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Feedback stabilization of a boundary layer equation, Part 2: nonhomogeneous state equations and numerical simulations

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Abstract: We study the feedback stabilization of a fluid flow over a flat plate, around a stationary solution, in the presence of known perturbations. The feedback law is determined by solving a Linear-Quadratic optimal control problem. The observation is the laminar-to-turbulent transition location linearized about its stationary position, the control is a suction velocity through a small slot in the plate, the state equation is the linearized Crocco equation about its stationary solution. This paper is the continuation of [7] where we have studied the corresponding Linear-Quadratic control problem in the absence of perturbations. The solution to the algebraic Riccati equation determined in [7], together with the solution of an evolution equation taking into account the nonhomogeneous perturbations in the model, are used to define the feedback control law.

KEY WORDS Feedback control law, Riccati equation, boundary layer equation, unbounded control operator, degenerate parabolic system.

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1 Introduction.

We are interested in the feedback stabilization of a fluid flow over a flat plate, around a stationary solution, in the presence of known perturbations. The control variable is a suction velocity through a small slot near the leading edge of the plate. This paper is the continuation of [7] in which we have studied an algebraic Riccati equation that we are going to use to define our feedback control law.

Let Ω be the open set $(0,L)\times(0,1)$, $L>0$, and denote by γ an interval $(x_0,x_1)\subset(0,L)$. Let us consider the degenerate parabolic equation:

$$\left\{\begin{array}{ll}
\dfrac{\partial z}{\partial t} = -a(\eta)\dfrac{\partial z}{\partial \xi} + b(\xi,\eta)\dfrac{\partial^2 z}{\partial \eta^2} - c(\xi,\eta)z + f & (t,\xi,\eta)\in\ (0,\infty)\times\Omega,\\
z(0,\xi,\eta) = z_0(\xi,\eta) & (\xi,\eta)\in\ \Omega,\\
\sqrt{a(\eta)}\,z(t,0,\eta) = \sqrt{a(\eta)}\,z_b(t,\eta) & (t,\eta)\in(0,\infty)\times(0,1),\\
(bz)(t,\xi,1)=0, \quad \dfrac{\partial z}{\partial \eta}(t,\xi,0) = u(t,\xi)\,\mathbb{1}_\gamma(\xi) + g(t,\xi) & (t,\xi)\in(0,\infty)\times(0,L),
\end{array}\right. \tag{1}$$

where u is a control variable, νu corresponds to a suction velocity, the positive constant ν is the viscosity of the fluid, and $\mathbb{1}_\gamma$ is the characteristic function of $\gamma=(x_0,x_1)\subset(0,L)$. Equation (1) comes from the linearization of a boundary layer equation (Prandtl' equations written in the so-called Crocco variables) about a stationary solution w_s. This linearized model is introduced in [6] and [7]. We refer to [18] for the corresponding nonlinear equation and the Crocco transformation. The function w_s is the solution to the (nonlinear) stationary Crocco equation:

$$\left\{\begin{array}{lll}
U_\infty^s\,\eta\,\dfrac{\partial w_s}{\partial \xi} - \nu w_s^2\dfrac{\partial^2 w_s}{\partial \eta^2} = 0 & \text{in} & (0,L)\times(0,1),\\
\nu\left(w_s\dfrac{\partial w_s}{\partial \eta}\right)(\xi,0) = v_s(\xi)\,w_s(\xi,0), \quad \lim\limits_{\eta\to 1} w_s(\xi,\eta)=0 & \text{for} & \xi\in(0,L),\\
w_s(0,\eta) = w_b(\eta) & \text{for} & \eta\in(0,1).
\end{array}\right. \tag{2}$$

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Here $(0, L)$ represents a part of the plate where the flow is laminar, $(0, 1)$ is the thickness of the boundary layer in the Crocco variables, $U_\infty^s > 0$ is a constant corresponding to the first component of the incident velocity of the flow, w_b is the velocity profile in Crocco variables at $\xi = 0$, v_s is a stationary suction velocity through the plate. Recall that the transformation used to rewrite the Prandtl equations into the Crocco equation is

$$\xi = x, \qquad \eta = \frac{u_s(x,y)}{U_\infty^s}, \qquad w(\xi,\eta) = \frac{1}{U_\infty^s}\frac{\partial u_s}{\partial y}(x,y)\,, \tag{3}$$

see [18], when (u_s, v_s) is the stationary solution of the Prandtl system, and $(x, y) \in (0, L) \times (0, \infty)$. Assuming that the regularity and compatibility conditions between w_b and v_s stated in [18, Theorem 3.3.2]) are satisfied, the stationary equation (2) admits a unique solution w_s in the class of functions satisfying

$$\begin{aligned} & w \in C_b(\Omega), \quad K_1|1-\eta| \le w(\xi,\eta) \le K_2|1-\eta|, \quad \left|\frac{\partial w}{\partial \xi}\right| \le K_3|1-\eta|, \\ & \frac{\partial w}{\partial \eta} \in L^\infty(\Omega), \qquad w\frac{\partial^2 w}{\partial \eta^2} \in L^\infty(\Omega), \qquad \frac{\partial w}{\partial \xi} \in L^\infty(\Omega)\,, \end{aligned} \tag{4}$$

where K_1, K_2, and K_3 are positive constants. The coefficients a, b, c depend on the stationary solution w_s of the Crocco equation, and are defined by:

$$a = U_\infty^s \eta, \quad b = \nu(w_s)^2, \quad c = -2w_s\frac{\partial^2 w_s}{\partial \eta^2}.$$

The aim of this paper is to provide a feedback control law able to stabilize the so-called laminar-to-turbulent transition location of the flow, around its stationary position X_{LT}^s, when the incident velocity is of the form $(U_\infty(t), 0)$, with

$$U_\infty(t) = U_\infty^s + u_\infty(t).$$

The instationary perturbation $u_\infty(t)$ is supposed to be small with respect to the constant U_∞^s. In the linearized model (1), f and g are related to the perturbation $u_\infty(t)$ in the following way:

$$f(t,\xi,\eta) = u_\infty(t)d(\xi,\eta) + \frac{u_\infty'(t)}{U_\infty^s}e(\xi,\eta), \qquad g(t,\xi) = -\frac{1}{\nu w_s(\xi,0)}\frac{u_\infty'(t)}{U_\infty^s},$$

with

$$d = -\eta\,\frac{\partial w_s}{\partial \xi} \quad \text{and} \quad e = -w_s - (1-\eta)\frac{\partial w_s}{\partial \eta}.$$

The laminar-to-turbulent transition location at time t, denoted by $X_{LT}(t)$, depends in a nonlinear and complicated way on $u_\infty(t)$. However a first order Taylor expansion of $X_{LT}(t)$ gives an accurate approximation when $u_\infty(t)$ is small with respect to U_∞^s. It can be shown that this approximation of $X_{LT}(t)$ is of the form

$$\int_\Omega \phi(\xi,\eta)\, z(t,\xi,\eta)\, d\xi d\eta + c_0 u_\infty(t) = Cz(t) + c_0 u_\infty(t),$$

where the function $\phi \in L^2(\Omega)$ and the constant c_0 can be calculated accurately [1, 5]. Therefore we intend to determine a feedback control law, able to stabilize $|X_{LT}(t) - X_{LT}^s|$, by solving the Linear-Quadratic optimal control problem:

$$(P) \qquad \mathrm{Inf}\Big\{J(z,u) \;\mid\; (z,u) \in L^2(0,\infty;L^2(\Omega)) \times L^2(0,\infty;L^2(0,L)),\; (z,u) \text{ satisfies } (1)\Big\},$$

where

$$J(z,u) = \frac{1}{2}\int_0^\infty |Cz(t) + c_0 u_\infty(t)|^2\,dt + \frac{1}{2}\int_0^\infty\int_0^L |u|^2\,.$$

In [7], we have already shown that equation (1) can be rewritten in the form

$$z' = \mathcal{A}z + f + B(\mathbb{1}_\gamma u + g) + (-\mathcal{A})Dz_b, \qquad z(0) = z_0\,,$$

where the operator $\mathcal{A}$, B, D are precisely defined in section 2 (see also [7]), and that (P) admits a unique solution $(\bar z, \bar u)$ characterized by the optimality system:

$$\left\{\begin{array}{ll} \bar z' = \mathcal{A}\bar z + f + B(\mathbb{1}_\gamma \bar u + g) + (-\mathcal{A})Dz_b, & \bar z(0) = z_0, \\ -p' = \mathcal{A}^* p + C^*(C\bar z + c_0 u_\infty), & p(\infty) = 0, \\ \bar u = -\mathbb{1}_\gamma B^* p. & \end{array}\right. \tag{5}$$

Even if the obtention of the optimality system is quite standard [7], studying the algebraic Riccati equation associated with (P) is much more complicated because of the degenerate character of the operator $\mathcal{A}$ (see [7]). This Riccati equation is obtained by considering the particular case where $z_b = 0$ and $u_\infty = 0$ in the definition of (P) (thus $f = 0$ and $g = 0$). In that case the optimality system is homogeneous. In the present paper we want to find the optimal control $\bar{u}$, in feedback form when the optimality system (5) is nonhomogeneous, which corresponds to the interesting case for applications.

Let us finally mention that u_∞ is supposed to be small and known. Actually it is quite standard to take into account, in feedback control laws, such nonhomogeneous terms appearing in the state equation and the cost functional (see e.g. [3]). We can observe, in test 3 of section 6, that the tracking variable (taking into account the nonhomogeneous terms of the control problem) may significantly influence the optimal solution. In an experiment in a wind tunnel, it may correspond to a given variation of the inflow velocity.

2 Assumptions an preliminary results

In this section we recall some notations and results already introduced in [6] and [7], but which are necessary in what follows. In particular we want to define the operators $\mathcal{A}$, B, and D.

2.1 Assumptions

As in [6, 7], we make the following assumptions on the coefficients a, b, and c.

(H_1) $a(\eta) = U_\infty^s\,\eta$, and $b \in W^{1,\infty}(\Omega)$. There exist positive constants C_i, $i = 1$ to 4, such that

$$\begin{aligned} &C_1|1-\eta|^2 \le b(\xi,\eta) \le C_2|1-\eta|^2, \\ &\left|\frac{\partial b}{\partial \eta}(\xi,\eta)\right| \le C_3|1-\eta| \qquad \text{and} \qquad \left|\frac{\partial b}{\partial \xi}(\xi,\eta)\right| \le C_4|1-\eta|^2 \qquad \text{for all } (\xi,\eta)\in\Omega. \end{aligned} \tag{6}$$

(H_2) The coefficient c belongs to $L^\infty(\Omega)$, and we denote by C_0 a positive constant such that

$$\|c\|_{L^\infty(\Omega)} \le C_0. \tag{7}$$

The nonhomogeneous terms f, g, z_b, u_∞, and the initial condition z_0 satisfy

(H_3) $z_0 \in L^2(\Omega)$, $z_b \in L^2(0,\infty; L^2(0,1))$, $u_\infty \in L^2(0,\infty)$, and $g \in L^2(0,\infty; L^2(0,L))$.
(H_4) $f \in L^2(0,\infty; L^2(\Omega))$.

2.2 The operators $\mathcal{A}$ and $\mathcal{A}^*$

Let us recall some notation introduced in [6]. Let $H^1(0,1;d)$ be the closure of $C^\infty([0,1])$ in the norm:

$$\|z\|_{H^1(0,1;d)} = \left(\int_0^1 |z|^2 + |1-\eta|^2 \left|\frac{\partial z}{\partial \eta}\right|^2 d\eta\right)^{1/2}. \tag{8}$$

To take the Dirichlet boundary condition $bz(\xi,1,t) = 0$ into account, we denote by $H^1_{\{1\}}(0,1;d)$ the closure of $C_c^\infty([0,1))$ in the norm $\|\cdot\|_{H^1(0,1;d)}$. According to Triebel [21, Theorem 2.9.2]

$$H^1(0,1;d) = H^1_{\{1\}}(0,1;d).$$

Let us set

$$\Gamma_0 = \big([0,L)\times\{0\}\big) \cup \big(\{0\}\times(0,1)\big), \qquad \Gamma_1 = \big(\{L\}\times(0,1)\big) \cup \big((0,L]\times\{1\}\big).$$

If the vector field $\left(az, -b\dfrac{\partial z}{\partial \eta}\right)$ belongs to $(L^2(\Omega))^2$, and its divergence belongs to $L^2(\Omega)$, the normal trace on the boundary Γ of the vector field $\left(az, -b\dfrac{\partial z}{\partial \eta}\right)$ belongs to $H^{-1/2}(\Gamma)$. We denote this normal trace by $T\Big(az, -b\frac{\partial z}{\partial \eta}\Big)$. We can define $T_0\Big(az, -b\frac{\partial z}{\partial \eta}\Big)$ as an element in $(H^{1/2}_{00}(\Gamma_0))'$ in the following way

$$\left\langle T_0\Big(az, -b\frac{\partial z}{\partial \eta}\Big), \phi\right\rangle_{(H^{1/2}_{00}(\Gamma_0))', H^{1/2}_{00}(\Gamma_0)} = \left\langle T\Big(az, -b\frac{\partial z}{\partial \eta}\Big), \phi\right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for all $\phi \in H^{1/2}_{00}(\Gamma_0)$.

Similarly, if the vectorfield $\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$ belongs to $(L^2(\Omega))^2$, and its divergence belongs to $L^2(\Omega)$, the normal trace on the boundary Γ of the vectorfield $\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$, denoted by $T\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$, belongs to $H^{-1/2}(\Gamma)$, and we can define $T_1\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$ as an element in $(H^{1/2}_{00}(\Gamma_1))'$ by

$$\Big\langle T_1\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big), \phi\Big\rangle_{(H^{1/2}_{00}(\Gamma_1))', H^{1/2}_{00}(\Gamma_1)} = \Big\langle T\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big), \phi\Big\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for all $\phi \in H^{1/2}_{00}(\Gamma_1)$. We refer to [7] and to Dautray-Lions [12, Chapter 7, Section 2, Remark 1] for a precise definition of the spaces $H^{1/2}_{00}(\Gamma_0)$ and $H^{1/2}_{00}(\Gamma_1)$.

