Chapter 3

Optimal Control of Evolution Equations with Bounded Control Operators

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Introduction to the optimal control of evolution equations

Distributed control of the heat equation

Existence of optimal controls

Characterization of optimal controls

Distributed control of the wave equation

A general control problem

Control of a first order hyperbolic system

Optimal control of evolution equations

Setting of the problem

We consider equations of the form

$$(E) \qquad y' = Ay + Bu + f, \qquad y(0) = y_0.$$

Assumptions

Y and U are Hilbert spaces.
The unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y. This semigroup will be denoted by $(e^{tA})_{t\geq 0}$.
The operator B belongs to $\mathcal{L}(U;Y)$.

The control problem

$$(P)\ \ \text{inf}\{J(y,u) \mid u \in L^2(0,T;U),\ (y,u) \text{ obeys } (E)\},$$

$$J(y,u)$$

$$= \frac{1}{2}\int_0^T |Cy(t) - z_d(t)|_Z^2 + \frac{1}{2}|Dy(T) - z_T|_{Z_T}^2$$

$$+\frac{1}{2}\int_0^T |u(t)|_U^2.$$

Assumption

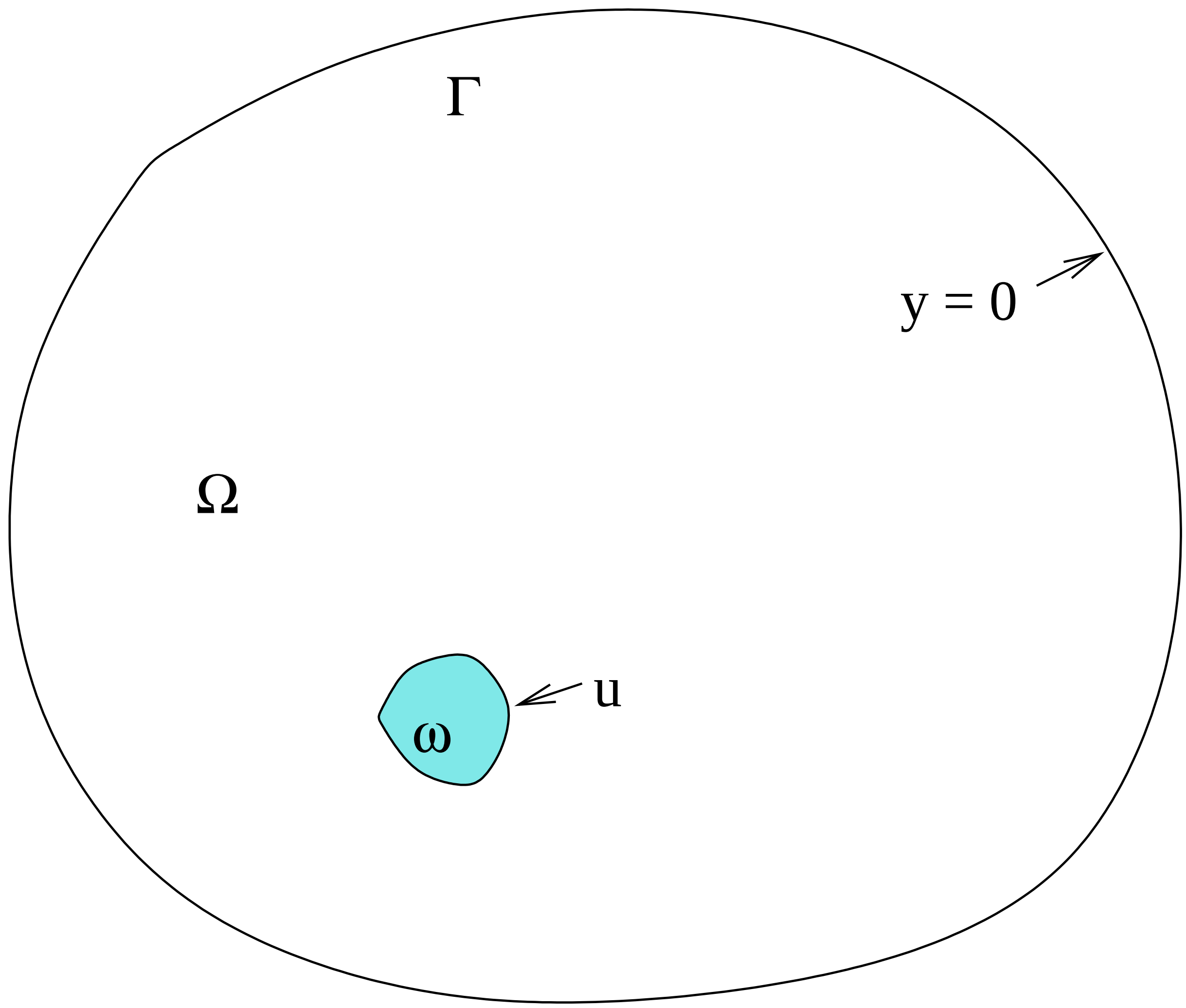
Z and Z_T are Hilbert spaces.
The operator C belongs to $\mathcal{L}(Y;Z)$, and the operator D belongs to $\mathcal{L}(Y;Z_T)$. The function z_d belongs to $L^2(0,T;Z)$ and $z_T \in Z_T$.

