Chapter 2

Evolution Equations

Introduction to Semigroup Theory

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Introduction to Evolution Equations

Differential Equations in Banach Spaces

Consider the equation

$$(E_1) \qquad y' = Ay + f, \qquad y(0) = y_0,$$

with $f \in C([0,T];Y)$, $y_0 \in Y$, Y is a Banach space.

If $A \in \mathcal{L}(Y)$, then equation (E_1) admits a unique solution in $C^1(\mathbb{R};Y)$ given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A} f(s)ds,$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad \forall t \in \mathbb{R}.$$

The 1-D Heat Equation

Consider the heat equation in $(0,L)\times(0,T)$

$$
\begin{aligned}
& y \in L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L)), \\
& y_t - y_{xx} = 0 \qquad \text{in } (0,L)\times(0,T), \\
& y(0,t) = y(L,t) = 0 \qquad \text{in } (0,T), \\
& y(x,0) = y_0(x) \qquad \text{in } (0,L),
\end{aligned}
$$

where $T > 0$, $L > 0$, et $y_0 \in L^2(0,L)$.

We can rewrite the equation in the form

$$
\begin{aligned}
& y \in L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L)) \\
& \frac{dy}{dt} \in L^2(0,T;H^{-1}(0,L)), \\
& \frac{dy}{dt} = Ay \qquad \text{in } L^2(0,T;H^{-1}(0,L)), \\
& y(0) = y_0 \qquad \text{in } L^2(0,L),
\end{aligned}
$$

where $A \in \mathcal{L}(H_0^1(0,L);H^{-1}(0,L))$ is defined by

$$\langle Ay, z \rangle = -\int_0^L y_x \cdot z_x \, dx.$$

The operator A can be defined as an unbounded operator in $L^2(0,L)$ by setting

$$D(A) = H^2(0,L) \cap H_0^1(0,L), \qquad Ay = y_{xx}.$$

We would like to write the solution y in the form

$$y(t) = e^{tA} y_0.$$

Observe that

$$H_0^1(0,L) \stackrel{A}{\longmapsto} H^{-1}(0,L),$$

$$H^3(0,L) \cap \{y \mid y_{xx}(0) = y_{xx}(L) = 0\} \stackrel{A}{\longmapsto} H_0^1(0,L).$$

To find an other definition for e^{tA}, we introduce

$$\phi_k = \sqrt{\frac{2}{L}}\sin\Big(\frac{k\pi x}{L}\Big).$$

The family $(\phi_k)_{k\geq 1}$ is a Hilbertian basis of $L^2(0,L)$, and ϕ_k is an eigenfunction of the operator $(A, D(A))$:

$$\phi_k \in D(A), \qquad A\phi_k = \lambda_k \phi_k, \quad \lambda_k = -\frac{k^2\pi^2}{L^2}.$$

We look for y in the form

$$y(x,t) = \sum_{k=1}^{\infty} g_k(t)\phi_k(x).$$

If $$y_0(x) = y(x,0) = \sum_{k=1}^{\infty} g_k(0)\phi_k(x),$$

and if the P.D.E. is satisfied in the sense of distributions in $(0,L)\times(0,T)$, then g_k obeys

$$g_k' + (k^2\pi^2)/(L^2)g_k = 0 \qquad \text{in } (0,T),$$
$$g_k(0) = y_{0k} = (y_0, \phi_k).$$

We have $$g_k(t) = y_{0k}e^{-\frac{k^2\pi^2 t}{L^2}}.$$

The function $y \in L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L))$

$$y(x,t) = \Sigma_{k=1}^{\infty}\ y_{0k}e^{-\frac{k^2\pi^2 t}{L^2}}\phi_k(x)$$

is the solution of the heat equation.

Remark. y is not defined for $t < 0$.