The differential operators A and A^* are defined by

$$Az = -a\,\frac{\partial z}{\partial\xi} + b\,\frac{\partial^2 z}{\partial\eta^2} - cz\,, \qquad A^*z = a\frac{\partial z}{\partial\xi} + \frac{\partial^2(bz)}{\partial\eta^2} - cz\,.$$

Associated with the above differential operators A and A^*, we define two unbounded operators in $L^2(\Omega)$ as follows:

$$\begin{aligned}
&D(\mathcal{A}) = \Big\{z \in L^2(0,L; H^1(0,1;d)) \mid Az \in L^2(\Omega),\ T_0\Big(az, -b\frac{\partial z}{\partial\eta}\Big) = 0\Big\},\\
&\mathcal{A}z = Az \quad \text{for all } z \in D(\mathcal{A}),\\
&D(\mathcal{A}^*) = \Big\{z \in L^2(0,L; H^1(0,1;d)) \mid A^*z \in L^2(\Omega),\ T_1\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big) = 0\Big\},\\
&\mathcal{A}^*z = A^*z \quad \text{for all } z \in D(\mathcal{A}^*)\,.
\end{aligned}$$

According to [6, Theorem 5.9], $(\mathcal{A}^*, D(\mathcal{A}^*))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$. Moreover we have the following theorem.

Theorem 2.1. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous exponentially stable semigroup on $L^2(\Omega)$.*

2.3 The Dirichlet and Neumann operators

Let v belong to $L^2(0,L)$, and $z_b \in L^2(0,1)$. We can define the solution to the Neumann problem

$$Aw = 0 \text{ in } \Omega, \quad \sqrt{a}\,w(0,\cdot) = 0 \text{ in } (0,1), \quad (bw)(\cdot,1) = 0 \ \text{ and } \ \frac{\partial w}{\partial\eta}(\cdot,0) = v \quad \text{in } (0,L), \tag{9}$$

and to the Dirichlet problem

$$A\zeta = 0 \text{ in } \Omega, \quad \sqrt{a}\zeta(0,\cdot) = \sqrt{a}z_b \text{ in } (0,1), \quad (bw)(\cdot,1) = 0 \ \text{ and } \ \frac{\partial w}{\partial\eta}(\cdot,0) = 0 \quad \text{in } (0,L), \tag{10}$$

by the transposition method (see [7, Definition 2.2]). For existence and regularity results to equations (9) and (10) we refer to [7, Theorem 2.4].

We denote by N and D the operators defined by

$$Nv = w, \qquad Dz_b = \zeta,$$

where w is the solution to equation (9), and ζ is the solution to equation (10).

Recall that N belongs to $\mathcal{L}(L^2(0,L); L^2(0,L;H^1(0,1;d)))$, and that D belongs to $\mathcal{L}(L^2(0,1); L^2(0,L;H^1(0,1;d)))$. Moreover, we have (see [7])

$$N^*A^*p = -b(\xi,0)p(\xi,0) \qquad \text{and} \qquad D^*A^*p = -a(\eta)p(0,\eta) \qquad \text{for all } p \in D(\mathcal{A}^*).$$

Using the extrapolation method the semigroup $(e^{t\mathcal{A}})_{t\in\mathbb{R}^+}$ can be extended to $(D(\mathcal{A}^*))'$. Denoting the corresponding semigroup by $(e^{t\widehat{\mathcal{A}}})_{t\in\mathbb{R}^+}$, the generator $(\widehat{\mathcal{A}}, D(\widehat{\mathcal{A}}))$ of this semigroup is an unbounded operator in $(D(\mathcal{A}^*))'$ with domain $D(\widehat{\mathcal{A}}) = L^2(\Omega)$.

In [7] we have shown that equation (1) can be rewritten in the form

$$z' = \widehat{\mathcal{A}}z + f + (-\widehat{\mathcal{A}})Ng + (-\widehat{\mathcal{A}})N(\mathbb{1}_\gamma\, u) + (-\widehat{\mathcal{A}})Dz_b, \qquad z(0) = z_0. \tag{11}$$

In the following, we make the abuse of notation consisting in replacing $\widehat{\mathcal{A}}$ by $\mathcal{A}$ and we set $B = (-\mathcal{A})N$. For the definition of weak solutions to equation (1), for estimates and regularity results, we refer to [6] and to [7].

2.4 The optimal control problem

Let us recall results obtained in [7] for the control problem (P).

Theorem 2.2 ([7], Theorems 4.1 and 4.2)**.** *Assume that* (H_1)–(H_4) *are fulfilled. Then problem* (P) *admits a unique solution* $(\bar z, \bar u)$*. The optimal control* $\bar u$ *is given by*

$$\bar u = \mathbb{1}_\gamma\, b\, p,$$

where p *is the solution to the equation*

$$-p' = \mathcal{A}^* p + C^*(Cz + c_0 u_\infty) \quad \textit{in } (0,\infty), \quad p(\infty) = 0.$$

Conversely if a pair $(z,p) \in \big(L^2(0,\infty; H^1(0,1;d))\big)^2$ *obeys the system*

$$\begin{cases} z' = \mathcal{A}z + f + B(\mathbb{1}_\gamma\, b(\cdot,0)p(\cdot,0) + g) + (-\mathcal{A})Dz_b & \textit{in } (0,\infty), \quad z(0) = z_0, \\ -p' = \mathcal{A}^* p + C^*(Cz + c_0 u_\infty) & \textit{in } (0,\infty), \quad p(\infty) = 0, \end{cases} \tag{12}$$

then the pair $(z, \mathbb{1}_\gamma\, b\, p)$ *is the optimal solution to problem* (P)*.*

We are going to see that $\bar u$ obeys a feedback formula of the form

$$\bar u = -\mathbb{1}_\gamma\, B^*(\Pi z(t) + r(t)). \tag{13}$$

The operator Π has already been studied in [7], where we have shown that it is the unique solution to an algebraic Riccati equation. In the following section we recall results concerning the operator Π, while the equation satisfied by r is studied in section 4.

3 Algebraic Riccati equation

In [7] the operator $\Pi \in \mathcal{L}(L^2(\Omega))$ has been characterized by a kernel $\pi \in L^2(\Omega\times\Omega)$. For clarity we have written $\Omega\times\Omega$ in the form $\Omega_X \times \Omega_\Xi$. The current point $(X,\Xi) \in \Omega_X\times\Omega_\Xi$ corresponds to $X = (x,y) \in \Omega_X$ and $\Xi = (\xi,\eta) \in \Omega_\Xi$. Similarly, A_X^* (resp. A_Ξ^*) corresponds to the operator A^* written in X-variable (resp. in Ξ-variable), that is:

$$A_X^* p = a\frac{\partial p}{\partial x} + \frac{\partial^2 (bp)}{\partial y^2} - c\,p.$$

With this notation Π and π are related by the identity

$$\Pi z(X) = \int_\Omega \pi(X,\Xi) z(\Xi) d\Xi\,.$$

Let us set $\mathcal{O} = \Omega_X\times\Omega_\Xi$. If $z \in L^2(\Omega)$ and $\zeta \in L^2(\Omega)$, we denote by $z\otimes\zeta$ the function belonging to $L^2(\mathcal{O})$ defined by

$$z\otimes\zeta \;:\; (X,\Xi) \longmapsto z(X)\zeta(\Xi)\,.$$

Lemma 3.1 ([7], Lemma 5.3)**.** *For* $t \ge 0$*, let* $S^*(t) \in \mathcal{L}(L^2(\mathcal{O}))$ *be defined by*

$$S^*(t) \;:\; \psi \longmapsto e^{t\mathcal{A}_X^*} e^{t\mathcal{A}_\Xi^*}\psi\,.$$

The family $(S^*(t))_{t\ge 0}$ *is a strongly continuous exponentially stable semigroup on* $L^2(\mathcal{O})$*.*

We denote by $L_s^2(\mathcal{O})$ the space of functions $\pi \in L^2(\mathcal{O})$ satisfying:

$$\pi(X,\Xi) = \pi(\Xi,X) \qquad \text{for almost all } (X,\Xi) \in \Omega_X\times\Omega_\Xi\,.$$

The restriction of $\mathcal{A}_{X,\Xi}^*$ to $L_s^2(\mathcal{O})$ is an unbounded operator in $L_s^2(\mathcal{O})$ whose domain is defined by

$$D(\mathcal{A}_{X,\Xi}^{s\,*}) = D(\mathcal{A}_{X,\Xi}^*) \cap L_s^2(\mathcal{O})\,.$$

Theorem 3.2 ([7], Theorem 5.2)**.** *The operator* $(\mathcal{A}_{X,\Xi}^{s\,*}, D(\mathcal{A}_{X,\Xi}^{s\,*}))$ *is the infinitesimal generator of an exponentially stable semigroup on* $L_s^2(\mathcal{O})$*.*

We denote by $L^2_+(\mathcal{O})$ the cone in $L^2_s(\mathcal{O})$ of functions π satisfying:

$$\int_{\mathcal{O}} \pi\, z\otimes z \geq 0 \qquad \text{for all } z\in L^2(\Omega)\,.$$

Theorem 3.3 ([7], Theorem 6.2)**.** *The algebraic Riccati equation*

$$\pi\in D(\mathcal{A}^{s\ *}_{X,\Xi})\cap L^2_+(\mathcal{O}),\quad \mathcal{A}^*_X\pi+\mathcal{A}^*_\Xi\pi-\int_\gamma |b(s,0)|^2\pi(s,0,\Xi)\pi(X,s,0)\,ds\,+\phi(X)\phi(\Xi)=0\,, \tag{14}$$

admits a unique solution.

In [7, Theorem 6.4] we have shown that the infinitesimal generator of the closed loop system obtained by solving the homogeneous control problem is the operator $(\mathcal{A}_\pi, D(\mathcal{A}_\pi))$ whose domain is

$$D(\mathcal{A}_\pi)=\Big\{z\in L^2(0,L;H^1(0,1;d))\ |\\ Az\in L^2(\Omega),\ T_0\Big(az,-b\frac{\partial z}{\partial\eta}\Big)=-\mathbb{1}_\gamma\, b(s,0)^2\int_\Omega \pi(X,s,0)\,z(X)\,dX\Big\}$$

and such that the equation

$$z\in D(\mathcal{A}_\pi),\qquad \mathcal{A}_\pi z=\psi,$$

when $\psi\in L^2(\Omega)$, is equivalent to the variational problem

$$\begin{aligned}&z\in L^2(0,L;H^1(0,1;d)),\\ &Az=\psi\quad\text{in }\Omega,\qquad T_0\Big(az,-b\frac{\partial z}{\partial\eta}\Big)=-\mathbb{1}_\gamma\, b(s,0)^2\int_\Omega \pi(X,s,0)\,z(X)\,dX.\end{aligned}$$

4 Equation satisfied by r

We are going to see that the function r in (13) obeys the equation

$$\left\{\begin{aligned}&-r'(t,\Xi)=\mathcal{A}^*_\pi r(t,\Xi)+\int_\Omega \pi(X,\Xi)f(t,X)\,dX\\ &\qquad\qquad -\int_0^L \pi(x,0,\Xi)b(x,0)g(t,x)\,dx+\int_0^1 \pi(0,y,\Xi)a(y)z_b(t,y)\,dy+c_0\,u_\infty(t)\,\phi(\Xi),\\ &r(\infty,\Xi)=0.\end{aligned}\right. \tag{15}$$

Equation (15) is of the form

$$-r'(t,\Xi)=\mathcal{A}^*_\pi r(t,\Xi)+h(t,\Xi),\qquad r(\infty,\Xi)=0, \tag{16}$$

where h belongs to $L^2(0,\infty;L^2(\Omega))$. Since $(\mathcal{A}_\pi, D(\mathcal{A}_\pi))$ is the infinitesimal generator of an exponentially stable semigroup on $L^2(\Omega)$, we can claim that $(\mathcal{A}^*_\pi, D(\mathcal{A}^*_\pi))$ is the infinitesimal generator of an exponentially stable semigroup on $L^2(\Omega)$, that the solution r to equation (16) is defined by

$$r(t)=\int_t^\infty e^{(\tau-t)\mathcal{A}^*_\pi}h(\tau)d\tau,$$

and that it obeys

$$\|r\|_{L^2(0,\infty;L^2(\Omega))}\leq C\|h\|_{L^2(0,\infty;L^2(\Omega))}. \tag{17}$$

But in order to prove the feedback formula (13), we have to prove some regularity results for r, and we have to give some explicit expression for $\mathcal{A}^*_\pi$. More precisely we would like to show that the solution r to equation (16) also obeys the equation

$$-r'(t,\Xi)=\mathcal{A}^*r(t,\Xi)-\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds+h(t,\Xi),\qquad r(\infty,\Xi)=0. \tag{18}$$

For that we characterize the operator $(\mathcal{A}^*_\pi, D(\mathcal{A}^*_\pi))$ in the next section.

4.1 Adjoint operator of $(\mathcal{A}_\pi, D(\mathcal{A}_\pi))$

Let us define the unbounded operator $(\mathcal{A}_\pi^\sharp, D(\mathcal{A}_\pi^\sharp))$ by

$$D(\mathcal{A}_\pi^\sharp) = D(\mathcal{A}^*) \quad \text{and} \quad \mathcal{A}_\pi^\sharp \phi = \mathcal{A}^*\phi - \int_\gamma b(s,0)^2 \phi(s,0)\pi(s,0,\cdot)\, ds.$$

Lemma 4.1. *For all $z \in D(\mathcal{A}_\pi)$ and $\phi \in D(\mathcal{A}_\pi^\sharp)$ the following Green formula holds*

$$\int_\Omega \mathcal{A}_\pi^\sharp \phi\, z = \int_\Omega \phi\, \mathcal{A}_\pi z.$$

Proof. We can use a regularization argument, as in [6, Theorem 4.14], and the characterization of $D(\mathcal{A}_\pi)$.