Optimal control of the heat equation

The state equation

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T > 0$, set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider the heat equation with a distributed control

$$(HE) \qquad \begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u \quad \text{in } Q, \\ &y = 0 \quad \text{on } \Sigma, \quad y(x,0) = y_0 \quad \text{in } \Omega. \end{aligned}$$


Γ
y = 0
Ω
u
ω

The control problem

$$(P) \qquad \begin{aligned} &\inf\{J(y,u) \mid u \in L^2(\omega \times (0,T))), \\ &(y,u) \text{ obeys } (HE)\}, \end{aligned}$$

where

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2$$

$$+\frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2,$$

$\beta > 0$ and $y_d \in C([0,T]; L^2(\Omega))$.

Estimate for the state variable

$$\|y\|_{C([0,T];L^2(\Omega))}$$

$$\leq C\big(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} + \|u\|_{L^2(\omega\times(0,T))}\big).$$

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\omega \times (0,T))$, that is

$$\lim_{n\to\infty} F(u_n) = \inf_{u \in L^2(\omega\times(0,T))} F(u).$$

Let y_n the solution of (HE) corresponding to u_n, suppose that $(u_n)_n$ is bounded in $L^2(\omega \times (0,T))$, and that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\omega \times (0,T)).$$

2. Let $\bar{y} = y(\bar{u})$.

The operator

$$\Lambda \;:\; u \longrightarrow y(u)$$

is affine and continuous from $L^2(\omega \times (0,T))$ to $L^2(Q)$, and

$$\Lambda_T \;:\; u \longrightarrow y(u)(T)$$

is affine and continuous from $L^2(\omega \times (0,T))$ to $L^2(\Omega)$.

The sequence $(y_n)_n$ converges to $\bar{y}$ for the weak topology of $L^2(Q)$, and $(y_n(T))_n$ converges to $\bar{y}(T)$ for the weak topology of $L^2(\Omega)$.

3. Using the weakly lower semicontinuity of $\|\cdot\|^2_{L^2(Q)}$, $\|\cdot\|^2_{L^2(\Omega)}$, $\|\cdot\|^2_{L^2(\omega\times(0,T))}$, we obtain

$$\int_{\omega\times(0,T)} \bar{u}^2 \leq \text{liminf}_{n\to\infty} \int_{\omega\times(0,T)} u_n^2,$$

$$\int_Q |\bar{y}-y_d|^2 \leq \text{liminf}_{n\to\infty} \int_Q |y_n-y_d|^2,$$

and

$$\int_\Omega |\bar{y}(T)-y_d(T)|^2 \leq \text{liminf}_{n\to\infty} \int_\Omega |y_n(T)-y_d(T)|^2.$$

Combining these results, we have

$$F(\bar{u}) \leq \text{liminf}_{n\to\infty} F(u_n) = m.$$

Thus $\bar{u}$ is a solution to (P).

Uniqueness. Recall that the mappings

$$u \longrightarrow y(u) \quad \text{and} \quad u \longrightarrow y(u)(T)$$

are affine. Thus

$$u \longrightarrow \frac{1}{2}\int_Q |y(u) - y_d|^2 + \frac{1}{2}\int_\Omega |y(u)(T) - y_d(T)|^2$$

is convex. The mapping

$$u \longrightarrow \frac{\beta}{2}\int_Q \chi_\omega u^2$$

is stricly convex. Thus the uniqueness follows from the strict convexity of F.

Optimality conditions

Derivative of the state variable

Equation satisfied by $z_\lambda = y(u+\lambda v) - y(u)$

$$\frac{\partial z}{\partial t} - \Delta z = \lambda \chi_\omega v \quad \text{in } Q,$$

$$z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = 0 \quad \text{in } \Omega.$$

From the estimate for (HE) it follows that

$$\|z_\lambda\|_{C([0,T];L^2(\Omega))} \le C|\lambda|\|v\|_{L^2(\omega\times(0,T))}.$$

Thus

$$y(u+\lambda v) \quad \stackrel{C([0,T];L^2(\Omega))}{\longrightarrow} \quad y(u).$$

$$F'(u)v = \text{lim}_{\lambda\searrow 0}\frac{F(u+\lambda v)-F(u)}{\lambda}.$$

By a classical calculation we have

$$F'(u)v = \int_Q (y(u)-y_d)z(v)$$

$$+\int_\Omega (y(u)(T)-y_d(T))z(v)(T) + \beta\int_{\omega\times(0,T)} uv,$$

where $z(v)$ is the solution of

$$\frac{\partial z}{\partial t} - \Delta z = \chi_\omega v \quad \text{in } Q,$$

$$z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u)-y_d)z(v) + \int_\Omega [(y(u)-y_d)z(v)](T) = \int_{\omega\times(0,T)} q\, v.$$