Setting

$$S(t)y_0 = \sum_{k=1}^{\infty} (y_0, \phi_k) e^{-\frac{k^2\pi^2 t}{L^2}} \phi_k(x),$$

we have

$$(i) \quad S(0) = I,$$

$$(ii) \quad S(t) \in \mathcal{L}(L^2(0, L)) \quad \text{for all } t \geq 0,$$

$$(iii) \quad S(t+s)y_0 = S(t) \circ S(s)\, y_0 \qquad \forall t \geq 0,\ \forall s \geq 0,$$

$$(iv) \quad \lim_{t \searrow 0} \|S(t)y_0 - y_0\|_{L^2(0,L)} = 0.$$

When $A \in \mathcal{L}(Y)$, the family $(e^{tA})_{t\in\mathbb{R}}$ satisfies:

$$(i) \quad e^{0A} = I,$$

$$(ii) \quad e^{tA} \in \mathcal{L}(Y) \quad \text{for all } t \in \mathbb{R},$$

$$(iii) \quad e^{(s+t)A} = e^{sA} \circ e^{tA} \qquad \forall t \in \mathbb{R},\ \forall s \in \mathbb{R},$$

$$(iv) \quad \lim_{t\to 0}\|e^{tA} - I\|_{\mathcal{L}(Y)} = 0,$$

$$(v) \quad Ay = \lim_{t\to 0}\frac{1}{t}\Big(e^{tA}y - y\Big) \quad \forall y \in Y.$$

M-Dissipative Operators

Unbounded Operators

Definition. An unbounded linear operator on a Banach space Y is defined by a couple $(A, D(A))$, where $D(A)$ is a linear subspace of Y, and A is a linear mapping from $D(A) \subset Y$ into Y. The subspace $D(A)$ is called the domain of the operator A.

In a similar way, an unbounded linear operator from Y into Z is defined by a couple $(A, D(A))$, where $D(A)$ is a linear subspace of Y, and A is a linear mapping from $D(A) \subset Y$ into Z.

Definition. An unbounded linear operator $(A, D(A))$ on Y is a closed operator if its graph $G(A) = \{(y, Ay) \mid y \in D(A)\}$ is closed in $Y \times Y$.

Definition. Let $(A, D(A))$ be an unbounded linear operator $(A, D(A))$ on Y. We say that $(A, D(A))$ is a densely defined operator in Y, or that $(A, D(A))$ is an operator with dense domain in Y, if $D(A)$ is dense in Y.

Definition. Let $(A, D(A))$ be a densely defined operator in Y. The adjoint operator of A is the operator $(A^*, D(A^*))$ defined by

$$D(A^*) = \{z \in Y' \mid \exists c \geq 0 \text{ such that } \langle Ay, z\rangle_{Y,Y'} \leq c\|y\|_Y \text{ for all } y \in D(A)\},$$

and

$$\langle y, A^* z\rangle_{Y,Y'} = \langle Ay, z\rangle_{Y,Y'}$$

for all $y \in D(A)$ and all $z \in D(A^*)$.

Theorem. Let $(A, D(A))$ be an unbounded linear operator with dense domain in Y. Suppose that Y is a reflexive Banach space and that A is closed. Then $D(A^*)$ is dense in Y'.

Example 1. Suppose that Ω is a bounded regular subset of $\mathbb{R}^n$. The boundary of Ω is denoted by Γ. Set

$$Y = L^2(\Omega), \quad D(A) = H^2 \cap H_0^1(\Omega), \quad Ay = \Delta y.$$

A **is a closed operator.** Let $(y_n)_n \subset D(A)$ such that

$$y_n \xrightarrow{L^2(\Omega)} y \quad \text{and} \quad Ay_n \xrightarrow{L^2(\Omega)} f.$$

We know that

$$\Delta y_n \xrightarrow{\mathcal{D}'(\Omega)} \Delta y = f.$$

Therefore

$$y \in H(\Delta;\Omega) = \{y \in L^2(\Omega) \mid \Delta y \in L^2(\Omega)\}.$$

Since

$$\gamma_0 \in \mathcal{L}(H(\Delta;\Omega); H^{-1/2}(\Gamma)),$$

we have

$$\gamma_0 y = 0.$$

The problem

$$y \in H(\Delta;\Omega), \qquad \Delta y = f \text{ in } \Omega, \quad \gamma_0 y = 0 \text{ on } \Gamma,$$

admits a unique solution. From elliptic existence results and elliptic regularity results it follows that $y \in D(A)$.

