-current $\tilde{T} \geq 0$ on $\tilde{X}$ such that $\tilde{T}$ is of $L^1_{\pi^*\omega}$.

□

Recall the definition of pluriclosed star split metric:

Definition 4.3.7. *[Pop23] Let X be a complex manifold with $dim_{\mathbb{C}}X = n$ and ω a Hermitian metric on X. Let $\star$ be the Hodge star operator induced by ω and ρ_ω the unique $(1,1)$-form such that $i\partial\bar{\partial}\omega_{n-2} = \omega_{n-2} \wedge \rho_\omega$. The metric ω is said to be pluriclosed star split if $\partial\bar{\partial}(\star\rho_\omega) = 0$.*

According to [Pop23], we have $\star\rho_\omega = g\omega_{n-1} - i\partial\bar{\partial}\omega_{n-2}$ for some real-valued C^∞ function g on X satisfying:

$$g > 0 \text{ on } X \quad \text{or} \quad g < 0 \text{ on } X \quad \text{or} \quad g \equiv 0.$$

We have $g \equiv 0$ if and only if $\omega \wedge \partial\bar{\partial}\omega_{n-2} = 0$ if and only if ρ_ω is primitive.

Definition 4.3.8. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$. A metric ω on X is said to be* **pluriclosed star split hyperbolic** *if ω is pluriclosed star split $\pi^*(\star\rho_\omega) = \partial\bar{\Gamma} + \bar{\partial}\Gamma$ on $\tilde{X}$ with Γ $\tilde{\omega}$-bounded.*

Theorem 4.3.9. *Let X be a compact complex manifold of dimension n. If X is pluriclosed star split hyperbolic with ρ_ω not primitive, then X is divisorially hyperbolic.*

Proof. Suppose there exists a holomorphic map $f : \mathbb{C}^{n-1} \to X$ non-degenerate at some point and has subexponential growth.

Let $\pi : \tilde{X} \longrightarrow X$ be the universal cover of X. There exists a $\pi^*\omega$-bounded $(2n-3)$-form Γ on $\tilde{X}$, such that $\pi^*(\omega_{n-1}) = \partial\bar{\Gamma} + \bar{\partial}\Gamma$.

Then $\tilde{f}^*\Gamma$ is $f^*\omega$-bounded:

$$\begin{aligned}\left|\tilde{f}^*\Gamma(v_1, \cdots v_{2n-3})\right| &= |\Gamma(f_*v_1, \cdots, f_*v_{2n-3})| \\ &\leqslant C\left|\tilde{f}_*v_1\right|_{\pi^*\omega} \cdots \left|\tilde{f}_*v_{2n-3}\right|_{\pi^*\omega} \\ &= C|v_1|_{f^*\omega} \cdots |v_{2n-3}|_{f^*\omega}\end{aligned}$$

for any tangent vectors $v_1, \cdots, v_{2n-3}$ in $\mathbb{C}^{n-1}$.

Because of the compactness of X, $\frac{i\bar{\partial}\omega_{n-2}}{g}$ is bounded. Therefore, we have

$$\begin{aligned}
\mathrm{Vol}_{\omega,f}(B_r) &= \int_{B_r} f^*(\omega_{n-1}) \\
&= (n-1)\int_{B_r} f^*\left(\frac{1}{g}\star\rho_\omega + \frac{i}{g}\partial\bar{\partial}\omega_{n-2}\right) \\
&= (n-1)\int_{B_r} d\left(\tilde{f}^*\left(\frac{\Gamma+\bar{\Gamma}}{g}\right) + f^*\left(\frac{i\bar{\partial}\omega_{n-2}}{g}\right)\right) \\
&\leqslant C\int_{S_r} d\sigma_{\omega,f,r},
\end{aligned}$$

where $d\sigma_{\omega,f,r} = \left(\star_{f^*\omega}\left(\frac{d\tau}{|d\tau|_{f^*\omega}}\right)\right)|_{S_r}$.

By Lemma 4.2.7, the proof is complete. □

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