Let p be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and p:

$$\int_{\omega\times(0,T)} v\, p = \int_Q (z_t - \Delta z)p$$
$$= \int_Q z(-p_t - \Delta p) + \int_\Omega z(T)p(T) - \int_\Sigma \frac{\partial z}{\partial n} p$$

Identification with

$$\int_Q (y(u) - y_d) z + \int_\Omega [(y(u) - y_d) z](T) = \int_{\omega \times (0,T)} q\, v.$$

We set

$$-\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \quad \text{in } Q,$$

$$p = 0 \quad \text{on } \Sigma, \quad p(x,T) = (y(u) - y_d)(T) \quad \text{in } \Omega,$$

and we have

$$F'(u)v = \int_{\omega \times (0,T)} (p + \beta u) v,$$

if the above calculation are justified.

The adjoint equation

Let $g \in L^2(Q)$, $p_T \in L^2(\Omega)$. The terminal boundary value problem

$$(AE) \qquad \begin{aligned} -\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \quad p(x,T) = p_T \quad \text{in } \Omega, \end{aligned}$$

is well posed.

$$\|p\|_{C([0,T];L^2(\Omega))} \leq C(\|g\|_{L^2(Q)} + \|p_T\|_{L^2(\Omega)}).$$

Proof. A weak solution in $L^2(0,T;L^2(\Omega))$ to (AE) is a function $p \in L^2(0,T;L^2(\Omega))$ such that, for all $z \in H^2 \cap H_0^1(\Omega)$, the mapping

$$t \longmapsto \langle p(t), z\rangle$$

belongs to $H^1(0,T)$ and obeys

$$-\frac{d}{dt}\langle p(t), z\rangle = \langle y(t), A^* z\rangle + \langle g(t), z\rangle,$$
$$\langle p(T), z\rangle = \langle p_T, z\rangle.$$

The function p is a weak solution to (AE) if and only if the function q defined by

$$q(x,t) = p(x, T-t)$$

is the solution to the equation

$$\begin{aligned} &\frac{\partial q}{\partial t} - \Delta q = \tilde{g} \quad \text{in } Q, \\ &q = 0 \quad \text{on } \Sigma, \quad q(x,0) = p_T \quad \text{in } \Omega, \end{aligned}$$

where $\tilde{g}(x,t) = g(x, T-t)$.

Integration by parts between z and p

Theorem. Suppose that $\phi \in L^2(Q)$, $g \in L^2(Q)$, and $p_T \in L^2(\Omega)$. Then the solution z of equation

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in } Q, \qquad z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = 0 \quad \text{in } \Omega,$$

and the solution p of (AE) satisfy the following formula

$$\int_Q \phi\, p = \int_Q z\, g + \int_\Omega z(T) p_T.$$

Proof. If $p_T \in H_0^1(\Omega)$, due to a Theorem of Chapter 2, z and p belong to $L^2(0,T;D(A))) \cap H^1(0,T;L^2(\Omega))$. In that case, with the Green formula we have

$$\int_\Omega -\Delta z(t)p(t)\,dx = \int_\Omega -\Delta p(t)z(t)\,dx$$

for almost every $t \in [0,T]$, and

$$\int_0^T \int_\Omega \frac{\partial z}{\partial t} p = -\int_0^T \int_\Omega \frac{\partial p}{\partial t} z + \int_\Omega z(T)p_T.$$

Thus the IBP formula is established in the case when $p_T \in H_0^1(\Omega)$. If $(p_{Tn})_n$ is a sequence in $H_0^1(\Omega)$ converging to p_T in $L^2(\Omega)$, due to the '$C([0,T];L^2(\Omega))$-estimate', $(p_n)_n$ - where p_n is the solution to (AE) corresponding to p_{Tn} - converges to p (the solution of (AE) associated with p_T) in $C([0,T];L^2(\Omega))$ when n tends to infinity. Thus, in the case when $p_T \in L^2(\Omega)$, the IBP formula can be deduced by passing to the limit in the formula satisfied by p_n.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)}$, where p is the solution to the adjoint equation corresponding to $\bar{y}$:

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{y} - y_d \quad \text{in } Q,$$

$$p = 0 \quad \text{on } \Sigma, \quad p(x, 0) = \bar{y}(T) - y_d(T) \quad \text{in } \Omega.$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$
\begin{aligned}
&\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = f - \frac{1}{\beta}\chi_\omega \tilde{p} \quad \text{in } Q, \\
&\tilde{y} = 0 \quad \text{on } \Sigma, \quad \tilde{y}(x,0) = \bar{y}_0 \quad \text{in } \Omega, \\
&-\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = \tilde{y} - y_d \quad \text{in } Q, \\
&p = 0 \quad \text{on } \Sigma, \quad \tilde{p}(T) = \tilde{y}(T) - y_d(T) \quad \text{in } \Omega,
\end{aligned}
$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem (P).