$A = A^*$. First prove that

$$D(A) \subset D(A^*).$$

Let $z \in D(A)$. For every $y \in D(A)$, we have

$$\langle Ay, z\rangle_{Y,Y'} = \int_\Omega \Delta y\, z = \int_\Omega y\, \Delta z \le C\|y\|_{L^2(\Omega)}.$$

Thus $z \in D(A^*)$ and $A^*z = \Delta z$.

Prove the reverse inclusion. Let $z \in D(A^*)$. We know that

$$|(y, A^*z)_{L^2}| = |(Ay, z)_{L^2}| = |\int_\Omega \Delta y\, z| \le C\|y\|_{L^2(\Omega)}$$

for all $y \in D(A)$. Thus

$$A^*z \in L^2(\Omega).$$

Set $f = A^* z$. Denote by $\tilde{z} \in D(A)$ the solution of

$$\tilde{z} \in D(A), \qquad \Delta \tilde{z} = f.$$

We have

$$(Ay, z - \tilde{z})_{L^2} = (y, A^* z - \Delta \tilde{z})_{L^2} = 0$$

for all $y \in D(A)$. For every $g \in L^2(\Omega)$,

$$(g, z - \tilde{z})_{L^2} = 0.$$

This means that $z = \tilde{z}$.

Example 2. Let $(0, L)$ be an open bounded interval in $\mathbb{R}$. Set

$$Y = L^2(0, L), \quad D(A) = \{y \in H^1(0, L) \mid y(0) = 0\},$$

$$Ay = y_x \quad \forall y \in D(A).$$

A **is a closed operator.** Let $(y_n)_n \subset D(A)$ such that

$$y_n \overset{L^2(0,L)}{\longrightarrow} y \quad \text{and} \quad Ay_n \overset{L^2(0,L)}{\longrightarrow} f.$$

We know that

$$\frac{dy_n}{dx} \overset{\mathcal{D}'(0,L)}{\longrightarrow} \frac{dy}{dx} = f.$$

Thus $y \in H^1(0, L)$. Since $(y_n)_n$ is bounded in $H^1(0, L)$, we can prove that $y_n(0) \to y(0)$. Therefore $y \in D(A)$.

Characterization of A^*. Let us prove that

$$D(A^*) = \{y \in H^1(0,L) \mid y(L) = 0\}, \quad A^*y = -y_x.$$

First prove that

$$\{y \in H^1(0,L) \mid y(L) = 0\} \subset D(A^*).$$

Let $z \in \{y \in H^1(0,L) \mid y(L) = 0\}$. For every $y \in D(A)$, we have

$$(Ay, z)_{L^2} = \int_0^L y_x\, z = -\int_0^L y\, z_x \leq C\|y\|_{L^2(0,L)}.$$

Thus $z \in D(A^*)$ and $A^*z = -z_x$.

Prove the reverse inclusion. Let $z \in D(A^*)$. We know that

$$|(y, A^* z)_{L^2}| = |(Ay, z)_{L^2}| = |\int_0^L y_x \, z| \le C\|y\|_{L^2(0,L)}.$$

for all $y \in D(A)$. Thus

$$z_x \in L^2(0, L).$$

Set $f = -z_x$. Denote by $\tilde{z} \in \{y \in H^1(0, L) \mid y(L) = 0\}$ the solution of

$$\tilde{z} \in H^1(0, L), \quad \tilde{z}(L) = 0, \qquad -\tilde{z}_x = f.$$

We have

$$(Ay, z - \tilde{z})_{L^2} = (y, A^* z - \tilde{z}_x)_{L^2} = 0$$

for all $y \in D(A)$.

For every $g \in L^2(0, L)$,

$$(g, z - \tilde{z})_{L^2} = 0.$$

This means that $z = \tilde{z}$.

M-Dissipative Operators on Hilbert spaces

From now on we suppose that Y is a Hilbert space.

