Filtered, cellokr and Clo algebras • Idea: Repeat the topological construction of CW-complexes but with algebras over an operad. A CW - songlex is (successively) constructed

by attaching cells:

55'-58

 $\int \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ 

Now, if Kis analgebra are an opened G



 $DD'' \rightarrow X$ 1 in Alg

$$x \leq -1 \qquad (-1)$$

X is closed (monoragi We have a tersor product, a shift 11ES and an internal han Han (-,-) ES satisfying

Hon (Xoy Z) = Hon(X, Hon (YZ)) Sympty: Depending on a parameter (EE(1,2,..., ~) the monoidal category is Symmetric ((1,2)) braided ((1=2)) pron-sym ((1=1)) This mans that if legz there is a braining rated

isomorphism B: Xoy = Yox who inerse is B if 6>2. YX We assume that h is fixed and depending on it there is an implicit notion of "symmetry".

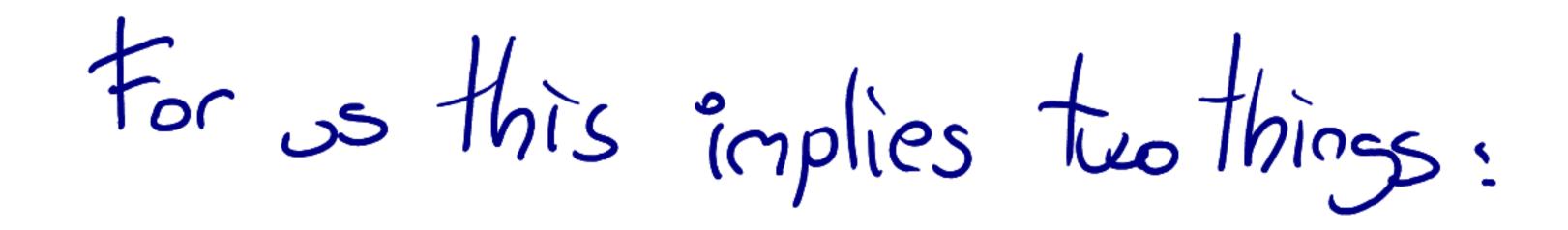
\* S is simplicially enriched This is that Sis a sSet-enriched category

Inparticular there is a simplicial set of rophian

Map(X,Y) esset for X,Y eS

\* Sis complete and cocomplete in the enriched

Sense Kall SSet-indexed colimits and limits exist



\*\* Sis complete and cocomplete the Shas a coponering and a powering The coponering is represented as  $-x - : sSet \times S \to S$ The powering is represented as

 $(-)^{-}: sSet \times S \rightarrow S$ and satisfies for K, Lesset and X=S  $K \times (L \times X) \cong (K \times L) \times X,$ 

 $(X^{\mu})^{\mu} = X^{\mu}$ 

and they are adjoints (in the enriched sense)

95 simplicial sets

The competibility of the structures sives us a tontor s: sSet -> S Kr Kx 1 strong considel  $S(KrL) \cong S(K) \otimes S(L)$ 

IF S is pointed, enriched are set a toratically the terrinal and initial object agree t = kimplies enriched ar SSet

<u>Examples</u>:

\* sSet

Map (X, T) = Hors (X × 1) (Y) sset Cartesia product as both X. @ s: sset is sset

\* SSet\* (similarly with the small product) \* Top ( Compaty generated weakly Hausdorff Spaces)

S=1-1: SSet Jop (gametric realization)

& is the catesian product Horn(X, C) = Horn (K, Y) with the compact-open Top Top Topology \*Top (similarly with the small products) \* sMod (Simplicial lk-modules, for the com cing) KYES16dk, Map (X,Y) = Hom (Xolk[J],Y) stbd 14 Moduk Kesset leveluise K×X=lk[k]@X X = Map (K, X) with the simplicid sset Ik-module inherted from X