Proof. (i) The necessary optimality condition is already proved.
(ii) The sufficient optimality condition can be proved with a theorem stated in Chapter 1.

Optimal control of the wave equation

The state equation

The assumptions on Ω, Γ, ω, T, Q, Σ are the ones of the previous section. We consider

(WE)

$$y'' - \Delta y = f + \chi_\omega u \quad \text{in } Q, \qquad y = 0 \text{ on } \Sigma,$$

$$y(x,0) = y_0 \quad \text{and} \quad y'(x,0) = y_1 \quad \text{in } \Omega,$$

with $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $f \in L^2(Q)$, and $u \in L^2(\omega \times (0,T))$.

The operator

$$(f + \chi_\omega u, y_0, y_1) \mapsto y(f + \chi_\omega u, y_0, y_1)$$

is linear and continuous from $L^2(Q) \times H_0^1(\Omega) \times L^2(\Omega)$ into $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

The family of control problems

$$(P_i) \qquad \text{inf}\{J_i(y,u) \mid (y,u) \text{ obeys } (WE),\ u \in L^2\},$$

with, for $i = 1, \ldots, 3$, the functionals J_i are defined by

$$J_1(y,u) = \frac{1}{2}\int_Q |y - y_d|^2 + \frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2,$$

$$J_2(y,u) = \frac{1}{2}\int_\Omega |\nabla y(T) - \nabla y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2,$$

$$J_3(y,u) = \frac{1}{2}\int_\Omega \left|y'(T) - y_d'(T)\right|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2,$$

where the function $y_d \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$.

Theorem. Assume that $f \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $y_d \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. For $i = 1, \ldots, 3$, problem (P_i) admits a unique solution $(\bar{y}_i, \bar{u}_i)$.

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\omega \times (0,T))$, that is

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in L^2(\omega\times(0,T))} F(u).$$

We suppose that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\omega \times (0,T)).$$

Let y_n the solution of (WE) corresponding to u_n, suppose that $(u_n)_n$ is bounded in $L^2(\omega \times (0,T))$, and that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\omega \times (0,T)).$$

Passage to the limit in the equation.

Let $\bar{y} = y(\bar{u})$. The operator

$$\Lambda \ : \ u \longrightarrow \Big(y(u), y(u)(T), y(u)'(T)\Big)$$

is affine and continuous from $L^2(\omega \times (0,T))$ to $L^2(Q) \times H_0^1(\Omega) \times L^2(\Omega)$.

We may conclude that, for $i = 1, \ldots, 3$, problem (P_i) admits a unique solution $(\bar{y}_i, \bar{u}_i)$.

Optimality conditions for (P_1)

$$J_1(y,u)$$
$$= \frac{1}{2}\int_Q |y-y_d|^2 + \frac{1}{2}\int_\Omega |y(T)-y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2,$$

By a classical calculation we have

$$F'(u)v = \int_Q (y(u)-y_d)z(v)$$
$$+ \int_\Omega (y(u)(T)-y_d(T))z(v)(T) + \beta\int_{\omega\times(0,T)} uv,$$

where $z(v)$ is the solution of

$$z'' - \Delta z = \chi_\omega v \quad \text{in } Q, \qquad z = 0 \quad \text{on } \Sigma,$$
$$z(x,0) = 0 \text{ and } z'(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u)-y_d)z(v)+\int_\Omega [(y(u)-y_d)z(v)](T) = \int_{\omega\times(0,T)} q\, v.$$

Let p be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and p:

$$\int_{\omega\times(0,T)} v\, p = \int_Q (z'' - \Delta z)p$$

$$= \int_Q z(p'' - \Delta p) + \int_\Omega z'(T)p(T)$$

$$- \int_\Omega z(T)p'(T) - \int_\Sigma \frac{\partial z}{\partial n} p$$

Identification with

$$\int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_{\omega\times(0,T)} q\, v.$$

We set

$$p'' - \Delta p = y(u) - y_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma,$$

$$p(x,T) = 0 \text{ and } \; p'(x,T) = (y(u) - y_d)(T) \quad \text{in } \Omega.$$

and we have

$$F'(u)v = \int_{\omega\times(0,T)} (p + \beta u)v,$$

if the above calculation are justified.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P_1) then $\bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)}$, where p is the solution to:

$$p'' - \Delta p = \bar{y} - y_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma,$$

$$p(x, T) = 0, \quad p'(x, T) = \bar{y}(T) - y_d(T) \quad \text{in } \Omega,$$

(ii) Conversely, if $(\tilde{y}, \tilde{p}) \in (C([0, T]; L^2(\Omega)))^2$ obeys:

$$\tilde{y}'' - \Delta \tilde{y} = f - \frac{1}{\beta} \chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma,$$

$$\tilde{y}(x, 0) = y_0, \quad \tilde{y}'(x, 0) = y_1, \quad \text{in } \Omega,$$

$$\tilde{p}'' - \Delta \tilde{p} = \tilde{y} - y_d \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,$$

$$\tilde{p}(T) = 0, \quad \tilde{p}'(T) = y(T) - y_d(T) \quad \text{in } \Omega,$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to (P_1).