Definition. An unbounded linear operator $(A, D(A))$ on Y, is dissipative if and only if

$$\forall y \in D(A), \qquad (Ay, y)_Y \leq 0.$$

For a complex Hilbert space the previous condition is replaced by

$$\forall y \in D(A), \qquad \mathrm{Re}(Ay, y)_Y \leq 0.$$

Remark. If Y is a Banach space, an unbounded linear operator $(A, D(A))$ on Y, is dissipative if and only if

$$\forall y \in D(A),\ \forall \lambda > 0, \qquad \|\lambda y - Ay\| \geq \lambda \|y\|.$$

Definition. An unbounded linear operator $(A, D(A))$ on Y, is m-dissipative if and only if

- A is dissipative,
- $\forall f \in Y,\ \forall \lambda > 0, \qquad \exists y \in D(A) \quad \text{such that}$

$$\lambda y - Ay = f.$$

Theorem. If $(A, D(A))$ is an m-dissipative operator then, for all $\lambda > 0$, the operator $(\lambda I - A)$ admits an inverse, $(\lambda I - A)^{-1} f$ belongs to $D(A)$ for all $f \in Y$, and $(\lambda I - A)^{-1}$ is a linear bounded operator on Y satisfying

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{1}{\lambda}.$$

Theorem. Let $(A, D(A))$ be an unbounded dissipative operator on Y. The operator A is m-dissipative if and only if

$$\exists \lambda_0 > 0 \quad \text{such that } \forall f \in Y,$$

$$\exists y \in D(A) \quad \text{satisfying} \quad \lambda_0 y - Ay = f.$$

Theorem. If A is an m-dissipative then A is closed and $D(A)$ is dense in Y.

Remark. If $(A, D(A))$ is an unbounded operator on Y, the mapping

$$y \longmapsto \|y\|_Y + \|Ay\|_Y$$

is a norm on $D(A)$. We denote it by $\| \cdot \|_{D(A)}$.

Corollary. Let A be an m-dissipative operator. Then $(D(A), \| \cdot \|_{D(A)})$ is a Banach and $A|_{D(A)} \in \mathcal{L}(D(A); Y)$.

Theorem. If A is a dissipative operator with dense domain in Y. Then A is m-dissipative if and only if A is closed and A^* is dissipative.

Definition. An unbounded linear operator $(A, D(A))$, with dense domain in Y is selfadjoint if $A = A^*$. It is skew-adjoint if $A = -A^*$.

Example 1. The heat operator in $L^2(\Omega)$.

Let Ω be a bounded regular subset of $\mathbb{R}^n$, with a boundary Γ of class C^2. Set

$$Y = L^2(\Omega), \quad D(A) = H^2 \cap H_0^1(\Omega), \quad Ay = \Delta y.$$

A is dissipative.

$$(Ay, y)_{L^2(\Omega)} = \int_\Omega \Delta y \, y = -\int_\Omega \nabla y \cdot \nabla y \leq 0.$$

A is m-dissipative. Let $\lambda > 0$. For all $f \in L^2(\Omega)$, the equation

$$\lambda y - \Delta y = f$$

admits a unique solution in $D(A)$.

Example 2. A convection operator in $L^2(\mathbb{R}^n)$.

Let $\vec{V} \in \mathbb{R}^n$. Set

$$Y = L^2(\mathbb{R}^n), \quad D(A) = \{y \in L^2(\mathbb{R}^n) \mid \vec{V} \cdot \nabla y \in L^2(\mathbb{R}^n)\},$$

$$Ay = -\vec{V} \cdot \nabla y \quad \forall y \in D(A).$$

A is dissipative.

$$(Ay, y)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} -(\vec{V} \cdot \nabla y)\, y = \int_{\mathbb{R}^n} y\, (\vec{V} \cdot \nabla y) \leq 0.$$

A is m-dissipative. Let $\lambda > 0$. For all $f \in L^2(\mathbb{R}^n)$, consider the equation

$$\lambda y + \vec{V} \cdot \nabla y = f.$$

Let us prove that

$$y(x) = \int_0^\infty e^{-\lambda s} f(x - s\vec{V}) ds$$

is the unique solution to the above equation in $D(A)$.