Theorem 4.2. *The operator $(\mathcal{A}_\pi^\sharp, D(\mathcal{A}_\pi^\sharp))$ is the adjoint of $(\mathcal{A}_\pi, D(\mathcal{A}_\pi))$.*

Proof. For $k > 0$, we define $(\mathcal{A}_{k,\pi}, D(\mathcal{A}_{k,\pi}))$ by

$$D(\mathcal{A}_{k,\pi}) = D(\mathcal{A}_\pi) \quad \text{and} \quad \mathcal{A}_{k,\pi} z = \mathcal{A}_\pi z - kaz.$$

Similarly, we define $(\mathcal{A}_{k,\pi}^\sharp, D(\mathcal{A}_{k,\pi}^\sharp))$ by

$$D(\mathcal{A}_{k,\pi}^\sharp) = D(\mathcal{A}_\pi^\sharp) \quad \text{and} \quad \mathcal{A}_{k,\pi}^\sharp z = \mathcal{A}_\pi^\sharp z - kaz.$$

To prove the theorem it is necessary and sufficient to show that, for some $k > 0$, $(\mathcal{A}_{k,\pi}^\sharp, D(\mathcal{A}_{k,\pi}^\sharp))$ is the adjoint of $(\mathcal{A}_{k,\pi}, D(\mathcal{A}_{k,\pi}))$.

Step 1. First prove that we can choose $k > 0$ big enough so that

$$\big(-\mathcal{A}_{k,\pi}^\sharp z, z\big) \geq \frac{C_1}{2}\|z\|^2_{L^2(0,L;H^1(0,1;d))}. \tag{19}$$

Let $C_\gamma > 0$ be such that $|z(0)| \leq C_\gamma \|z\|_{H^1(0,1;d)}$. Observe that

$$\begin{aligned}
&\Big|\int_\Omega \int_\gamma b(s,0)^2 z(s,0)\pi(s,0,\Xi)\, ds\, z(\Xi)\, d\Xi\Big| \\
&\leq C_2^2 C_\gamma^2 \|z\|_{L^2(0,L;H^1(0,1;d))} \|\pi\|_{L^2(\Omega_\Xi, L^2(0,L;H^1(0,1;d)))} \|z\|_{L^2(\Omega)} \\
&\leq C_2^2 C_\gamma^2 \|\pi\|^2_{L^2(\Omega_\Xi, L^2(0,L;H^1(0,1;d)))} \frac{\varepsilon}{2} \|z\|^2_{L^2(0,L;H^1(0,1;d))} + \frac{C_2^2 C_\gamma^2}{2\varepsilon}\|z\|^2_{L^2(\Omega)}
\end{aligned}$$

We set $C_5 = C_2^2 C_\gamma^2 \|\pi\|^2_{L^2(\Omega_\Xi, L^2(0,L;H^1(0,1;d)))}$. Arguing as in [6], we can write

$$\begin{aligned}
\big(-\mathcal{A}_{k,\pi}^\sharp z, z\big) &= \big(-\mathcal{A}^* z, z\big) + \int_\Omega \int_\gamma b(s,0)^2 z(s,0)\pi(s,0,\Xi)\, ds\, z(\Xi)\, d\Xi + \int_\Omega k\ a\, z^2\, d\Xi \\
&= \int_\Omega \Big(b\Big|\frac{\partial z}{\partial \eta}\Big|^2 + \frac{\partial b}{\partial \eta}\frac{\partial z}{\partial \eta} z + (c+ka)z^2\Big) + \int_\Omega \int_\gamma b(s,0)^2 z(s,0)\pi(s,0,\Xi)\, ds\, z(\Xi)\, d\Xi. \\
&\geq \Big(C_1 - \frac{C_3\varepsilon}{2} - \frac{C_5\varepsilon}{2}\Big)\|z\|^2_{L^2(0,L;H^1(0,1;d))} \\
&\qquad + \int_\Omega \left(-C_0 + ka - \frac{1}{2}\left(2C_1 + \frac{C_3}{\varepsilon} + \frac{C_2^2 C_\gamma^2}{\varepsilon}\right)\right)|z|^2 d\Xi.
\end{aligned}$$

We choose $\varepsilon > 0$ such that $\frac{C_1}{2} - \frac{C_3\varepsilon}{2} - \frac{C_5\varepsilon}{2} \geq 0$. Next we choose $k > 0$ such that

$$\left(-C_0 + ka - \frac{1}{2}\left(2C_1 + \frac{C_3}{\varepsilon} + \frac{C_2^2 C_\gamma^2}{\varepsilon}\right)\right) \geq 0,$$

and (19) is proved.

Step 2. From (19) and from Lax-Milgram Theorem it follows that, for all $f \in L^2(\Omega)$, the equation

$$-\mathcal{A}_{k,\pi}^\sharp z = f,$$

admits a unique solution z in $L^2(0,L;H^1(0,1;d))$.

Step 3. Let us show that $D(\mathcal{A}^\sharp_{k,\pi}) = D(\mathcal{A}^*_{k,\pi})$. For all $\phi \in D(\mathcal{A}^\sharp_{k,\pi})$, we have

$$\int_\Omega \mathcal{A}^\sharp_{k,\pi}\phi\, z\, d\Xi = \int_\Omega \mathcal{A}^*_k \phi\, z\, d\Xi - \int_\Omega\int_\gamma b(s,0)^2\phi(s,0)\pi(s,0,\Xi)\, ds\, z(\Xi)\, d\Xi \le C\|z\|_{L^2(\Omega)}.$$

Thus $D(\mathcal{A}^\sharp_{k,\pi}) \subset D(\mathcal{A}^*_{k,\pi})$. Let us prove the reverse inclusion. Let ϕ belong to $D(\mathcal{A}^*_{k,\pi}) = D(\mathcal{A}^*_\pi)$. From the definition of $D(\mathcal{A}^*_{k,\pi})$, it follows that $\mathcal{A}^*_{k,\pi}\phi$ can be identified with an element of $L^2(\Omega)$. Let ψ be this element, that is the unique function in $L^2(\Omega)$ satisfying

$$\int_\Omega \phi\,\mathcal{A}_{k,\pi} z = \int_\Omega \psi\, z \quad \text{for all } z \in D(\mathcal{A}_{k,\pi}).$$

Due to Step 2, there exists a unique $\tilde\phi$ satisfying

$$\tilde\phi \in D(\mathcal{A}^\sharp_{k,\pi}), \qquad \mathcal{A}^\sharp_{k,\pi}\tilde\phi = \psi.$$

For all $z \in D(\mathcal{A}_{k,\pi})$, we have

$$\int_\Omega \phi\,\mathcal{A}_{k,\pi} z = \int_\Omega \mathcal{A}^\sharp_{k,\pi}\tilde\phi\, z = \int_\Omega \tilde\phi\,\mathcal{A}_{k,\pi} z.$$

Thus $\phi = \tilde\phi$. We have proved that ϕ belongs to $D(\mathcal{A}^\sharp_{k,\pi})$, and that

$$\mathcal{A}^*_{k,\pi}\phi = \mathcal{A}^\sharp_{k,\pi}\phi.$$

The proof is complete.

4.2 Equation satisfied by r

We first consider the case when the support of h is compact, e.g. we assume that the support of h is included in $(0,T)$ for some $T>0$. In that case both equations (16) and (15) can be stated with the terminal condition $r(T)=0$ in place of $r(\infty)=0$.

Definition 4.3. *Assume that the support of h is included in $(0,T)\times\Omega$ for some $T>0$. A function $r \in L^2(0,\infty;L^2(\Omega))$ is a weak solution to equation* (16) *if and only if*

$$\int_0^T\int_\Omega r\,\phi\,\theta'\, d\Xi\, dt = \int_0^T\int_\Omega r\mathcal{A}_\pi\phi\,\theta\, d\Xi\, dt + \int_0^T\int_\Omega h\,\phi\,\theta\, d\Xi\, dt,$$

for all $\theta \in C^1_c((0,T])$, and all $\phi \in D(\mathcal{A}_\pi)$.

Definition 4.4. *Assume that the support of h is included in $(0,T)\times\Omega$ for some $T>0$. A function $r \in L^2(0,\infty;L^2(0,L;H^1(0,1,d)))$ is a weak solution to equation* (18) *if and only if*

$$\int_0^T\int_\Omega r\,\phi\,\theta'\, d\Xi\, dt = \int_0^T\int_\Omega r\mathcal{A}\phi\,\theta\, d\Xi\, dt - \int_0^T\int_\Omega\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)\, ds\,\phi(\Xi)\,\theta\, d\Xi\, dt$$
$$+\int_0^T\int_\Omega h\,\phi\,\theta\, d\Xi\, dt,$$

for all $\theta \in C^1_c((0,T])$, and all $\phi \in D(\mathcal{A})$.

Theorem 4.5. *Assume that the support of h is included in $(0,T)\times\Omega$ for some $T>0$. If $r \in L^2(0,\infty;L^2(0,L;H^1(0,1,d)))$ is a weak solution to equation* (18)*, then r is a weak solution to equation* (16).

Proof. Step 1. We first construct sequences in $D(\mathcal{A})$ approximating functions of $D(\mathcal{A}_\pi)$. Let ϕ belong to $D(\mathcal{A}_\pi)$, and set $\psi = \mathcal{A}_\pi\phi$. Set

$$k(\xi) = b(\xi,0)^2\int_\Omega \pi(\xi,0,X)\phi(X)dX \quad\text{and}\quad k_n(\xi,\eta) = nk(\xi)\mathbb{1}_{(0,\frac{1}{n})}(\eta),$$

where $\mathbb{1}_{(0,\frac{1}{n})}$ is the characteristic function of the interval $(0,\frac{1}{n})$. Let $\phi_n \in D(\mathcal{A})$ be the solution to the equation

$$\mathcal{A}\phi_n = \psi - k_n. \tag{20}$$

We can show that $(\phi_n)_n$ converges to ϕ in $L^2(\Omega)$ and weakly in $L^2(0,L;H^1(0,1,d))$. Moreover ϕ_n obeys

$$\int_\Omega \psi\, z\, d\Xi = -\int_\Omega \Big(a\frac{\partial \phi_n}{\partial \xi}z + b\frac{\partial \phi_n}{\partial \eta}\frac{\partial z}{\partial \eta} + \frac{\partial b}{\partial \eta}\frac{\partial \phi_n}{\partial \eta}z + c\,\phi_n\, z\Big)d\Xi + \int_\Omega k_n(\Xi)z(\Xi)\, d\Xi, \tag{21}$$

for all $z \in L^2(0,L;H^1(0,1,d))$. By passing to the limit when n tends to infinity, we show that ϕ obeys

$$\int_\Omega \psi\, z\, d\Xi = -\int_\Omega \Big(a\frac{\partial \phi}{\partial \xi}z + b\frac{\partial \phi}{\partial \eta}\frac{\partial z}{\partial \eta} + \frac{\partial b}{\partial \eta}\frac{\partial \phi}{\partial \eta}z + c\,\phi\, z\Big)d\Xi + \int_\gamma z(s,0)b(s,0)^2\int_\Omega \pi(s,0,X)\,\phi(X)\,dX\,ds, \tag{22}$$

for all $z \in L^2(0,L;H^1(0,1,d))$.

Step 2. Since $r \in L^2(0,\infty;L^2(0,L;H^1(0,1,d)))$ is a weak solution to equation (18), taking ϕ_n, defined in Step 1, as test function in the definition of weak solution of equation (18), with (20) and (21) we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega r\,\phi_n\,\theta'\,d\Xi\,dt\\
&=\int_0^T\int_\Omega r\mathcal{A}\phi_n\,\theta\,d\Xi\,dt - \int_0^T\int_\Omega\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds\,\phi_n(\Xi)\,\theta\,d\Xi\,dt + \int_0^T\int_\Omega h\,\phi_n\,\theta\,d\Xi\,dt\\
&=-\int_0^T\theta\int_\Omega\Big(a\frac{\partial \phi_n}{\partial \xi}r + b\frac{\partial \phi_n}{\partial \eta}\frac{\partial r}{\partial \eta} + \frac{\partial b}{\partial \eta}\frac{\partial \phi_n}{\partial \eta}r + c\,\phi_n\, r\Big)d\Xi\,dt\\
&\qquad-\int_0^T\int_\Omega\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds\,\phi_n(\Xi)\,\theta\,d\Xi\,dt + \int_0^T\int_\Omega h\,\phi_n\,\theta\,d\Xi\,dt,
\end{aligned}$$

for all $\theta \in C_c^1((0,T])$. By passing to the limit when n tends to infinity, it follows that

$$\begin{aligned}
&\int_0^T\int_\Omega r\,\phi\,\theta'\,d\Xi\,dt\\
&=\int_0^T\theta\int_\Omega\Big(-a\frac{\partial \phi}{\partial \xi}r - b\frac{\partial \phi}{\partial \eta}\frac{\partial r}{\partial \eta} - \frac{\partial b}{\partial \eta}\frac{\partial \phi}{\partial \eta}r - c\,\phi\, r\Big)d\Xi\,dt\\
&\qquad-\int_0^T\int_\Omega\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds\,\phi(\Xi)\,\theta\,d\Xi\,dt + \int_0^T\int_\Omega h\,\phi\,\theta\,d\Xi\,dt.
\end{aligned}$$

Due to (22) we finally get

$$\int_0^T\int_\Omega r\,\phi\,\theta'\,d\Xi\,dt = \int_0^T\int_\Omega r\mathcal{A}_\pi\phi\,\theta\,d\Xi\,dt + \int_0^T\int_\Omega h\,\phi\,\theta\,d\Xi\,dt,$$

which completes the proof.