Optimality conditions for (P_2)

Recall that

$$J_2(y,u) = \frac{1}{2}\int_\Omega |\nabla y(T) - \nabla y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2 \,.$$

Theorem. (i) If $(\bar{y},\bar{u})$ is the solution to (P_2) then $\bar{u} = -\frac{1}{\beta}p|_{\omega\times(0,T)}$, where p is the solution to the adjoint equation

$$p'' - \Delta p = 0 \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma,$$

$$p(T) = 0 \text{ and } \; p'(T) = -\Delta(\bar{y}(T) - y_d(T)) \quad \text{in } \Omega.$$

$$(p,p') \in C([0,T];L^2(\Omega)) \times C([0,T];H^{-1}(\Omega)).$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$\tilde{y}'' - \Delta \tilde{y} = f - \frac{1}{\beta}\chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma,$$

$$\tilde{y}(x,0) = y_0, \quad \tilde{y}'(x,0) = y_1, \quad \text{in } \Omega,$$

$$\tilde{p}'' - \Delta \tilde{p} = 0 \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,$$

$$\tilde{p}(T) = 0, \quad p'(T) = -\Delta(\tilde{y}(T) - y_d(T)) \quad \text{in } \Omega,$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to (P_2).

Remark 1. We set

$$F_2(u) = J_2(y(u), u).$$

We have

$$F_2'(u)v = \int_\Omega \Big(\nabla y(T) - \nabla y_d(T)\Big) \cdot \nabla z(T) + \beta \int_{\omega \times (0,T)} u\, v\,,$$

where z is the solution to

$$z'' - \Delta z = \chi_\omega v \quad \text{in } Q, \qquad z = 0 \quad \text{on } \Sigma,$$

$$z(x,0) = 0, \qquad z'(x,0) = 0, \qquad \text{in } \Omega.$$

Moreover

$$\int_\Omega \Big(\nabla y(T) - \nabla y_d(T)\Big) \cdot \nabla z(T)$$
$$= \Big\langle z(T), (-\Delta)(y(T) - y_d(T))\Big\rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.$$

This is why we have

$$p'(x,T) = -\Delta(\bar{y}(T) - y_d(T))$$

in the adjoint equation.

Remark 2. If $\tilde{y} \in C([0,T]; H_0^1(\Omega))$, then $\Delta\tilde{y}(T)$ belongs to $H^{-1}(\Omega)$. Thus the adjoint equation is stated with $p'(T)$ in $H^{-1}(\Omega)$. We are going to prove that the wave equation is well posed with an initial condition in $L^2(\Omega) \times H^{-1}(\Omega)$.

Let us recall a result from chapter 2. Set $Y = H_0^1(\Omega) \times L^2(\Omega)$ and endow Y with the inner product

$$\Big(u, v\Big)_Y = \int_\Omega \nabla u_1 \cdot \nabla v_1 + \int_\Omega u_2 \, v_2,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Set $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and

$$Ay = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

In chapter 2 we have proved that $(A, D(A))$ and $(-A, D(A))$ are m-dissipative in Y.

Now we set $\widehat{Y} = L^2(\Omega) \times H^{-1}(\Omega)$. We equip $\widehat{Y}$ with the inner product

$$\Big(u, v\Big)_{\widehat{Y}} = \int_\Omega u_1 \cdot v_1 + \Big\langle (-\Delta)^{-1} u_2, v_2 \Big\rangle_{H_0^1(\Omega), H^{-1}(\Omega)},$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Set $D(\widehat{A}) = H_0^1(\Omega) \times L^2(\Omega)$ and

$$\widehat{A}y = \widehat{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}.$$

We can prove that $(\widehat{A}, D(\widehat{A}))$ and $(-\widehat{A}, D(\widehat{A}))$ are m-dissipative in $\widehat{Y}$.

Optimality conditions for (P_3)

The functional is

$$J_3(y,u) = \frac{1}{2}\int_\Omega \left| y'(T) - y_d'(T) \right|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} u^2 \,.$$

Theorem. (i) If $(\bar{y},\bar{u})$ is the solution to (P_3) then $\bar{u} = -\frac{1}{\beta} p|_{\omega\times(0,T)}$, where p is the solution to the adjoint

$$p'' - \Delta p = 0 \quad \text{in } Q, \qquad p = 0 \quad \text{on } \Sigma,$$
$$p(T) = (\bar{y}' - y_d')(T) \text{ and } \; p'(T) = 0 \quad \text{in } \Omega.$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$\tilde{y}'' - \Delta \tilde{y} = f - \frac{1}{\beta}\chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma,$$

$$\tilde{y}(x,0) = y_0, \quad \tilde{y}'(x,0) = y_1, \quad \text{in } \Omega,$$

$$\tilde{p}'' - \Delta \tilde{p} = 0 \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,$$

$$\tilde{p}(T) = (\tilde{y}' - y_d')(T), \quad \tilde{p}'(T) = 0 \quad \text{in } \Omega,$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to (P_3).