We first prove this result when $f \in \mathcal{D}(\mathbb{R}^n)$. In this case

$$\vec{V} \cdot \nabla y(x) = \int_0^\infty e^{-\lambda s} \vec{V} \cdot \nabla f(x - s\vec{V}) ds.$$

But

$$\vec{V} \cdot \nabla f(x - s\vec{V}) = -\frac{d}{ds}[f(x - s\vec{V})].$$

With an integration by parts

$$\vec{V} \cdot \nabla y(x)$$
$$= -\lambda \int_0^\infty e^{-\lambda s} f(x - s\vec{V}) ds + \Big[- e^{-\lambda s} f(x - s\vec{V}) \Big]_0^\infty$$
$$= -\lambda \int_0^\infty e^{-\lambda s} f(x - s\vec{V}) ds + f(x)$$
$$= -\lambda y(x) + f(x) \, .$$

To prove that $y \in D(A)$, let us establish the following estimate

$$\|y\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^n)}.$$

From Cauchy-Schwarz inequality it follows that

$$\begin{aligned}
&|y(x)| \\
&\leq \Big(\int_0^\infty e^{-\lambda s} ds\Big)^{1/2} \Big(\int_0^\infty e^{-\lambda s} |f(x - s\vec{V})|^2 ds\Big)^{1/2} \\
&\leq \Big(\frac{1}{\lambda}\Big)^{1/2} \Big(\int_0^\infty e^{-\lambda s} |f(x - s\vec{V})|^2 ds\Big)^{1/2}.
\end{aligned}$$

Thus

$$\|y\|^2_{L^2(\mathbb{R}^n)} \leq \frac{1}{\lambda}\int_{\mathbb{R}^n}\int_0^\infty e^{-\lambda s}|f(x - s\vec{V})|^2 ds\, dx$$
$$\leq \frac{1}{\lambda}\ \|f\|^2_{L^2(\mathbb{R}^n)}\ \frac{1}{\lambda}.$$

The estimate is proved. From the estimate and the equation, we deduce that

$$\|\vec{V}\cdot\nabla y\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

Thus $y \in D(A)$.

Existence when $f \in L^2(\mathbb{R}^n)$.

Let $(f_n)_n$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^n)$ converging to f in $L^2(\mathbb{R}^n)$. With the above estimates we prove that

$$y_n(x) = \int_0^\infty e^{-\lambda s} f_n(x - s\vec{V})ds,$$

converges to

$$y(x) = \int_0^\infty e^{-\lambda s} f(x - s\vec{V})ds,$$

in $D(A)$ and that y is a solution to equation

$$\lambda y + \vec{V} \cdot \nabla y = f.$$

Uniqueness.

Let $y \in D(A)$ obeying

$$\lambda y + \vec{V} \cdot \nabla y = 0.$$

Let us prove that $y = 0$. Let ρ_ε be a mollifier

$$\rho_\varepsilon(x) = \begin{cases} k\varepsilon^{-n}\text{exp}(-\varepsilon^{-2}/(\varepsilon^{-2} - |x|^2)), & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon, \end{cases}$$

and

$$k^{-1} = \int_{|x|<1} \text{exp}(-1/(1 - |x|^2))dx.$$

Set

$$y_\varepsilon = \rho_\varepsilon * y.$$

We have

$$\lambda y_\varepsilon + \vec{V} \cdot \nabla y_\varepsilon = 0.$$

For x fixed set

$$h(t) = e^{\lambda t} y_\varepsilon(x + \vec{V} t).$$

For x fixed, we have

$$h'(t) = e^{\lambda t}\Big(\lambda y_\varepsilon(x + \vec{V} t) + \vec{V} \cdot \nabla y_\varepsilon(x + \vec{V} t)\Big) = 0.$$

Thus h is a constant function. Letting t tend to $-\infty$, we obtain $h = 0$ because y_ε is bounded. Thus $y_\varepsilon(x) = 0$. Since x is arbitrary, $y_\varepsilon = 0$. We finally obtain $y = 0$.

Semigroup on a Banach space

We are interested in equation

$$(E) \qquad\qquad y' = Ay, \qquad y(0) = y_0,$$

$y_0 \in Y$, Y is a Banach space, A is an unbounded operator on Y.

When equation (E) does admit a solution in $C^1(\mathbb{R}; Y)$ given by

$$y(t) = S(t)y_0 \qquad \forall t \in \mathbb{R}\ ?$$

Definition. A family of bounded linear operators $(S(t))_{t\geq 0}$ on Y is a strongly continuous semigroup on Y when the following conditions hold:

(i) $S(0) = I$,

(ii) $S(t+s) = S(t) \circ S(s)\ \forall t \geq 0, \quad \forall\ s \geq 0$,

(iii) $\lim_{t\searrow 0} \|S(t)y - y\|_Y = 0$ for all $y \in Y$.