Lemma 4.6. *Assume that the support of h is included in $(0,T)\times\Omega$ for some $T>0$. Equation* (18) *admits a unique solution in the space*

$$E = \Big\{r \in L^2(0,T;L^2(0,L;H^1(0,1;d)))\mid r\in C_c([0,T);L^2(\Omega))\quad and\quad \sqrt{a}\,r\in L^\infty(0,L;L^2(0,T))\Big\},$$

Proof. The proof relies on a fixed point method as in [17, Proposition 2.6]. For that, for a given $\rho \in E$, we consider the equation

$$-r'(t,\Xi) = \mathcal{A}^* r(t,\Xi) - \int_\gamma \rho(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds + h(t,\Xi), \qquad r(T,\Xi)=0. \tag{23}$$

Since the mapping

$$(t,\Xi)\longmapsto \int_\gamma \rho(t,s,0)b(s,0)^2\pi(s,0,\Xi)\,ds$$

belongs to $L^2(0,T;L^2(\Omega))$, equation (23) admits a unique solution in E. Moreover, following [7, Proof of Theorem 5.5], we have

$$\begin{aligned}\left\|\int_\gamma \rho(\cdot,s,0)b(s,0)^2\pi(s,0,\cdot)\,ds\right\|_{L^2(T-\bar t,T;L^2(\Omega))} &\le C|\bar t|^{\frac{2-\varepsilon}{2(4-\varepsilon)}}\|b\|_\infty^2\|\rho\|^{2/(4-\varepsilon)}_{L^2(T-\bar t,T;L^2(0,L;H^1(0,1;d)))}\\ &\quad \|\rho\|^{(2-\varepsilon)/(4-\varepsilon)}_{L^\infty(T-\bar t,T;L^2(\Omega))}\|\pi\|_{L^2(\Omega;L^2(0,L;H^1(0,1;d)))}.\end{aligned}$$

Next as in [17, Proposition 2.6] it can be shown that the mapping $\rho\mapsto r_\rho$, where r_ρ is the solution to equation (23), is a contraction in

$$E_{\bar t}=\Big\{r\in L^2(T-\bar t,T;L^2(0,L;H^1(0,1;d)))\mid r\in C_c([T-\bar t,T);L^2(\Omega)),\ \sqrt{a}\,r\in L^\infty(0,L;L^2(T-\bar t,T))\Big\},$$

provided that $\bar t$ is small enough. Next, as in [17, Proposition 2.6], we can iterate this process to prove the existence of a unique solution in E to equation (18).

Theorem 4.7. *If h belongs to $L^2(0,\infty;L^2(\Omega))$, then the weak solution r to equation* (16) *satisfies*

$$r\in L^2(0,\infty;L^2(0,L;H^1(0,1;d)))\cap L^\infty(0,\infty;L^2(\Omega)),$$
$$\sqrt{a}r\in L^\infty(0,L;L^2((0,\infty)\times(0,1))),$$

and

$$\|r\|_{L^\infty(0,\infty;L^2(\Omega))}+\|\sqrt{a}r\|_{L^\infty(0,L;L^2((0,\infty)\times(0,1)))}+\|r\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))}\le C\|h\|_{L^2((0,\infty)\times\Omega)}. \tag{24}$$

Proof. We already know that the solution r to equation (16) obeys (17). Let h belong to $L^2((0,\infty)\times\Omega)$, let $(h_n)_n$ be a sequence in $C_c([0,\infty);L^2(\Omega))$ converging to h in $L^2((0,\infty)\times\Omega)$, and let r_n be the solution to equation (16) corresponding to h_n. From estimate (17), its follows that

$$\|r_n\|_{L^2(0,\infty;L^2(\Omega))}\le C\|h\|_{L^2((0,\infty)\times\Omega)}\quad\text{for all } n\in\mathbb{N}, \tag{25}$$

and that the sequence $(r_n)_n$ converges to r in $L^2(0,\infty;L^2(\Omega))$. Due to Lemma 4.6, for $h=h_n$, equation (18) admits a unique solution in the space

$$\Big\{r\in L^2(0,\infty;L^2(0,L;H^1(0,1;d)))\mid r\in C_c([0,\infty);L^2(\Omega))\quad\text{and}\quad\sqrt{a}\,r\in L^\infty(0,L;L^2((0,\infty)))\Big\}.$$

Therefore, from Theorem 4.5 it follows that r_n is the solution of equation (18) corresponding to h_n. Using the estimate established in [6, Theorem 6.2], we can show that r_n satisfies:

$$\begin{aligned}&\frac12\int_0^1\int_x^L r_n(\xi,y,t)^2\,d\xi\,dy+\frac12\int_t^T\int_0^1 a\,r_n(x,y,\tau)^2\,dy\,d\tau\\ &+\int_t^T\int_0^1\int_x^L\Big(b\left|\frac{\partial r_n}{\partial y}\right|^2+\frac{\partial b}{\partial y}\frac{\partial r_n}{\partial y}r_n+(c+ka)r_n^2\Big)\,d\xi\,dy\,d\tau\\ &\le\int_t^T\int_0^1\int_x^L\Big(h_n\,r_n\,d\xi\,dy+k\,a\,r_n\Big)\,d\tau+\int_t^T\int_0^1\int_x^L\int_\gamma r(t,s,0)b(s,0)^2\pi(s,0,\Xi)ds\,r_n(t,X)dX,\end{aligned} \tag{26}$$

for all $t\in(0,T)$ and all $x\in[0,L]$, where T is such that $\mathrm{supp}(h)\subset[0,T)\times\Omega$, and $k>0$ is such that (19) is satisfied. With (25) and (26) we can show that

$$\begin{aligned}\|r_n\|_{L^\infty(0,T;L^2(\Omega))}+\|\sqrt{a}r_n\|_{L^\infty(0,L;L^2((0,T)\times(0,1)))}+\|r_n\|_{L^2(0,T;L^2(0,L;H^1(0,1;d)))}&\\ \le C\|h_n\|_{L^2((0,T)\times\Omega)},&\end{aligned}$$

where C is independent of T. Thus the same estimate holds true over the time interval $(0,\infty)$, and estimate (24) is obtained by passing to the limit when n tends to infinity.

Theorem 4.8. *Let $g \in L^2((0,\infty)\times(0,L))$, $f \in L^2((0,\infty)\times\Omega)$, $z_b \in L^2((0,\infty)\times(0,1))$, and $u_\infty \in L^2(0,\infty)$. The system (15) admits a unique weak solution r such that*

$$r \in L^2(0,\infty; L^2(0,L; H^1(0,1;d))) \cap L^\infty(0,\infty; L^2(\Omega)),$$
$$\sqrt{a} r \in L^\infty(0,L; L^2((0,\infty)\times(0,1))).$$

Moerover r obeys:

$$\begin{aligned}\|r\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{a}r\|_{L^\infty(0,L;L^2((0,\infty)\times(0,1)))} + \|r\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))}\\ \leq C\Big(\|f\|_{L^2((0,\infty)\times\Omega)} + \|z_b\|_{L^2((0,\infty)\times(0,1))} + \|g\|_{L^2(0,\infty,L^2(0,L))} + \|u_\infty\|_{L^2(0,\infty)}\Big).\end{aligned} \tag{27}$$

Proof. Setting

$$\begin{aligned}h(t,\Xi) = \int_\Omega \pi(X,\Xi)f(t,X)\,dX - \int_0^L \pi(x,0,\Xi)b(x,0)g(t,x)\,dx\\ + \int_0^1 \pi(0,y,\Xi)a(y)z_b(t,y)\,dy + c_0u_\infty(t)\phi(\Xi),\end{aligned} \tag{28}$$

the theorem is an immediate consequence of Theorem 4.7.

5 Feedback formula

Lemma 5.1. *Let π be the solution to the Riccati equation (14), $u \in L^2(0,\infty; L^2(0,L))$, $z_0 \in L^2(\Omega)$, and let z be the solution to equation*

$$z' = \mathcal{A}z + f + B(\mathbb{1}_\gamma u + g) + (-\mathcal{A})Dz_b, \qquad z(0) = z_0.$$

Then z satisfies the following identity:

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{O}} (\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi(X,\Xi)\, z(t)\otimes z(t)\\ &= -\int_{\mathcal{O}} \pi(X,\Xi)\, z_0\otimes z_0 + 2\int_0^\infty \int_\gamma b(s,0)u(t,s)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,d\Xi\,ds\,dt\\ &\quad -2\int_0^\infty \int_{\mathcal{O}} f(t,X)\,\pi(X,\Xi)\,z(t,\Xi)d\Xi\,dX\,dt - 2\int_0^\infty \int_0^1 a(y)z_b(t,y)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,dy\,dt\\ &\quad +2\int_0^\infty \int_0^L b(s,0)g(t,s)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,ds\,dt.\end{aligned} \tag{29}$$

Proof. It is sufficient to adapt the proof of [7, Lemma 6.3].

Lemma 5.2. *Let us assume that the assumptions of Theorem 4.8 are fulfilled and let r be the solution to equation (15). Let u belong to $L^2(0,\infty; L^2(0,L))$, z_0 belong to $L^2(\Omega)$, and z be the solution to equation*

$$z' = \mathcal{A}z + f + B(\mathbb{1}_\gamma u + g) + (-\mathcal{A})Dz_b, \qquad z(0) = z_0.$$

Then z and r satisfy the following identity:

$$\begin{aligned}&\int_\Omega z_0\, r(0)\\ &= -\int_0^\infty \int_\gamma b(s,0)^2 r(t,s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,d\Xi\,ds\,dt + \int_0^\infty \int_\gamma b(s,0)r(t,s,0)u(t,s)\,ds\,dt\\ &\quad + \int_0^\infty \int_{\mathcal{O}} \pi(X,\Xi)f(t,X)z(t,\Xi)\,d\Xi\,dX\,dt - \int_0^\infty \int_0^L b(s,0)g(t,s)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,d\Xi\,ds\,dt\\ &\quad + \int_0^\infty \int_0^1 a(y)z_b(t,y)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)\,d\Xi\,dy\,dt + \int_0^\infty \int_\Omega c_0u_\infty(t)\,\phi(X)\,z(t,X)\,dX\,dt\\ &\quad + \int_0^\infty \int_0^L b(s,0)r(t,s,0)\,g(t,s)\,ds\,dt\\ &\quad - \int_0^\infty \int_0^1 a(y)z_b(t,y)r(t,0,y)\,dy\,dt - \int_0^\infty \int_\Omega f(t,X)\,r(t,X)\,dX\,dt\,.\end{aligned} \tag{30}$$

Proof. Due to [6, Theorem 6.6], if the support of h is included in $[0,T)\times\infty$, the solution r to equation (16) and z obey the formula

$$\begin{aligned}
&\int_\Omega z_0\, r(0)\\
&= -\int_0^\infty\int_\gamma b(s,0)^2 r(t,s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,d\Xi\,ds\,dt + \int_0^\infty\int_\gamma b(s,0)r(t,s,0)u(t,s)\,ds\,dt\\
&\quad + \int_0^\infty\int_\Omega h(t,\Xi)z(t,\Xi)\,d\Xi\,dt\\
&\quad + \int_0^\infty\int_0^L b(s,0)r(t,s,0)\,g(t,s)\,ds\,dt\\
&\quad - \int_0^\infty\int_0^1 a(y)z_b(t,y)r(t,0,y)\,dy\,dt - \int_0^\infty\int_\Omega f(t,X)\,r(t,X)\,dX\,dt\,.
\end{aligned}$$

From a density argument we can claim that this formula holds true for $h\in L^2(0,\infty;L^2(\Omega))$. Now if h is given by (28), we obtain (30).

Lemma 5.3. *Let π be the solution to (14). Let us assume that the assumptions of Theorem 4.8 are fulfilled and let r be the solution to equation (15), $u\in L^2(0,\infty;L^2(0,L))$, $z_0\in L^2(\Omega)$, and z be the solution to equation*

$$z' = \mathcal{A}z + f + B(\mathbb{1}_\gamma u + g) + (-\mathcal{A})Dz_b, \qquad z(0)=z_0\,.$$

Then the cost function satisfies

$$\begin{aligned}
J(z,u) \;=\;& \frac12\int_{\mathcal{O}}\pi\, z_0\otimes z_0 + \int_\Omega z_0\, r(0) + \frac12\int_0^\infty |c_0 u_\infty(t)|^2 dt\\
&+\frac12\int_0^\infty\int_\gamma \left|u(t,s) - b(s,0)\int_\Omega \pi(s,0,X)z(t,X)dX - b(s,0)r(t,s,0)\right|^2 ds\,dt\\
&-\frac12\int_0^\infty\int_\gamma |b(s,0)r(t,s,0)|^2\,ds\,dt - \int_0^\infty\int_0^L b(s,0)r(t,s,0)\,g(t,s)\,ds\,dt\\
&+\int_0^\infty\int_0^1 a(y)z_b(t,y)r(t,0,y)\,dy\,dt + \int_0^\infty\int_\Omega f(t,X)\,r(t,X)\,dX\,dt\,.
\end{aligned}\tag{31}$$

Proof. With Lemma 5.1 and equation (14), we can write

$$\begin{aligned}
&-\int_{\mathcal{O}}\pi(X,\Xi)\, z_0\otimes z_0 + 2\int_0^\infty\int_\gamma b(s,0)u(t,s)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,d\Xi\,ds\,dt\\
&\qquad -2\int_0^\infty\int_{\mathcal{O}} f(t,X)\,\pi(X,\Xi)\,z(t,\Xi) - 2\int_0^\infty\int_0^1 a(y)z_b(t,y)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,dy\,dt\\
&\qquad +2\int_0^\infty\int_0^L b(s,0)g(t,s)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,ds\,dt\\
&= \int_0^\infty\int_{\mathcal{O}} (\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi(X,\Xi)\,z(t)\otimes z(t)\\
&= \int_0^\infty\int_\gamma \left|b(s,0)\int_\Omega \pi(s,0,X)z(t,X)dX\right|^2 ds\,dt - \int_0^\infty\left|\int_\Omega \phi(X)z(t,X)dX\right|^2 dt\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
J(z,u) \;=\;& \frac12\int_0^\infty\left|\int_\Omega\phi(X)z(t,X)dX + c_0u_\infty(t)\right|^2 dt + \frac12\int_0^\infty\int_\gamma |u|^2 ds\,dt\\
=\;& \frac12\int_{\mathcal{O}}\pi\, z_0\otimes z_0 + \frac12\int_0^\infty\int_\gamma |u|^2 ds\,dt - \int_0^\infty\int_\gamma b(s,0)u(t,s)\left(\int_\Omega \pi(s,0,X)z(t,X)dX\right) ds\,dt\\
&+\frac12\int_0^\infty\int_\gamma\left|b(s,0)\int_\Omega \pi(s,0,X)z(t,X)dX\right|^2 ds\,dt + \frac12\int_0^\infty |c_0u_\infty(t)|^2 dt\\
&+\int_0^\infty\int_\Omega c_0u_\infty(t)\phi(X)z(t,X)\,dX\,dt\\
&+\int_0^\infty\int_{\mathcal{O}} f(t,X)\,\pi(X,\Xi)\,z(t,\Xi)\,d\Xi\,dX\,dt + \int_0^\infty\int_0^1 a(y)z_b(t,y)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,dy\,dt\\
&-\int_0^\infty\int_0^L b(s,0)g(t,s)\int_\Omega \pi(0,y,\Xi)z(t,\Xi)d\Xi\,ds\,dt.
\end{aligned}$$