Optimal control of evolution equations

The state equation

$$(SE) \qquad y' = Ay + Bu + f, \qquad y(0) = y_0.$$

Assumptions

Y and U are Hilbert spaces.
The unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Z. This semigroup will be denoted by $(e^{tA})_{t\geq 0}$.
The operator B belongs to $\mathcal{L}(U;Y)$.

The control problem
(P)

$$\inf\{J(y,u) \mid u \in L^2(0,T;U),\ (y,u) \text{ obeys } (SE)\},$$

with

$$J(y,u) = \frac{1}{2}\int_0^T |Cy(t) - z_d(t)|_Z^2$$

$$+\frac{1}{2}|Dy(T) - z_T|_{Z_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2.$$

Assumption

Z and Z_T are Hilbert spaces.
The operator C belongs to $\mathcal{L}(Y;Z)$, and the operator D belongs to $\mathcal{L}(Y;Z_T)$. The function z_d belongs to $L^2(0,T;Z)$ and $z_T \in Z_T$.

Existence of a unique optimal control

If the assumptions on B, C, D are satisfied. Problem (P) admits a unique solution (y, u).

The proof is based on the existence of a minimizing sequence $(u_n)_n$, bounded in $L^2(0, T; U)$, and on the fact that the operator

$$\Lambda \ : \ u \longrightarrow \Big(Cy(u) - z_d, Dy(u)(T) - z_T\Big)$$

is affine and continuous from $L^2(0, T; U)$ to $L^2(0, T; Z) \times Z_T$.

Optimality conditions

The adjoint equation for (P) will be of the form

$$(AE) \qquad -p' = A^* p + g, \qquad p(T) = p_T.$$

From chapter 2, we know that $(A^*, D(A^*))$ is the infinitesimal generator of a strongly continuous semigroup on Y'. Thus (AE) is well posed if $p_T \in Y'$ and if $g \in L^1(0, T; Y')$. For simplicity we identify Y and Y'.

Integration by parts formula

We state an integration by parts formula between the adjoint state p and the solution z to the equation

$$(LE) \qquad z' = Az + f, \qquad z(0) = 0.$$

Theorem. For every $f \in L^2(0,T;Y)$, and every $(g, p_T) \in L^2(0,T;Y) \times Y$, the solution z to equation (LE) and the solution p to equation (AE) satisfy the following formula

$$\int_0^T \Big(f(t), p(t)\Big)_Y \, dt$$

$$= \int_0^T \Big(z(t), g(t)\Big)_Y \, dt + \Big(z(T), p_T\Big)_Y - \Big(z_0, p(0)\Big)_Y .$$

Proof. Suppose that f and g belong to $C^1([0,T];Y)$ and that p_T belongs to $D(A^*)$. In this case we can write

$$\int_0^T \Big(f(t),p(t)\Big)_Y\,dt = \int_0^T \Big(z'(t)-Az(t),p(t)\Big)_Y\,dt$$

$$= \int_0^T -\Big(z(t),p'(t)\Big)_Y\,dt + \Big(z(T),p_T\Big)_Y$$

$$-\Big(z_0,p(0)\Big)_Y - \int_0^T \Big(Az(t),p(t)\Big)_Y\,dt$$

$$= \int_0^T \Big(z(t),g(t)\Big)_Y\,dt + \Big(z(T),p_T\Big)_Y - \Big(z_0,p(0)\Big)_Y.$$

Thus, the IBP formula can be deduced from this case by using density arguments.

Optimality conditions

Theorem. If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = -B^*p$, where p is the solution to equation

$$-p' = A^*p + C^*(C\bar{y} - z_d), \qquad p(T) = D^*(D\bar{y}(T) - z_T).$$

Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T];Y) \times C([0,T];Y)$ obeys the system

$$\tilde{y}' = A\tilde{y} - BB^*\tilde{p} + f, \qquad \tilde{y}(0) = y_0,$$
$$-\tilde{p}' = A^*\tilde{p} + C^*(C\tilde{y} - z_d),$$
$$\tilde{p}(T) = D^*(D\tilde{y}(T) - z_T),$$

then the pair $(\tilde{y}, -B^*\tilde{p})$ is the optimal solution to problem (P).

Proof. Let $(\bar{y}, \bar{u})$ be the optimal solution to problem (P). Set $F(u) = J(y(u), u)$. For every $u \in L^2(0, T; U)$, we have

$$F'(\bar{u})u = \int_0^T \Big(C\bar{y}(t) - z_d, Cz(t)\Big)_Z$$

$$+\Big(D\bar{y}(T) - z_T, Dz(T)\Big)_{Z_T} + \int_0^T \Big(\bar{u}(t), u(t)\Big)_U$$

$$= \int_0^T \Big(C^*(C\bar{y}(t) - z_d), z(t)\Big)_Y$$

$$+\Big(D^*(D\bar{y}(T) - z_T), z(T)\Big)_Y + \int_0^T (\bar{u}(t), u(t))_U,$$

where z is the solution to

$$z' = Az + Bu, \qquad z(0) = 0.$$

Applying the IBP formula to p and z, we obtain

$$F'(\bar{u})u = \int_0^T (p(t), Bu(t))_Y + \int_0^T (\bar{u}(t), u(t))_U$$

$$= \int_0^T (B^* p(t) + \bar{u}(t), u(t))_U.$$

The first part of the Theorem is established. The second part follows from the sufficient optimality condition stated in Chapter 1.