Theorem. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y. Then there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(Y)} \leq Me^{\omega t} \qquad \text{for all } t \geq 0.$$

Corollary. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y. Then, for all $y \in Y$, the mapping

$$t \longmapsto S(t)y$$

is continuous from $[0,\infty)$ into Y.

Definition. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y. The infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ is the unbounded operator $(A, D(A))$ defined by

$$D(A) = \Big\{ y \in Y \mid \quad \lim_{t\searrow 0} \frac{S(t)y - y}{t} \quad \text{exists in } Y \Big\},$$

$$Ay = \lim_{t\searrow 0} \frac{S(t)y - y}{t} \quad \text{for all } y \in D(A).$$

Theorem. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y and let $(A, D(A))$ be its infinitesimal generator. The following properties are satisfied.

(i) For all $y \in Y$, we have

$$\lim_{t\searrow 0}\frac{1}{h}\int_t^{t+h} S(s)yds = S(t)y.$$

(ii) For all $y \in Y$ and all $t > 0$, $\int_0^t S(s)y\,ds$ belongs to $D(A)$ and

$$A\Big(\int_0^t S(s)y\,ds\Big) = S(t)y - y.$$

(iii) If $y \in D(A)$ then $S(t)y \in D(A)$ and

$$\frac{d}{dt}S(t)y = AS(t)y = S(t)Ay.$$

(iv) If $y \in D(A)$ then

$$S(t)y - S(s)y = \int_s^t S(\tau)Ay\,d\tau = \int_s^t AS(\tau)y\,d\tau$$

Corollary. If $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y, $(S(t))_{t\geq 0}$, then $D(A)$ is dense in Y, and A is closed.

Theorem. Let $(A, D(A))$ be the infinitesimal generator of $(S(t))_{t\geq 0}$, a strongly continuous semigroup on Y. For all $y_0 \in D(A)$, $y(t) = S(t)y_0$ is the unique solution of the problem

$$y \in C([0,\infty); D(A)) \cap C^1([0,\infty); Y),$$
$$y'(t) = Ay(t) \quad \text{for all } t \geq 0, \qquad y(0) = y_0.$$

Proof. Let $y_0 \in D(A)$ and set $y(t) = S(t)y_0$. We know that

$$AS(t)y_0 = S(t)Ay_0.$$

Since the mapping

$$t \longmapsto S(t)Ay_0$$

is continuous from $[0, \infty)$ into Y, $y \in C([0, \infty); D(A))$. Moreover

$$\frac{d}{dt}S(t)y_0 = AS(t)y_0 = S(t)Ay_0.$$

Thus $y \in C^1([0, \infty); Y)$ and $y' = Ay$.

Uniqueness. Let $t > 0$ be arbitrarily fixed. Let $u \in C([0,\infty); D(A)) \cap C^1([0,\infty); Y)$ be an other solution of the problem. Set

$$v(s) = S(t-s)u(s) \qquad \text{for } 0 \leq s \leq t.$$

We have

$$\frac{dv}{dt}(s) = -AS(t-s)u(s) + S(t-s)Au(s) = 0.$$

Consequently $v(s) = v(0)$ for all $s \in [0,t]$. In particular $v(t) = u(t)$ and $v(0) = y(t)$. Thus $u(t) = y(t)$.

Theorem. Let $(A, D(A))$ be the infinitesimal generator of $(S(t))_{t\geq 0}$, a strongly continuous semigroup on Y satisfying

$$\|S(t)\|_{\mathcal{L}(Y)} \leq Me^{\omega t}.$$

Then, for all $c \in \mathbb{R}$, $(A - cI, D(A))$ is the infinitesimal generator of the strongly continuous semigroup $(e^{-ct}S(t))_{t\geq 0}$ on Y.

Proof. It is easy to verify that $(e^{-ct}S(t))_{t\geq 0}$ is a strongly continuous semigroup on Y. To prove that $(A - cI, D(A))$ is its infinitesimal generator it is sufficient to show that

$$\frac{d}{dt}(e^{-ct}S(t))y = (A - cI)y$$

for all $y \in D(A)$.