Using Lemma 5.2 and the previous expression of the cost functional we obtain:

$$
\begin{aligned}
&J(z,u)\\
&= \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 + \frac{1}{2}\int_0^\infty \int_\gamma |u|^2 ds\, dt - \int_0^\infty \int_\gamma b(s,0)u(t,s)\left(\int_\Omega \pi(s,0,X)z(t,X)dX\right) ds\, dt\\
&\quad + \frac{1}{2}\int_0^\infty \int_\gamma \left| b(s,0)\int_\Omega \pi(s,0,X)z(t,X)dX\right|^2 ds\, dt + \frac{1}{2}\int_0^\infty |c_0 u_\infty(t)|^2 dt\\
&\quad + \int_\Omega z_0\, r(0) + \int_0^\infty \int_\gamma b(s,0)^2 r(t,s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\, d\Xi\, ds\, dt\\
&\quad - \int_0^\infty \int_\gamma b(s,0)r(t,s,0)u(t,s)\, ds\, dt - \int_0^\infty \int_0^L b(s,0)r(t,s,0)\, g(t,s)\, ds\, dt\\
&\quad + \int_0^\infty \int_0^1 a(y)z_b(t,y)r(t,0,y)\, dy\, dt + \int_0^\infty \int_\Omega f(t,X)\, r(t,X)\, dX\, dt\\
&= \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 + \int_\Omega z_0\, r(0) + \frac{1}{2}\int_0^\infty |c_0 u_\infty(t)|^2 dt\\
&\quad + \frac{1}{2}\int_0^\infty \int_\gamma \left| u(t,s) - b(s,0)\int_\Omega \pi(s,0,X)z(t,X)dX - b(s,0)r(t,s,0)\right|^2 ds\, dt\\
&\quad - \frac{1}{2}\int_0^\infty \int_\gamma |b(s,0)r(t,s,0)|^2\, ds\, dt - \int_0^\infty \int_0^L b(s,0)r(t,s,0)\, g(t,s)\, ds\, dt\\
&\quad + \int_0^\infty \int_0^1 a(y)z_b(t,y)r(t,0,y)\, dy\, dt + \int_0^\infty \int_\Omega f(t,X)\, r(t,X)\, dX\, dt\,.
\end{aligned}
$$

The proof is complete.

Theorem 5.4. *Let $(\bar{z},\bar{u})$ be the optimal solution to problem (P). The optimal control $\bar{u}$ obeys the feedback formula*

$$
\bar{u}(\tau,s) = b(s,0)\int_\Omega \pi(s,0,\Xi)\bar{z}(\tau,\Xi)\, d\Xi + b(s,0)r(\tau,s,0), \tag{32}
$$

where π is the solution to the algebraic Riccati equation (14), r is the solution to equation (15). The optimal cost is given by

$$
\begin{aligned}
J(\bar{z},\bar{u}) \quad = \quad & \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 + \int_\Omega z_0\, r(0) + \frac{1}{2}\int_0^\infty |c_0 u_\infty(t)|^2 dt\\
& - \frac{1}{2}\int_0^\infty \int_\gamma |b(s,0)r(t,s,0)|^2\, ds\, dt - \int_0^\infty \int_0^L b(s,0)r(t,s,0)\, g(t,s)\, ds\, dt\\
& + \int_0^\infty \int_0^1 a(y)z_b(t,y)r(t,0,y)\, dy\, dt + \int_0^\infty \int_\Omega f(t,X)\, r(t,X)\, dX\, dt\,.
\end{aligned}
$$

Proof. Due to Lemma 5.3, it is obvious that $(\bar{z},\bar{u})$, where $\bar{u}$ obeys (32), satisfies

$$
J(\bar{z},\bar{u}) \le J(z_u,u),
$$

where z_u is the solution to equation (1) with $u \in L^2(0,\infty;U)$. The optimal can be deduced from (31).

6 Numerical approximation and stabilization results

In this section, we present the discretization method for solving the linearized Crocco equation (1) and the nonlinear equation (50) controlled by the feedback control law determined in section 2.4. Before describing the numerical approximation of equation (1) let us mention the main difficulties associated to the operator A defined by

$$
\begin{gathered}
Az = -a(\eta)\,\frac{\partial z}{\partial \xi} + b(\xi,\eta)\,\frac{\partial^2 z}{\partial \eta^2} - c(\xi,\eta)z,\\
a(0) = 0, \quad b(\xi,1) = \frac{\partial b}{\partial \eta}(\xi,1) = 0, \quad c(\xi,\eta) \le 0.
\end{gathered}
$$

On the one hand, we have a stiff operator since the diffusion coefficient $b = \nu w_s^2$ is large near $\eta = 0$ and null at $\eta = 1$. On the other hand, in the neighbourhood of $\eta = 1$, the convection term dominates the diffusion term so a classical P_1 or Q_1 finite element method gives oscillatory solutions. To overcome these difficulties, we have used a discontinuous Galerkin method with respect to the variable ξ and a finite element method with respect to the variable η. The domain $\Omega = [0, 0.5] \times [0, 1]$ is divided into $N \times M$ quadrangular cells. The ξ variable is discretized with a P_1 (polynomials of degree 1) discontinuous Galerkin method (DG) with a monotone numerical flux [11]. A classical P_1 finite element method is used for the variable η. This method will be simply called the $P_1 - P_1$ Galerkin method. The DG methods are well adapted to solve convection equations with high-order accurate schemes [11]. Moreover, contrary to the finite difference schemes, no special numerical treatment of boundary conditions is required to conserve the global accuracy of the scheme. Up to now, we have not studied the convergence of this scheme because it was not our main goal and very few convergence results exist for DG method [9]. In our knowledge there is no numerical analysis provided for the approximation of degenerate operators of the type A. Concerning the approximation of the time part, due to the stiff nature of our system, the classical explicit schemes such that Runge-Kutta methods are not adapted (the CFL condition is too restrictive). For this reason, the time part is discretized with a robust implicit scheme, a Backward Differentiation Formula (BDF) of order 2. In numerical tests, we have verified that this scheme is unconditionally stable. A similar scheme will be used to solve the nonlinear Crocco equation (50).

In section 6.1, we describe the $P_1 - P_1$ Galerkin scheme used to discretize equation (1). The finite dimensional LQR problem is solved in section 6.2. The numerical order of accuracy of the $P_1 - P_1$ scheme is determined in section 6.3.1. As expected, we check by numerical simulations that this scheme is of order 2. To test the stabilization of equation (1) by the feedback law of Theorem 5.4, we make three experiments. In the first two tests, we linearize the instationary Crocco equation around an exact stationary solution corresponding to an *asymptotic suction profile* [19]. This solution satisfies the assumptions (4). The stationary solution is perturbed by acting on the upstream velocity U^s_∞. We show in section 6.3.2 that we can easily control two observations with the optimal feedback law. In the third test, the stationary solution corresponds to a well known *Blasius profile* [19]. This solution belongs to a different class of solutions [18, Section 3.3]. It is interesting since in that class we can consider a control problem with an observation corresponding to the laminar-turbulent transition location developed on a flat plate. We show that the optimal feedback law stabilizes the variations of the transition location. In section 6.3.3, we show that the feedback operator obtained with the linearized model stabilizes the nonlinear Crocco equation (50).

Throughout this section, we will take a control u constant with respect to the variable ξ so

$$u(t, \xi) = u(t).$$

The boundary control operator in the A.R.E (14) has to be modified accordingly.

6.1 Discretization of the linearized Crocco equation (1)

Let $(\xi_i)_{i=1}^{N+1}$ (resp. $(\eta_j)_{j=1}^{M+1}$) be a subdivision of $[0, 0.5]$ (resp. $[0, 1]$). The computational domain $\Omega = [0, 0.5] \times [0, 1]$ is divided into $N \times M$ quadrangles $\Omega_{ij} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$. We look for an approximation z^h of z in the form

$$z^h(t, \xi, \eta) = \sum_{i=1}^{N} \sum_{j=1}^{M+1} \sum_{\ell=0}^{1} z_{ij}^\ell(t) \varphi_{ij}^\ell(\xi, \eta), \tag{33}$$

with $z_{ij}^\ell \in C^1([0, T])$, $\varphi_{ij}^\ell(\xi, \eta) = \phi_i^\ell(\xi)\varphi_j(\eta)$ and

$$\phi_i^\ell \in V_h = \Big\{ v \in L^2(0, 0.5) \;\mid\; v|_{[\xi_i, \xi_{i+1}]} \in P_1([\xi_i, \xi_{i+1}]) \Big\},$$

$$\varphi_j \in W_h = \Big\{ w \in C^0([0, 1]) \;\mid\; w|_{[\eta_j, \eta_{j+1}]} \in P_1([\eta_j, \eta_{j+1}]) \Big\},$$

and where $P_1([a, b])$ is the space of polynomial functions of degree 1 on $[a, b]$. The basis functions have the following expressions:

$$\varphi_1(\eta) = \begin{cases} \dfrac{(\eta_2 - \eta)}{\eta_2 - \eta_1} & \eta \in [\eta_1, \eta_2], \\ 0 & \eta \geq \eta_2, \end{cases} \qquad \varphi_{M+1}(\eta) = \begin{cases} 0 & \eta \leq \eta_M, \\ \dfrac{(\eta - \eta_M)}{\eta_{M+1} - \eta_M} & \eta \in [\eta_M, \eta_{M+1}], \end{cases}$$

$$j=2,\dots,M,\quad \varphi_j(\eta)=\begin{cases}\dfrac{(\eta-\eta_{j-1})}{\eta_j-\eta_{j-1}} & \eta\in[\eta_{j-1},\eta_j],\\ \dfrac{(\eta_{j+1}-\eta)}{\eta_{j+1}-\eta_j} & \eta\in[\eta_j,\eta_{j+1}],\\ 0 & \text{otherwise,}\end{cases}$$

and, for $i\in\{1,\dots,N\}$,

$$\phi_i^0(\xi)=\begin{cases}1 & \xi\in[\xi_i,\xi_{i+1}],\\ 0 & \text{otherwise,}\end{cases}\qquad \phi_i^1(\xi)=\begin{cases}2\dfrac{(\xi-(\xi_{i+1}+\xi_i)/2)}{(\xi_{i+1}-\xi_i)} & \xi\in[\xi_i,\xi_{i+1}],\\ 0 & \text{otherwise.}\end{cases}$$

The finite element approximation of (1) consists in looking for z^h in the form (33) such that

$$\begin{aligned}\int_0^1\int_{\xi_i}^{\xi_{i+1}}\frac{\partial z_i^h}{\partial t}\varphi_{ij}^\ell \;=\;& \int_0^1\int_{\xi_i}^{\xi_{i+1}}\eta U_\infty^s z_i^h\frac{\partial\varphi_{ij}^\ell}{\partial\xi}-\int_0^1\eta U_\infty^s z_i^h(t,\xi_{i+1},\eta)\varphi_{ij}^\ell(\xi_{i+1},\eta)\\ &+\int_0^1\eta U_\infty^s z_{i-1}^h(t,\xi_i,\eta)\varphi_{ij}^\ell(\xi_i,\eta)-\int_0^1\int_{\xi_{i-1}}^{\xi_{i+1}} b\frac{\partial z_i^h}{\partial\eta}\frac{\partial\varphi_{ij}^\ell}{\partial\eta}\\ &-\int_{\xi_i}^{\xi_{i+1}} b(\xi,0)\left(u(t)\mathbb{1}_\gamma(\xi)+g(t,\xi)\right)\varphi_{ij}^\ell(\xi,0)-\int_0^1\int_{\xi_i}^{\xi_{i+1}}\frac{\partial b}{\partial\eta}\frac{\partial z_i^h}{\partial\eta}\varphi_{ij}^\ell\\ &-\int_0^1\int_{\xi_i}^{\xi_{i+1}} c z_i^h\varphi_{ij}^\ell+\int_0^1\int_{\xi_i}^{\xi_{i+1}} f\varphi_{ij}^\ell,\end{aligned}\tag{34}$$

for all $\varphi_{ij}^\ell\in V_h\times W_h$ and $t\geq 0$ where $z_i^h(t,\xi,\eta)=\displaystyle\sum_{j=1}^{M+1}\sum_{\ell=0}^{1} z_{ij}^\ell(t)\varphi_{ij}^\ell(\xi,\eta)$ for all $(\xi,\eta)\in[\xi_i,\xi_{i+1}]\times[0,1]$. The exact fluxes in ξ_i and ξ_{i+1} are approximated by the numerical upwind flux [11]. To take into account the Dirichlet boundary condition at $\xi=0$ in the variational formulation (34), we simply replace the term $z_0^h(t,0,\eta)$ by

$$z_b^h(t,\eta)=\sum_{j=1}^{M+1} z_b(t,\eta_j)\varphi_j(\eta),$$

what represents the projection of the boundary condition z_b onto W_h. The integrals are calculated exactly if possible or approximated by quadrature rules exact for polynomials of degree 3 in ξ and η [10].

The global numerical scheme.