By tensor product of K-modules leveluise

 $S: SSet \rightarrow SMod_{k} \text{ free } |k-rodule \; leveluoise.}$   $X \mapsto |k[X]$ 

\* Non-example: Ch: Chain condexes are lk There is no strong monical Enter S: SSet -> Chy so it is not a "good" categry. But Chy ~ Stlody by Dold-Kan theorem. \* Sp<sup>I</sup> (Symptic spectre)

Spear / En { n > of pointed simplicity sets with J-actions competible with the maps

 $E_{n}\Lambda S' \rightarrow E_{n+1}$ 

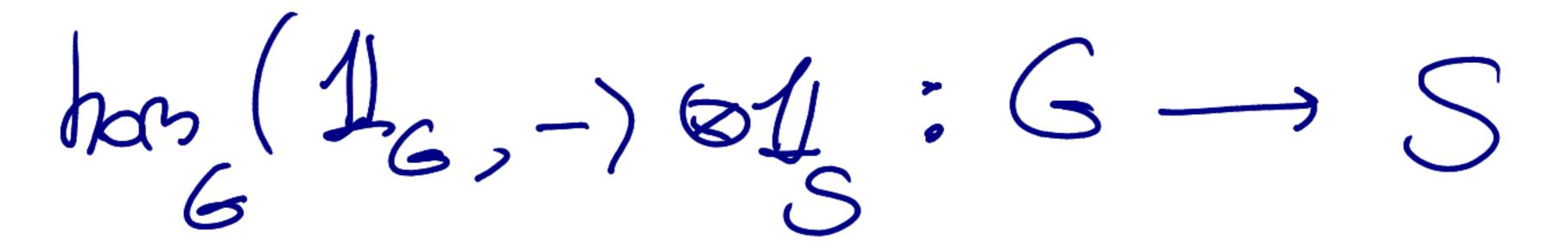
 $(K \times E) = E_{\eta} \Lambda K_{\eta}$  for Kesset KUX\*  $1 = \mathfrak{D} = \left( \mathfrak{D}^{n} = (\mathfrak{D}^{n})^{n} \right)_{n \neq 0}$   $\mathfrak{D} = \Lambda$  small product of symmetry. Spectry  $s: sSet \rightarrow Sp^{2}$  $K \rightarrow L^{\alpha} K_{+} = \left\{ S^{\alpha} K_{+} \right\}_{n \geq 0}$ \* Diagram categories  $\mathcal{B} = \mathcal{S}^{\mathcal{G}} = \mathcal{F}_{\mathcal{G}_n}(\mathcal{G}, \mathcal{S})$ Gis normally discrete or a groupoid. Proposition: 5 good and G (k-gametic) movider