The Hille-Yosida Theorem

Semigroups of contractions

Definition. A strongly continuous semigroup $(S(t))_{t\geq 0}$ on Y is a semigroup of contractions if

$$\|S(t)\| \leq 1 \qquad \text{for all } t \geq 0.$$

Theorem. (Hille-Yosida's Theorem in Banach spaces)

An unbounded linear operator $(A, D(A))$ in Y is the infinitesimal generator of a semigroup of contractions on Y if and only if the following conditions are satisfied:

(i) A is a closed operator,

(ii) $D(A)$ is dense in Y,

(iii) for all $\lambda > 0$, $(\lambda I - A)$ is a bijective mapping from $D(A)$ to Y, its inverse $(\lambda I - A)^{-1}$ is a bounded operator on Y obeying

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Theorem. (Hille-Yosida's Theorem in Hilbert spaces - Phillips' Theorem)

An unbounded linear operator $(A, D(A))$ in Y is the infinitesimal generator of a semigroup of contractions on Y if and only if A is m-dissipative in Y (or if and only if A^* is m-dissipative in Y').

Theorem. (Lumer-Phillips' Theorem in Hilbert spaces)

Let $(A, D(A))$ be an unbounded linear operator with dense domain in Y. If A is closed and if A and A^* are dissipative then A is the infinitesimal generator of a semigroup of contractions on Y.

A characterization of C^0-semigroups

Theorem. An unbounded linear operator $(A, D(A))$ in Y is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on Y obeying

$$\|S(t)\|_{\mathcal{L}(Y)} \leq M e^{\omega t} \qquad \forall t \geq 0,$$

if and only if the following conditions are satisfied:

(i) A is a closed operator,

(ii) $D(A)$ is dense in Y,

(iii) for all $\lambda > \omega$, $(\lambda I - A)$ is a bijective mapping from $D(A)$ to Y, its inverse $(\lambda I - A)^{-1}$ is a bounded operator on Y obeying

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}(Y)} \leq \frac{M}{(\lambda - \omega)^n}, \qquad \forall n \in \mathbb{N}.$$

Perturbations by bounded operators

Theorem. Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on Y obeying

$$\|S(t)\|_{\mathcal{L}(Y)} \leq Me^{\omega t} \qquad \forall t \geq 0.$$

If $B \in \mathcal{L}(Y)$, then $A + B$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on Y satisfying

$$\|T(t)\|_{\mathcal{L}(Y)} \leq Me^{(\omega + M\|B\|)t} \qquad \forall t \geq 0.$$

C^0-group on a Hilbert space

Definition. A family of bounded linear operators $(S(t))_{t\in\mathbb{R}}$ on Y is a strongly continuous group on Y when the following conditions hold:

(i) $S(0) = I$,

(ii) $S(t+s) = S(t) \circ S(s)\ \forall t \in \mathbb{R}, \quad \forall\ s \in \mathbb{R}$,

(iii) $\lim_{t\to 0}\|S(t)y - y\| = 0 \text{ for all } y \in Y$.

Definition. A strongly continuous group $(S(t))_{t\in\mathbb{R}}$ on Y is a unitary group if

$$\|S(t)y\|_Y = \|y\|_Y \qquad \forall y \in Y, \quad \forall t \in \mathbb{R}.$$

Theorem. (Stone's Theorem)

An unbounded linear operator $(A, D(A))$ on a complex Hilbert space Y is the infinitesimal generator of a unitary group on Y if and only if iA is self-adjoint.

Theorem. (Unitary group on a real Hilbert space)

Let $(A, D(A))$ be an m-dissipative operator on a real Hilbert space Y and let $(S(t))_{t\geq 0}$ be the C^0-semigroup on Y generated by A. Then is $(S(t))_{t\geq 0}$ is the restriction to $\mathbb{R}^+$ of a unitary group if and only if $-A$ is m-dissipative.