Let $z^n\in\mathbb{R}^{2N\times(M+1)}$ be the unknown vector defined by

$$z^n=[z_{11}^0,\dots,z_{N(M+1)}^0,z_{11}^1,\dots,z_{N(M+1)}^1]^T.$$

By assembling the $n=2N\times(M+1)$ local systems (34), we obtain the finite dimensional dynamical system:

$$E^n\frac{dz^n}{dt}=\tilde{A}^n z^n+\tilde{B}^n u+\tilde{E}_1^n u_\infty+\tilde{E}_2^n u_\infty',\quad z^n(0)=z_0^n,\tag{35}$$

where $E^n\in\mathbb{R}^{n\times n}$ is a mass matrix, $\tilde{A}^n\in\mathbb{R}^{n\times n}$, $\tilde{B}^n\in\mathbb{R}^{n\times 1}$ is the discrete control operator and $\tilde{E}_i^n\in\mathbb{R}^{n\times 1}$, $i=\{1,2\}$. We have implicitly assumed that the inflow condition z_b depends linearly of u_∞ and u_∞'. The initial condition z_0^n is given by the column vector

$$z_0^n=(E^n)^{-1}\left[\int_\Omega z_0\varphi_{11}^0,\dots,\int_\Omega z_0\varphi_{N(M+1)}^0,\int_\Omega z_0\varphi_{11}^1,\dots,\int_\Omega z_0\varphi_{N(M+1)}^1\right]^T.$$

To simplify the presentation of the next section, we will work with the system:

$$\frac{dz^n}{dt}=A^n z^n+B^n u+E_1^n u_\infty+E_2^n u_\infty',\quad z^n(0)=z_0^n,\tag{36}$$

where $A^n=(E^n)^{-1}\tilde{A}^n$, $B^n=(E^n)^{-1}\tilde{B}^n$ and $E_i^n=(E^n)^{-1}\tilde{E}_i^n$, $i=\{1,2\}$.

6.2 The LQR problem

In this section, we solve the linear-quadratic control problem:

$$(P^n) \qquad \mathrm{Inf}\Big\{J^n(z^n,u)\ \mid\ (z^n,u)\in L^2(0,\infty;\mathbb{R}^n)\times L^2(0,\infty;\mathbb{R}),\ (z^n,u)\ \text{satisfies}\ (36)\Big\},$$

where

$$J^n(z^n,u)=\frac{1}{2}\int_0^\infty |C^n z^n(t)+c_0u_\infty|^2\,dt+\frac{R}{2}\int_0^\infty |u(t)|^2\,dt,\quad R>0, \tag{37}$$

where $R>0$, $C^n\in\mathbb{R}^{1\times n}$ corresponds to the discretization of the observation operator C defined by the expression $Cz(t)=\int_\Omega \phi z(t)$ with $\phi\in L^2(\Omega)$. In the tests below, we shall take different values of R to test its influence and different functions ϕ. Under classical stabilizability and detectability assumptions that we have numerically checked, the optimal feedback control law is

$$u(t)=-R^{-1}(B^n)^T\big(\Pi^n z^n(t)+r^n(t)\big)=-K^n z^n(t)-R^{-1}(B^n)^T r^n(t), \tag{38}$$

where the gain matrix is defined by the expression

$$K^n=R^{-1}(B^n)^T\Pi^n, \tag{39}$$

$\Pi^n\in\mathcal{L}\left(\mathbb{R}^{n\times n}\right)$, $(\Pi^n)^T=\Pi^n\geq 0$ is the unique solution to the algebraic Riccati equation

$$(A^n)^T\Pi^n+\Pi^n A^n-\Pi^n B^n R^{-1}(B^n)^T\Pi^n+Q^n=0, \tag{40}$$

with $Q^n=(C^n)^TC^n$ and the tracking variable $r^n(t)$ satisfies the system:

$$-\frac{dr^n}{dt}=(A^n-B^nK^n)^T r^n+\Pi^n(E_1^n u_\infty+E_2^n u_\infty')+(C^n)^T c_0 u_\infty,\quad r^n(\infty)=0. \tag{41}$$

This system is obtained by writing the optimality conditions for the LQR problem (P^n). The closed loop system is:

$$\frac{dz^n}{dt}=(A^n-B^nK^n)z^n-B^nR^{-1}(B^n)^Tr^n+E_1^nu_\infty+E_2^nu_\infty',\quad z^n(0)=z_0^n. \tag{42}$$

Remark 6.1. *In the numerical tests, the function u_∞ is equal to 0 for $t\geq T$. Therefore the terminal condition $r(\infty)=0$ is replaced by $r(T)=0$. The tracking variable r^n obeying equation* (41)*, for the discrete control problem, is closely related to the tracking variable r obeying equation* (15) *in which f, g and z_b are expressed in terms of u_∞ and u'_∞. But equation* (41) *is not necessarily the discrete approximation of equation* (15)*. For the numerical experiments, we have prefered to take the tracking variable r^n of the discrete control problem rather than the solution of the approximate equation of the continuous problem.*

The solution of the algebraic Riccati equation (40) is determined by calculating an orthonormal basis $\begin{bmatrix}U_{1,1}\\U_{2,1}\end{bmatrix}\in\mathbb{R}^{4N(M+1)\times 2N(M+1)}(\mathbb{R})$ of the right invariant-subspace associated to eigenvalues, with negative real part, of the Hamiltonian matrix H defined by

$$H=\begin{bmatrix}A^n & -\beta B^nR^{-1}(B^n)^T\\ -Q^n/\beta & -(A^n)^T\end{bmatrix}.$$

Here $\beta=2^m$ is a scale factor such that $||-\beta B^nR^{-1}(B^n)^T||_1\approx||-Q^n/\beta||_1$, see [8]. This basis can be obtained with a classical Schur method applied to H, see [16]. The solution to the A.R.E. is $\Pi^n=\beta U_{2,1}U_{1,1}^{-1}$.

Remark 6.2. *Other methods to solve the A.R.E. such as the low rank Newton-ADI method [15, 4] or [14] can be used if $n>5000$. We have implemented the algorithm described in [14] to compute a low rank approximation of the solution to the A.R.E.. The convergence of the method is very fast. With this low rank approximation, Π^n is never explicitly built, we use only the factor $Z^n\in\mathbb{R}^{n\times\ell}$ where $\ell<<n$ such that $\Pi^n\approx Z^n(Z^n)^T$. Another algorithm so-called the Chandrasekhar method can be used to directly determined the feedback K^n, (see Banks [2] for the heat equation). Unfortunately, this last method converges slowly due to the stiff nature of our system.*

6.3 Simulation results

The numerical simulations are realized with MATLAB on a one Intel Quad core at $2.5GHz$ processor with $4Go\,RAM$. We recall that the domain of interest is $\Omega = [0, 0.5] \times [0, 1]$. In all simulations, we will take

$$U_\infty^s = 45\,[m/s], \quad \nu = \frac{1}{66000}\,[m^2/s], \quad L = 0.5\,[m], \quad Re = \frac{U_\infty^s L}{\nu} = 1.485\,10^6.$$

The control zone γ is the interval $[0.17, 0.22]$. These parameters correspond to experimental tests made at ONERA Toulouse for the feedback control of the transition location [20].

6.3.1 Validation test

In this section, we numerically compute the order of convergence of the $P_1 - P_1$ finite element methods. For this, we linearize equation (1) around $w_s(\xi, \eta) = 3300(1 - \eta)$ and we determine the right hand side f and the boundary conditions such that $z_{ex}(t, \xi, \eta) = \xi^2 \eta^2 \sin^2(4\pi t)$ is an exact solution to equation (1). We easily verify that z_{ex} is solution to equation:

$$\begin{cases} \dfrac{\partial z}{\partial t} + U_\infty^s \eta \dfrac{\partial z}{\partial \xi} - \nu \left(w_s(\xi, \eta)\right)^2 \dfrac{\partial^2 z}{\partial \eta^2} = f(t, \xi, \eta), & (t, \xi, \eta) \in\ (0, T) \times \Omega, \\ z(0, \xi, \eta) = 0 & (\xi, \eta) \in\ \Omega, \\ \sqrt{a}\, z(t, 0, \eta) = 0, & (t, \eta) \in (0, T) \times (0, 1), \\ \dfrac{\partial z}{\partial \eta}(t, \xi, 0) = 0 & (t, \xi) \in (0, T) \times (0, L), \\ (bz)(t, \xi, 1) = 0 & (t, \xi) \in (0, T) \times (0, L), \end{cases} \tag{43}$$

with

$$f = \frac{\partial z_{ex}}{\partial t} + U_\infty^s \eta \frac{\partial z_{ex}}{\partial \xi} - \nu \left(w_s\right)^2 \frac{\partial^2 z_{ex}}{\partial \eta^2}.$$

The simulations are stopped at $T = 0.3775\,s$. The step size is taken equal to 10^{-4} for all the tests. In Table 1 the errors in $L^1(\Omega), L^2(\Omega)$ and $L^\infty(\Omega)$ norms and the experimental order of convergence (EOC) are given for structured meshes of size $N \times N$ quadrangles with $N = 20, 40, 80, 160$. We recall that the EOC is defined by

$$EOC = \log_2 \left(\frac{||z_N^n - z_{ex}||}{||z_{2N}^n - z_{ex}||} \right).$$

We see that the EOC converges to 2. Other numerical experiments for several final times T have given similar results.

N=M	$\|\|z_N^n - z_{ex}\|\|_{L^1(\Omega)}$	EOC	$\|\|z_N^n - z_{ex}\|\|_{L^2(\Omega)}$	EOC	$\|\|z_N^n - z_{ex}\|\|_{L^\infty(\Omega)}$	EOC
20	4.6233E-05		2.5382E-05		1.0393E-04	
40	1.1543E-05	2.002	6.1503E-06	2.045	2.6009E-05	1.998
80	2.8854E-06	2	1.5062E-06	2.029	6.5038E-06	1.999
160	7.2116E-07	2	3.1787E-07	2.018	1.6260E-06	2

Table 1: L^1, L^2, L^∞ error and numerical convergence order at $T = 0.3775\,s$ with $dt = 0.0001$ for the $P_1 - P_1$ finite element method.

6.3.2 Stabilization of the linearized Crocco equation

In this section, we present numerical results of stabilization for the linearized Crocco equation (1). In all the tests, the ODE system (42) is solved with a Backward Difference Formula of order 2. At each step, the linear system is solved with a LU method. The time step is $dt = 0.0001$ and the simulations are stopped at $T = 3\,s$.

Test 1. We linearize the Crocco equation around the stationary solution defined by the following data:

$$\nu = \frac{1}{66000}\,[m^2.s^{-1}], \quad U^s_\infty = 45\,[m.s^{-1}], \quad v_s = -0.05\,[m.s^{-1}], \quad w_s(\xi,\eta) = -\frac{v_s}{\nu}(1-\eta)\,[m^{-1}]. \tag{44}$$

In physical variables, this solution corresponds to following velocity field

$$U(x,y) = U^s_\infty(1 - e^{\frac{v_s y}{\nu}}), \qquad V(x,y) = v_s.$$

Physically, this flow represents an *asymptotic suction profile* [19] i.e a flow over a flat plate with a uniform suction obtained when x tends to ∞. We perturbe U^s_∞ by $u_\infty(t) = 0.5\sin(4\pi t)^2 H(t)$ where H is the function defined by

$$H(t) = \begin{cases} 1 & \text{for} \quad t \in [0,1], \\ \psi(t) & \text{for} \quad t \in [1,2], \\ 0 & \text{for} \quad t \geq 2, \end{cases} \tag{45}$$

where $\psi \in C^1([1,2])$ is a decreasing function such that $\psi(1) = 1$ and $\psi(2) = 0$. The inflow boundary condition at $\xi = 0$ is defined by the expression

$$z_b(t,\eta) = -\frac{\alpha(t)}{v_s}(1-\eta), \quad \alpha(t) = \frac{u'_\infty(t)}{U^s_\infty}. \tag{46}$$

This function satisfies the compatibility conditions with the initial condition and the boundary conditions at $\eta = 0$ and $\eta = 1$.

To stabilize this flow, we minimize the criterion (37) with

$$Cz(t) + c_0 u_\infty(t) = \int_\Omega \mathbb{1}_{[0.07,0.37]}(\xi) z(t,\xi,\eta)\,d\xi d\eta - 0.01 u_\infty(t) = y_c(t). \tag{47}$$

We have denoted $Cz(t) + c_0 u_\infty(t)$ by $y_c(t)$ because in figures thereafter we have plotted $y_c(t)$. Let us explain what is the motivation for choosing $\phi(\xi,\eta) = \mathbb{1}_{[0.07,0.37]}(\xi)$. In Test 3 below, the criterion y_c represents the laminar-to-turbulent transition location in the boundary layer. In that case we notice that the support of the corresponding function ϕ is included in $[0.07, 0.37] \times [0,1]$.

The linear system (35) is discretized with a uniform mesh in the ξ and η directions. The domain is divided into 50×19 cells therefore the number of unknowns is equal to $n = 2000$. The feedback matrix K^n is given by (39). From the expression of the optimal feedback law

$$u(t) = -\int_\Omega k_R(\xi,\eta) z(t,\xi,\eta)\,d\xi\,d\eta - \frac{1}{R}\int_\gamma b(\xi,0) r(t,s,0)\,ds,$$

we can easily determine an approximation of k_R. The function k_R for $R = 10^{-2}, 1, 10^2, 10^4$ is plotted in Figure 1. We notice that for $R << 1$, $k_R(\xi,\eta) \approx -\dfrac{1}{\sqrt{R}}\mathbb{1}_{[0.07,0.37]}(\xi)$.

For this first stabilization test, the feedback gain is calculated with $R = 10^4$. The initial condition is $z^n(0) = 0$. In Figure 2, at left, we have represented the uncontrolled and controlled observations y_c defined by (47). At right, we visualize the quantity $\nu u(t)$ corresponding to a suction/blowing velocity where u represents the feedback law. The non optimal control $u(t) = -K^n z^n(t)$ (the integral term $\frac{1}{R}\int_\gamma br$ is cancelled) gives good stabilization results. With the optimal control we obtain 99% of reduction of maximum amplitude of the uncontrolled observation. It is interesting to notice that the perturbation u_∞ is of order of 10^{-2} and that the corresponding control is only of the order of 10^{-4}. For $t = 0.7\,s$, the solution of the uncontrolled linearized Crocco equation and the controlled one with the optimal control are plotted in Figure 3. We recall that the action zone is located in $\gamma = [0.17, 0.22]$ (see the beginning of section 6.3).