Exercise

Let $L > 0$ and a be a function in $H^1(0, L)$ such that $0 < c_1 \leq a(x)$ for all $x \in H^1(0, L)$. Consider the equation

(TE)

$$
\begin{aligned}
& y_t + a y_x = f + \chi_{(\ell_1,\ell_2)} u, \qquad \text{in } (0, L) \times (0, T), \\
& y(0, t) = 0, \qquad \text{in } (0, T), \\
& y(x, 0) = y_0, \qquad \text{in } (0, L),
\end{aligned}
$$

where $f \in L^2((0, L) \times (0, T))$, $\chi_{(\ell_1,\ell_2)}$ is the characteristic function of $(\ell_1, \ell_2) \subset (0, L)$, $u \in L^2((\ell_1, \ell_2) \times (0, T))$, and $y_0 \in L^2(0, L)$.

Prove that (TE) admits a unique solution in $C([0, T]; L^2(0, L))$ (use the Hille-Yosida theorem).

Study the control problem

$$(P) \qquad \begin{aligned} &\text{inf}\{J(y,u) \mid u \in L^2(0,T;L^2(\ell_1,\ell_2)),\\ &\ (y,u) \text{ satisfies } (TE)\}. \end{aligned}$$

with

$$J(y,u) = \frac{1}{2}\int_0^L (y(T) - y_d(T))^2 + \frac{1}{2}\int_0^T \int_{\ell_1}^{\ell_2} u^2,$$

where $y_d \in C([0,T];L^2(0,L))$. Prove the existence of a unique solution. Write first order optimality conditions.

Optimal control of a first order hyperbolic system

The state equation

Consider the first order hyperbolic system

$$\frac{\partial}{\partial t}\left[\begin{array}{c} z_1(x,t) \\ z_2(x,t) \end{array}\right] = \frac{\partial}{\partial x}\left[\begin{array}{c} m_1 z_1 \\ -m_2 z_2 \end{array}\right] - \left[\begin{array}{c} a_{11}z_1 + a_{12}z_2 + b_1 u_1 \\ a_{21}z_1 + a_{22}z_2 + b_2 u_2 \end{array}\right],$$

in $(0,\ell)\times(0,T)$, with the initial condition

$$z_1(x,0) = z_{01}(x), \qquad z_2(x,0) = z_{02}(x) \qquad \text{in} \ \ (0,\ell),$$

and the boundary conditions

$$z_1(\ell,t) = 0, \qquad z_2(0,t) = 0 \qquad \text{in} \ \ (0,T).$$

We refer to this system as the system (HE). This kind of systems intervenes in heat exchangers [9].

We suppose that the constant coefficients $m_1 > 0$, $m_2 > 0$, and that a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 are regular.

State equation

We set $Y = L^2(0,\ell) \times L^2(0,\ell)$, and we define the unbounded operator A in Y by

$$D(A) = \{z \in H^1(0,\ell)\times H^1(0,\ell) \mid z_1(\ell) = 0,\ z_2(0) = 0\}$$

and

$$Az = \begin{bmatrix} m_1 \dfrac{dz_1}{dx} \\ -m_2 \dfrac{dz_2}{dx} \end{bmatrix}.$$

We define the operator $L \in \mathcal{L}(Y)$ by

$$Lz = \begin{bmatrix} -a_{11}z_1 - a_{12}z_2 \\ -a_{21}z_1 - a_{22}z_2 \end{bmatrix}.$$

Theorem. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on Y.

Proof. The theorem relies the Hille-Yosida theorem.
(i) The operator A is dissipative in Y:

$$(Az, z) = \int_0^\ell m_1 \frac{dz_1}{dx} z_1 - \int_0^\ell m_2 \frac{dz_2}{dx} z_2$$
$$= -\frac{m_1}{2} z_1(0)^2 - \frac{m_2}{2} z_2(\ell)^2 \leq 0.$$

(ii) For $\lambda > 0$, $f \in L^2(0,\ell)$, $g \in L^2(0,\ell)$, consider the equation

$$z \in D(A), \qquad \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is

$$\begin{aligned} \lambda z_1 - m_1 \frac{dz_1}{dx} &= f \qquad \text{in } (0,\ell), \quad z_1(\ell) = 0, \\ \lambda z_2 + m_2 \frac{dz_2}{dx} &= g \qquad \text{in } (0,\ell), \quad z_2(0) = 0. \end{aligned}$$

This equation admits a unique solution $z \in D(A)$.

Theorem. The operator $(A+L, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y.