Example: The wave equation

To study the equation

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = 0 \quad \text{in } Q = \Omega \times (0,T),$$

$$z = 0 \quad \text{on } \Sigma = \Gamma \times (0,T),$$

$$z(x,0) = z_0 \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega,$$

with $(z_0, z_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, we transform the equation into a first order evolution equation. Set $y = (z, \frac{dz}{dt})$, the equation can be rewritten in the form

$$\frac{dy}{dt} = Ay, \quad y(0) = y_0,$$

where

$$Ay = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

Set $Y = H_0^1(\Omega) \times L^2(\Omega)$. The domain of A in Y is $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Let us prove that $(A, D(A))$ is m-dissipative on Y, when Y is equipped with the inner product

$$(u, v)_Y = \int_\Omega \nabla u_1 \cdot \nabla v_1 + \int_\Omega u_2\, v_2,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

A and $-A$ are dissipative.

$$(Ay, y)_Y = \int_\Omega \nabla y_2 \cdot \nabla y_1 + \int_\Omega \Delta y_1 \, y_2 = 0.$$

A is m-dissipative.

Let $(f, g) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\lambda > 0$. The equation

$$\lambda y - Ay = (f, g)$$

is equivalent to the system

$$\lambda y_1 - y_2 = f,$$

$$\lambda y_2 - \Delta y_1 = g.$$

Substituting $y_2 = \lambda y_1 - f$ into the second equation:

$$\lambda^2 y_1 - \Delta y_1 = \lambda f + g.$$

This equation admits a unique solution $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. Consequently $y_2 \in H_0^1(\Omega)$ is unique. Thus A is m-dissipative.

In the same way we prove that $-A$ is m-dissipative. Therefore $(A, D(A))$ is the generator of a semigroup of contractions on Y, and this semigroup can be extended to a unitary group on Y.

Weak solutions

Classical solutions to nonhomogeneous problems

We already know that equation

$$(E_2) \qquad y' = Ay, \qquad y(0) = y_0 \in D(A),$$

admits a unique classical solution y (i.e. $y \in C([0,\infty); D(A)) \cap C^1([0,\infty); Y)$) defined by

$$y(t) = S(t)y_0 \qquad \forall t \in \mathbb{R}.$$

We can extend this result to nonhomogeneous equations.

Theorem. Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on Y. If $y_0 \in D(A)$ and if $f \in C([0,T];Y) \cap L^1(0,T;D(A))$ or $f \in C([0,T];Y) \cap W^{1,1}(0,T;Y)$ then equation

$$(E_3) \qquad\qquad y' = Ay + f, \qquad y(0) = y_0,$$

admits a unique classical solution y defined by

$$y(t) = S(t)y_0 + \int_0^t S(t-s)f(s)\,ds \qquad \forall t \in \mathbb{R}^+.$$

Weak solutions

Definition. A weak solution to equation (E_3) in $L^p(0,T;Y)$ $(1 \leq p < \infty)$ is a function $y \in L^p(0,T;Y)$ such that, for all $z \in D(A^*)$, the mapping

$$t \longmapsto \langle y(t), z\rangle_{Y,Y'}$$

belongs to $W^{1,p}(0,T)$ and obeys

$$\frac{d}{dt}\langle y(t), z\rangle = \langle y(t), A^* z\rangle + \langle f(t), z\rangle,$$
$$\langle y(0), z\rangle = \langle y_0, z\rangle.$$

Theorem. If $y_0 \in Y$ and if $f \in L^p(0,T;Y)$, then equation (E_3) admits a unique weak solution in $L^p(0,T;Y)$. Moreover this solution belongs to $C([0,T];Y)$ and is defined by

$$y(t) = S(t)y_0 + \int_0^t S(t-s)f(s)ds, \qquad \text{for all } t \in [0,T].$$

Remark. From the variation of constant formula it follows that

$$\|y\|_{C([0,T];Y)} \leq C(\|y_0\|_Y + \|f\|_{L^1(0,T;Y)}).$$

Adjoint semigroup

Theorem. [8, Chapter 1, Corollary 10.6]

Let Y be a reflexive Banach space and let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y with infinitesimal generator A. Then the family $(S(t)^*)_{t\geq 0}$ is a semigroup, called the adjoint semigroup, which is strongly continuous on Y', whose infinitesimal generator is A^* the adjoint of A.

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