Test 2. We keep the same stationary solution as previously but we change y_c. We take now

$$y_c(t) = \int_\Omega \sin(2\pi\eta)\mathbb{1}_{[0.07,0.37]}(\xi) z(t,\xi,\eta)\,d\xi d\eta - 0.01 u_\infty(t).$$

Introducing the function $\sin(2\pi\eta)$ in the definition of the function ϕ, facilitates the evaluation of the role of the tracking variable in the optimal feedback control law. In Figure 4, we have plotted on the left the function $\phi(\xi,\eta) = \sin(\pi\eta)\mathbb{1}_{[0.07,0.37]}(\xi)$ and on the right the corresponding kernel of the feedback operator. Figure 5 shows that the feedback controls are able to reduce significantly the variations of the observation $y_c(t)$. The non optimal control law is less efficient that the optimal one but the stabilization results stay always acceptable.

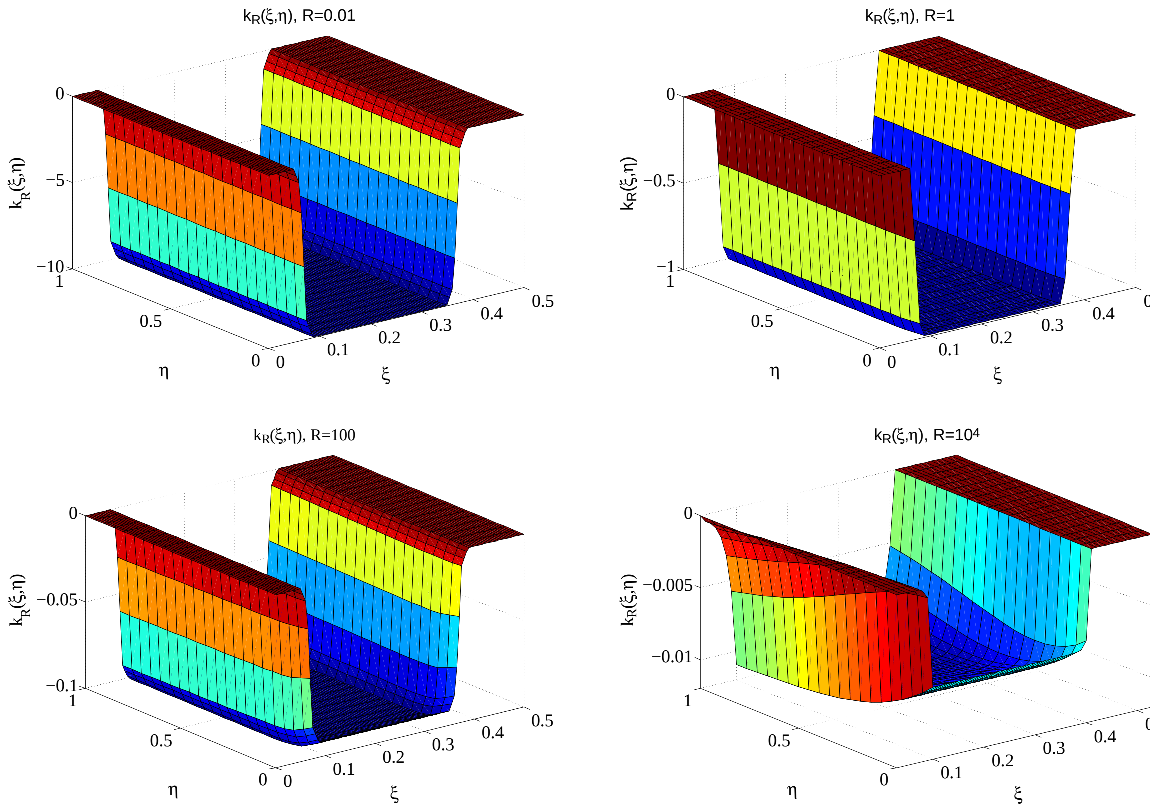

Figure 1: Kernel of feedback operator $k_R(\xi, \eta)$ for $R = 10^{-2}, 1, 10^2, 10^4$.

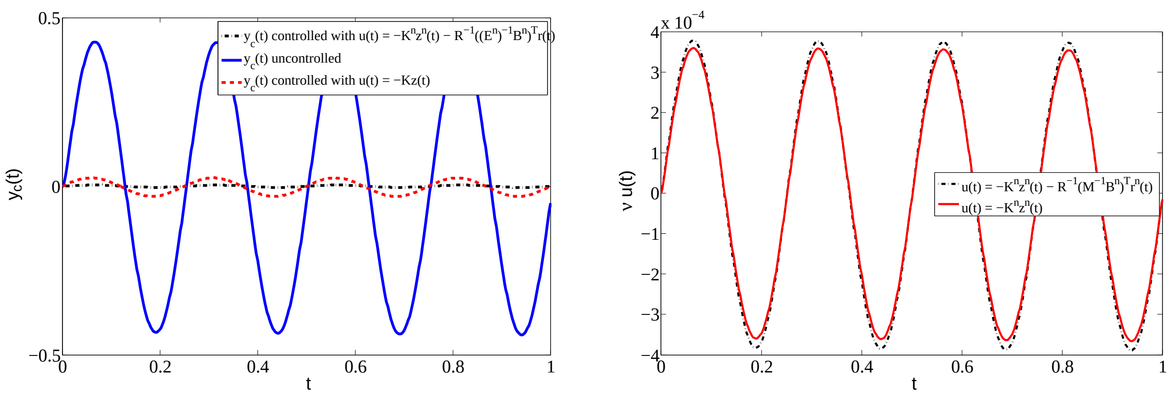

Figure 2: Controlled and uncontrolled observations (left) and control variation (right) with $R = 10^4$.

Test 3. Consider the so-called Blasius equation:

$$\begin{cases} h'''(\lambda) + h(\lambda)h''(\lambda) = 0, \quad \lambda = y\sqrt{\frac{U_\infty}{2\nu x}}, \\ h(0) = h'(0) = 0, \ h'(\lambda) \to 1, \quad \lambda \to \infty, \end{cases} \tag{48}$$

with $\nu = \frac{1}{66000}$, $U_\infty^s = 45$ and $v_s = 0$. For $U_\infty = U_\infty^s$ (resp. $U_\infty = U_\infty(t)$) the solution h to (48) will be denoted by h_b (resp. h_B). Let us notice that h_b only depends on λ while h_B depends on λ and t. We set

$$u_b(x, y) = U_\infty^s h_b'\left(y\sqrt{\frac{U_\infty^s}{2\nu x}}\right), \qquad u_B(t, x, y) = U_\infty(t) h_B'\left(y\sqrt{\frac{U_\infty(t)}{2\nu x}}\right),$$

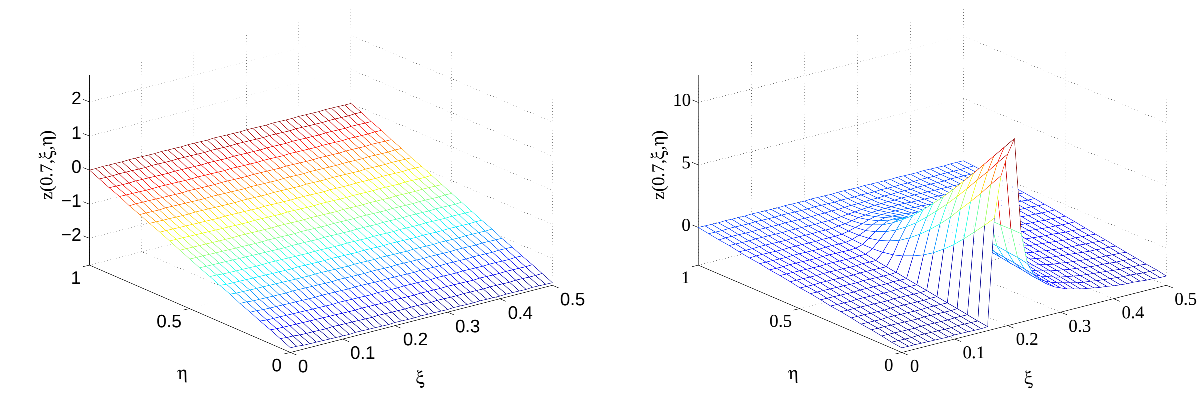

Figure 3: Solution of linearized Crocco equation uncontrolled (left) and controlled (right) at $t = 0.7\,s$.

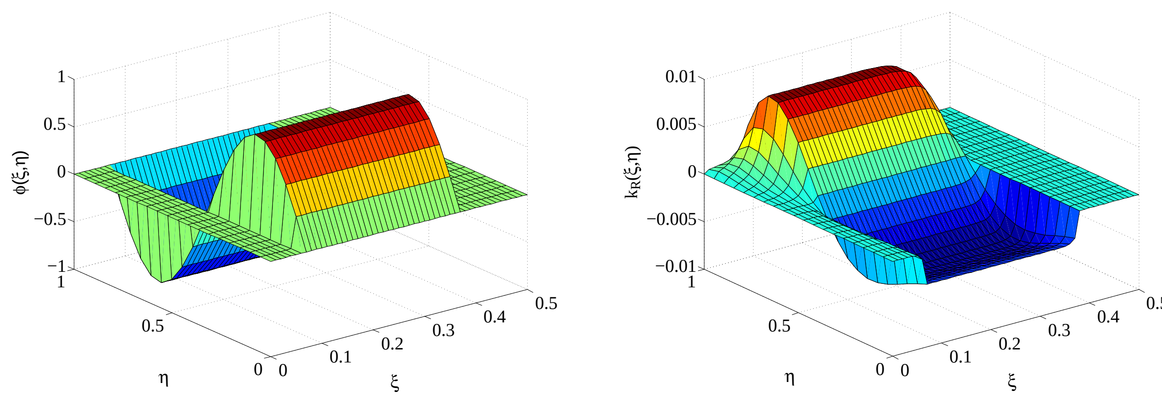

Figure 4: Functions $\phi(\xi, \eta)$ (left) and $k_R(\xi, \eta)$ with $R = 10^4$ (right).

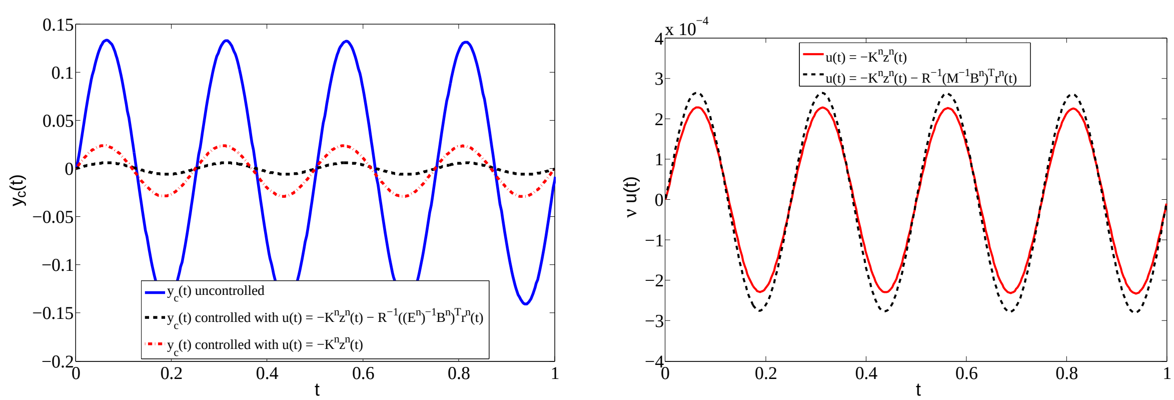

Figure 5: Controlled-uncontrolled observations (left) and corresponding controls (right).

and

$$w_b(\xi, \eta) = \frac{1}{U^s_\infty} \frac{\partial u_b}{\partial y}(x, y), \qquad w_B(t, \xi, \eta) = \frac{1}{U_\infty(t)} \frac{\partial u_B}{\partial y}(t, x, y), \tag{49}$$

where the variables (ξ, η) and (x, y) are related by (3). Let us recall that $U_\infty(t) = U^s_\infty + u_\infty(t)$ with $u_\infty(t) = 0.5 \sin(4\pi t)^2 H(t)$ and $H(t)$ is defined by (45). We linearize the Crocco equation around $w_s(\xi, \eta) = w_b(0.03 + \xi, \eta)$. Thanks to a Taylor expansion about U^s_∞ we obtain the inflow condition z_b in $\xi = 0$ such that

$$w_B(t, 0.03, \eta) - w_b(0.03, \eta) = z_b(t, \eta) + o\left(u_\infty, u'_\infty\right).$$

The Blasius equation is solved with a numerical method presented in [13, Section 11]. The coefficients of the linearized Crocco equation can be deduced from the expression of h and its derivatives, see [5]. The observation

y_c corresponds to the linearization of the transition location obtained with the parabola method associated with the e^n criterion [1]. For this Reynolds number, the transition location on a smooth flat plate can be estimated at $x_T = 40\,cm$ from the leading edge. Since we have not an explicit expression for the kernel ϕ of the observation operator C, we have determined it numerically [5]. Figure 6 shows the graph of ϕ for a stationary flow corresponding to the solution of the Blasius equation (48). To solve the algebraic Riccati equation, we take a uniform mesh in ξ with $N = 41$ and a geometric mesh in the direction of η with $M = 40$ such that $\frac{\eta_2-\eta_1}{\eta_{M+1}-\eta_M} = 10$. This choice is motivated by the existence of a boundary layer in a neighbourhood of $\eta = 1$. In the quadratic criterion (37), we take $R = 10$ and $c_0 = -0.002$. Figure 7 shows that the optimal control gives good stabilization results contrary to the non optimal control that does not take into account the nonhomogeneous terms in (35).