Theorem. For all $z_0 = (z_{10}, z_{20}) \in Y$, $u_1 \in L^2((0,\ell)\times(0,T))$, $u_2 \in L^2((0,\ell)\times(0,T))$, the system (HE) admits a unique weak solution in $L^2(0,T;L^2(0,\ell))$, this solution belongs to $C([0,T];Y)$ and satisfies

$$\|z\|_{C([0,T];Y)} \\ \leq C\Big(\|z_0\|_Y + \|u_1\|_{L^2((0,\ell)\times(0,T))} + \|u_2\|_{L^2((0,\ell)\times(0,T))}\Big).$$

The adjoint operator of $(A, D(A))$, with respect to the Y-topology, is defined by

$$D(A^*) = \Big\{ (\phi, \psi) \in H^1(0, \ell) \times H^1(0, \ell) \\ \mid \phi(0) = 0, \quad \psi(\ell) = 0 \Big\},$$

and

$$(A^* + L^*) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} -m_1 \dfrac{d\phi}{dx} - a_{11}\phi - a_{21}\psi \\ m_2 \dfrac{d\psi}{dx} - a_{12}\phi - a_{22}\psi \end{bmatrix}.$$

To study the system (HE), we define the operator $B \in \mathcal{L}((L^2(0, \ell))^2)$ by

$$B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 u_1 \\ b_2 u_2 \end{bmatrix}.$$

The (HE) is of the form

$$z' = (A + L)z + Bu, \qquad z(0) = z_0.$$

The control problem

We want to study the control problem

$$(P) \qquad \begin{array}{l} \text{inf}\{J(z,u) \mid \ (z,u) \text{ obeys } (HE), \\ \ u \in (L^2((0,\ell) \times (0,T)))^2\}, \end{array}$$

where

$$J(z,u) = \frac{1}{2}\int_0^\ell |z(T) - z_d(T)|^2 + \frac{\beta}{2}\int_0^T \int_0^\ell (u_1^2 + u_2^2),$$

and $\beta > 0$. We assume that $z_d \in C([0,T];Y)$.

Theorem. Problem (P) admits a unique solution $(\bar{z}, \bar{u})$. Moreover $\bar{u}$ is characterized by

$$\bar{u}_1(x,t) = -\frac{b_1}{\beta}\phi(x,t) \quad \text{and} \quad \bar{u}_2(x,t) = -\frac{b_2}{\beta}\psi(x,t),$$

in $(0,T)$, where (ϕ,ψ) is the solution to the adjoint system

$$-\frac{\partial}{\partial t}\begin{bmatrix} \phi(x,t) \\ \psi(x,t) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} -m_1\phi \\ m_2\psi \end{bmatrix} - \begin{bmatrix} a_{11}\phi + a_{21}\psi \\ a_{12}\phi + a_{22}\psi \end{bmatrix}$$

in $(0,\ell) \times (0,T)$, with the terminal condition

$$\phi(T) = \bar{z}_1(T) - z_{d,1}(T), \qquad \psi(T) = \bar{z}_2(T) - z_{d,2}(T)$$

in $(0,\ell)$, and the boundary conditions

$$\phi(0,t) = 0, \qquad \psi(\ell,t) = 0 \qquad \text{in } (0,T).$$

Proof. (i) The existence of a unique solution to (P) is classical and is left as exercice.

(ii) The state equation is of the form

$$z' = (A + L)z + Bu, \qquad z(0) = z_0,$$

and the cost functional

$$J(z,u) = \frac{1}{2}\|z(T) - z_d(T)\|^2_{L^2(0,\ell)} + \frac{\beta}{2}\int_0^T \|u(t)\|^2_{(L^2(0,\ell))^2}.$$

Thus the optimal control $\bar{u}$ is characterized by

$$\bar{u}(t) = -\frac{1}{\beta}B^* p(t),$$

where p is the solution to

$$-p' = (A + L)^* p, \qquad p(T) = \bar{z}(T) - z_d(T).$$

Set

$$p = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

We can verify that (ϕ, ψ) is the solution to the adjoint equation corresponding to $\bar{z}$.

We can prove that

$$B^*(\phi(t), \psi(t)) = (b_1\phi(x,t), b_2\psi(x,t)).$$

(iii) We can directly prove the optimality conditions for problem (P) by using the same method as for the heat and the wave equations. Setting $F(u) = J(z(z_0, u), u)$, where $z(z_0, u)$ is the solution to (HE), we have

$$F'(\bar{u})u = \int_0^\ell (\bar{z}_1(T) - z_{d1}(T))w_{u1}(T)$$

$$+ \int_0^\ell (\bar{z}_2(T) - z_{d2}(T))w_{u2}(T) + \beta \int_0^T (\bar{u}_1 u_1 + \bar{u}_2 u_2),$$

where $w_u = z(0, u)$, and $z(0, u)$ is the solution to (HE) for $z_0 = 0$.

We can establish an integration by parts formula between w_u and the solution (ϕ, ψ) to (AE) to complete the proof.

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