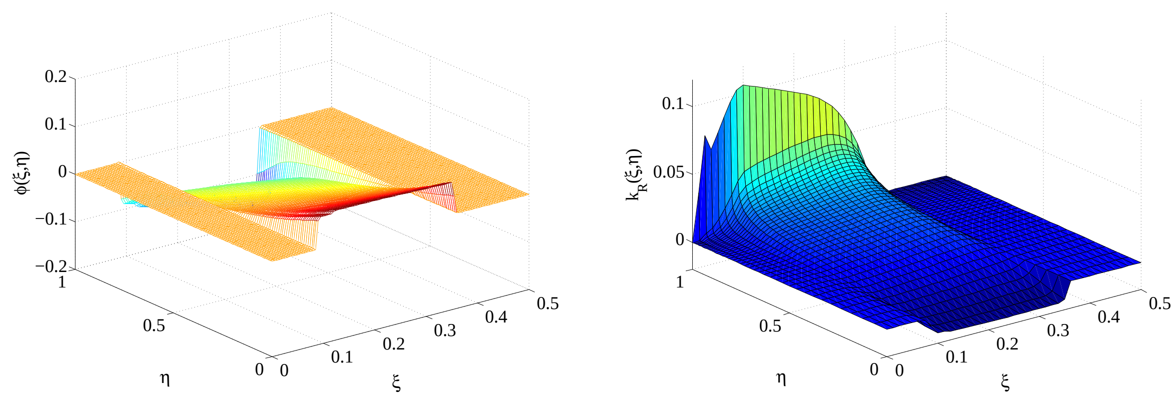

Figure 6: Graphs of ϕ (left) and k_R (right) for $R = 10$.

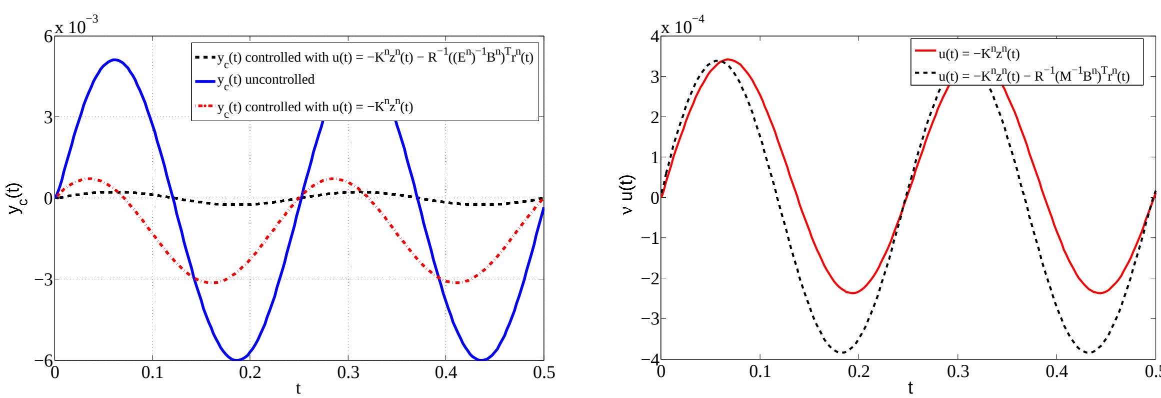

Figure 7: Controlled-uncontrolled y_c (left) and corresponding control (right).

6.3.3 *Stabilization of the nonlinear Crocco equation*

In this section, we verify by numerical simulations that the feedback operator determined with the linearized model is also able to stabilize the nonlinear Crocco equation. By substituting U (resp. v_0) by $U_\infty(t)$ (resp. $v_s(\xi) + u(t)\mathbb{1}_\gamma(\xi)$) in equation (4.4.1) stated in [18, p. 213] we obtain

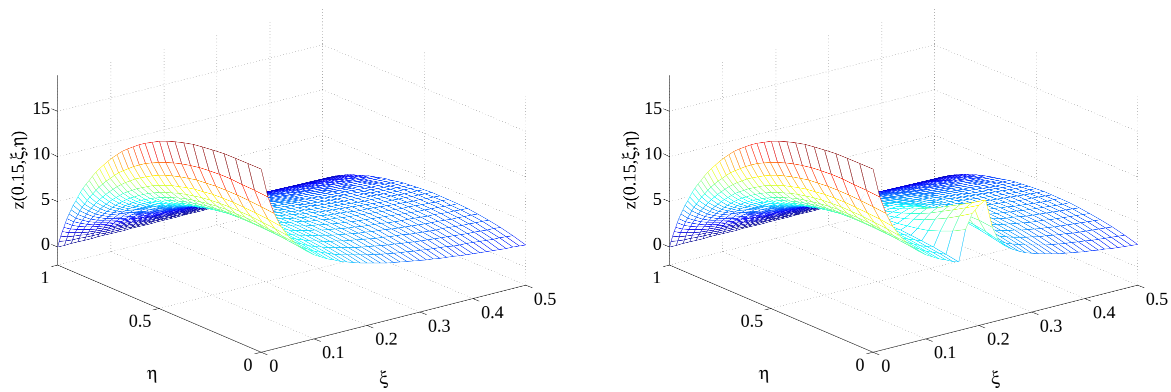

Figure 8: Solution of linearized Crocco equation uncontrolled (left) and controlled (right) at $t = 0.15\,s$.

$$\left\{\begin{array}{lll}
\dfrac{\partial w}{\partial t} - \dfrac{\partial}{\partial \eta}\left(\nu w^2 \dfrac{\partial w}{\partial \eta}\right) = -\eta\, U_\infty(t)\dfrac{\partial w}{\partial \xi} - \alpha(t)(1-\eta)\dfrac{\partial w}{\partial \eta} - \alpha(t) w - 2\nu w \left(\dfrac{\partial w}{\partial \eta}\right)^2 & \text{for} & (t,\xi,\eta)\in Q,\\
\nu\left(w\dfrac{\partial w}{\partial \eta}\right)(t,\xi,0) = \big(v_s(\xi) + \nu u(t) \mathbb{1}_\gamma(\xi)\big) w(t,\xi,0) - \alpha(t) & \text{for} & (t,\xi)\in (0,T)\times(0,L),\\
w(t,\xi,1) = 0 & \text{for} & (t,\xi)\in (0,T)\times(0,L),\\
w(t,0,\eta) = w_s(0,\eta) + z_b(t,\eta) & \text{for} & (t,\eta)\in (0,T)\times(0,1),\\
w(0,\xi,\eta) = w_s(\xi,\eta) & \text{for} & (\xi,\eta)\in\Omega,
\end{array}\right. \tag{50}$$

with $\alpha(t) = \frac{u'_\infty(t)}{U_\infty(t)}$, w_s a given stationary solution of system (2). To stabilize this equation, we replace $u(t)$ by

$$u(t) = -\int_\Omega k_R(\xi,\eta)\big[w(t,\xi,\eta) - w_s(\xi,\eta)\big]d\xi d\eta - \frac{1}{R}\int_\gamma b(\xi,0) r(t,s,0)\,ds,$$

where r is solution to equation (15),

$$k_R(\xi,\eta) = -\frac{1}{R}\int_\gamma b(s,0)\pi(\xi,\eta,s,0)\,ds,$$

and π solution to equation (14).

To solve equation (50) we use the $P_1 - P_1$ finite element method described previously. We look for w^h of the form

$$w^h(t,\xi,\eta) = \sum_{j=1}^{M+1}\sum_{i=1}^{N}\sum_{\ell=0}^{1} w_{ij}^\ell(t)\varphi_{ij}^\ell(\xi,\eta),$$

such that for all $\varphi_{ij}^\ell \in V_h \times W_h$ and $t \geq 0$,

$$\begin{aligned}
\int_0^1\int_{\xi_i}^{\xi_{i+1}} \frac{\partial w_i^h}{\partial t}\varphi_{ij}^\ell \;=\;& \int_0^1\int_{\xi_i}^{\xi_{i+1}} \eta U_\infty w_i^h \frac{\partial \varphi_{ij}^\ell}{\partial \xi} - \int_0^1 \eta U_\infty w_i^h(t,\xi_{i+1},\eta)\varphi_{ij}^\ell(\xi_{i+1},\eta)\\
&+\int_0^1 \eta U_\infty w_{i-1}^h(t,\xi_i,\eta)\varphi_{ij}^\ell(\xi_i,\eta) - \int_0^1\int_{\xi_i}^{\xi_{i+1}} \nu (w_i^h)^2 \frac{\partial w_i^h}{\partial \eta}\frac{\partial \varphi_{ij}^\ell}{\partial \eta}\\
&+\int_{\xi_i}^{\xi_{i+1}} \big(v_s(\xi) + \nu u(t)\mathbb{1}_\gamma(\xi))(w_i^h)^2(t,\xi,0) - \alpha(t) w_i^h(t,\xi,0)\big)\,\varphi_{ij}^\ell(\xi,0)\\
&-\int_0^1\int_{\xi_i}^{\xi_{i+1}} \left(\alpha(t) w_i^h + 2\nu w_i^h\left(\frac{\partial w_i^h}{\partial \eta}\right)^2 + \alpha(t)(1-\eta)\frac{\partial w_i^h}{\partial \eta}\right)\varphi_{ij}^\ell,
\end{aligned}$$

with $w_i^h(t,\xi,\eta) = \sum_{j=1}^{M+1}\sum_{\ell=0}^{p} w_{ij}^\ell(t)\varphi_{ij}^\ell(\xi,\eta)$ for all $(\xi,\eta)\in[\xi_i,\xi_{i+1}]\times[0,1]$. The term w_0^h is replaced by $\sum_{j=1}^{M+1}\big(w_b(\eta_j) + z_b(t,\eta_j)\big)\varphi_j(\eta)$. As the time part is discretized with a Backward Differentiation Formula of order 2 we have to solve a nonlinear system. For that, we use a quasi-Newton method. Each Newton step is carried out with a LU factorization. When $u \neq 0$, we use the kernel k_R determined with the linearized Crocco equation around w_s. The term $\frac{1}{R}\int_\gamma b(\xi,0)\,r(t,s,0)\,ds$ is directly replaced by $R^{-1}(B^n)^T r^n(t)$. These functions are interpolated on a finest mesh if necessary.

Test 4. We take the same parameters as in Test 1 of the previous section. The functions w_s and z_b are given by (44) and (46). Figure 9 shows the uncontrolled and controlled observation $y_c(t)$ obtained with the linearized Crocco equation (y_c^L) and the nonlinear Crocco equation (y_c^{NL}). We can see that the feedback operator determined with the linear model is able to stabilize the nonlinear Crocco equation. It seems natural since the solution of the linear model is a very good approximation of the nonlinear model.

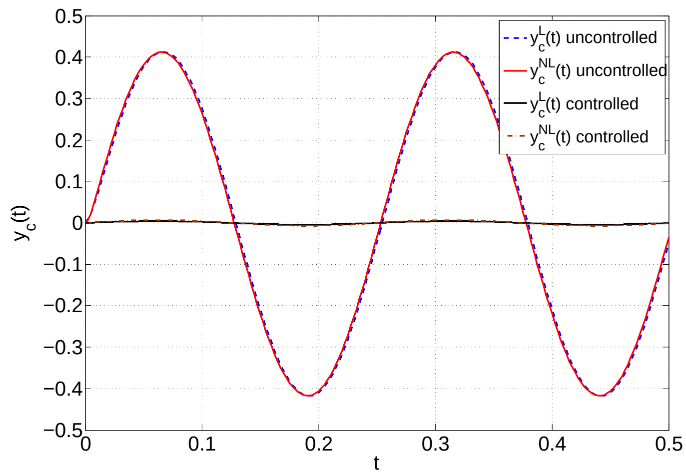

Figure 9: Uncontrolled and controlled observations obtained with the linear and the nonlinear models where the stationary solution corresponds to the class of asymptotic type solutions.

Test 5. We want to stabilize the nonlinear Crocco equation around the stationary Blasius solution given in Test 3 of the previous section. The perturbation z_b is taken equal to $w_B(t,0.03,\eta) - w_b(0.03,\eta)$ where w_B and w_b are defined by (49). The nonlinear Crocco equation is solved with a mesh divided into 100×100 quadrangles. We use a geometric mesh in the $\eta-$direction with $\frac{\eta_2-\eta_1}{\eta_{M+1}-\eta_M} = 10$ and a uniform mesh in the $\xi-$direction. Figure 10 shows the uncontrolled and controlled observations. We notice that for this class of solution the feedback law built with the linearized model is still able to stabilize the nonlinear model.

7 Conclusions and further works.

In this paper, we have considered the theoretical and numerical stabilization of the linearized Crocco equation (1).

We have supposed that the perturbation u_∞ is known. Therefore, the action of the perturbation on the observation that we want to control is taken into account in the feedback law with an extra term r solution of a backward equation.

Similar numerical results for the LQG problem based on the measurement of w on a part of the flat plate have been obtained in [5]. In this case, the longitudinal and vertical velocities in the boundary layer can be estimated.

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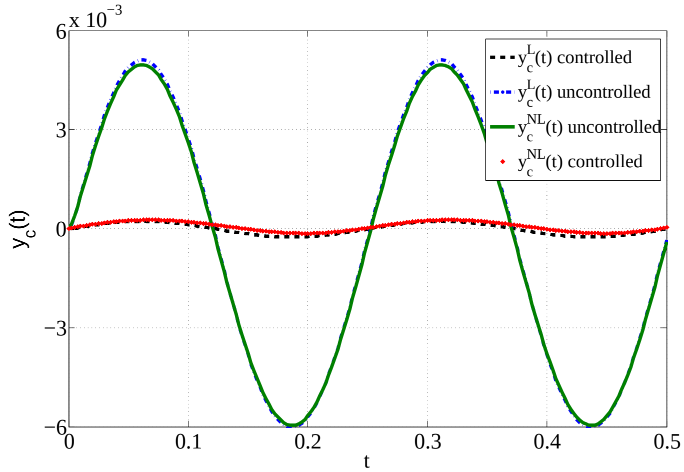

Figure 10: Uncontrolled and controlled observations obtained with the linear and the nonlinear models where the stationary solution corresponds to the solution of the Blasius equation.

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