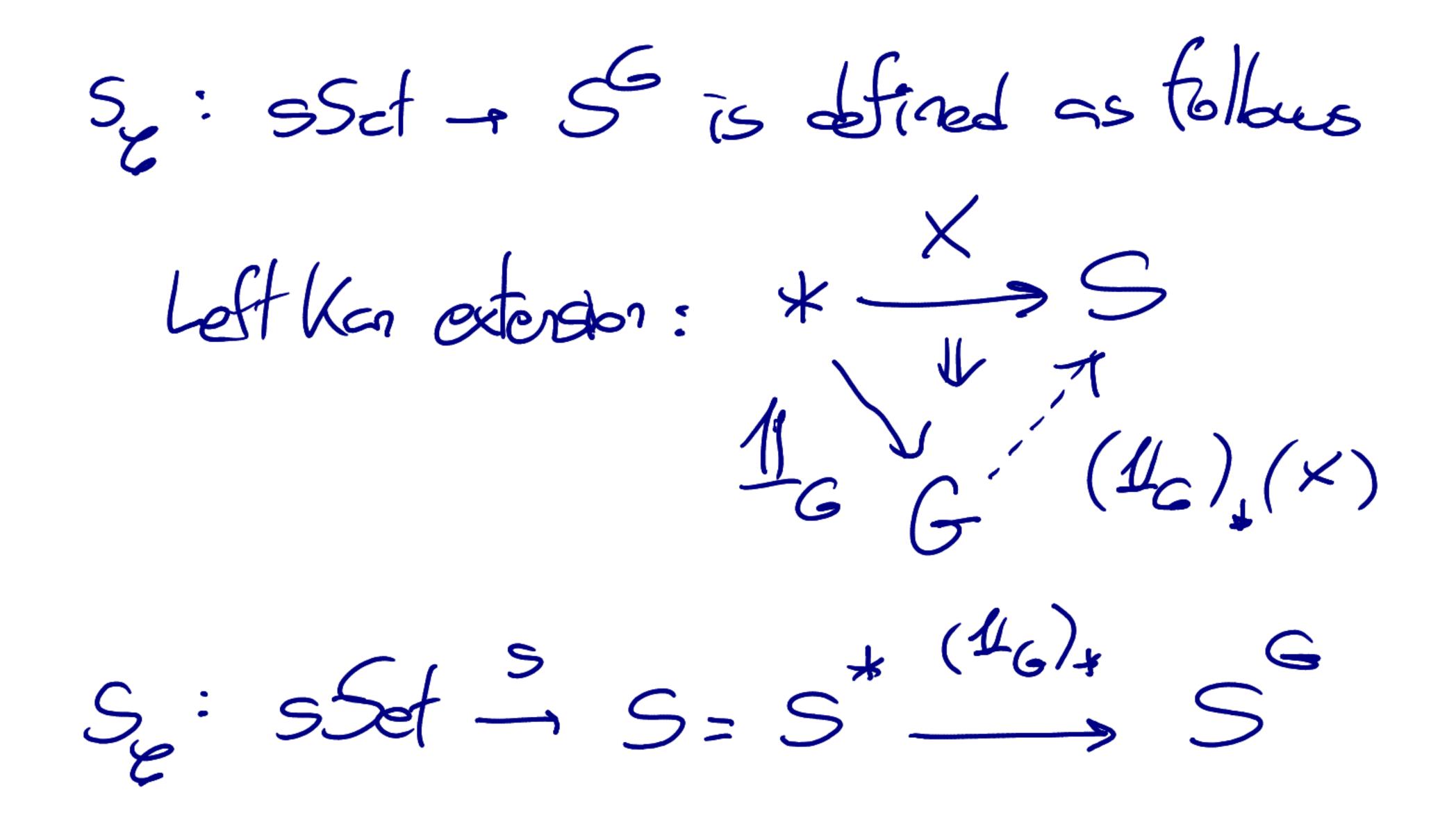
> 6=56 is good.



The tensor product in E is given by the Day Convolution: for X, YEE=SG G×G X×Y S×S -> S - It is a left Van extension.

Example: S= Vect with & and G=Z discrete XY: Z - Vect; to  $X \otimes Y : \mathbb{Z} \rightarrow lect (X \otimes Y)(k) = \bigoplus X(t) \otimes I(g)$  $t \in f_{g}=k$ The unit of the is





• Algebra de opende 6 opend in 6 (= 5) a good category. For le el (,2,..., oo) we write Gn=In (symptic grap) le >2 Gr=B (braid grap) k = 2 G\_= /12 (trivial grap) l = (

That Gisa collection of objects G(1) with Gastions, for 170 with corphisms \* Unit:  $1_G: 1_C \to G(1)$ \* Composition  $\mathcal{M}\left(n_{j}^{*}h_{i},...,h_{n}\right): \mathcal{O}(n) \otimes \mathcal{O}(h_{i}) \otimes \cdots \otimes \mathcal{O}(h_{n}) \rightarrow$ 

-> G(le, f--+ len)

which satisfies wit, associativity and equivariance

axions

An algebra X are the operad G is an object XEB together with comphisms

 $G(n) \otimes X \longrightarrow X$ 

satisfying unit associativity and equivariance axions.

Filtered algebras le mont a categorical approach of tiltrations and gradings.

Definition:

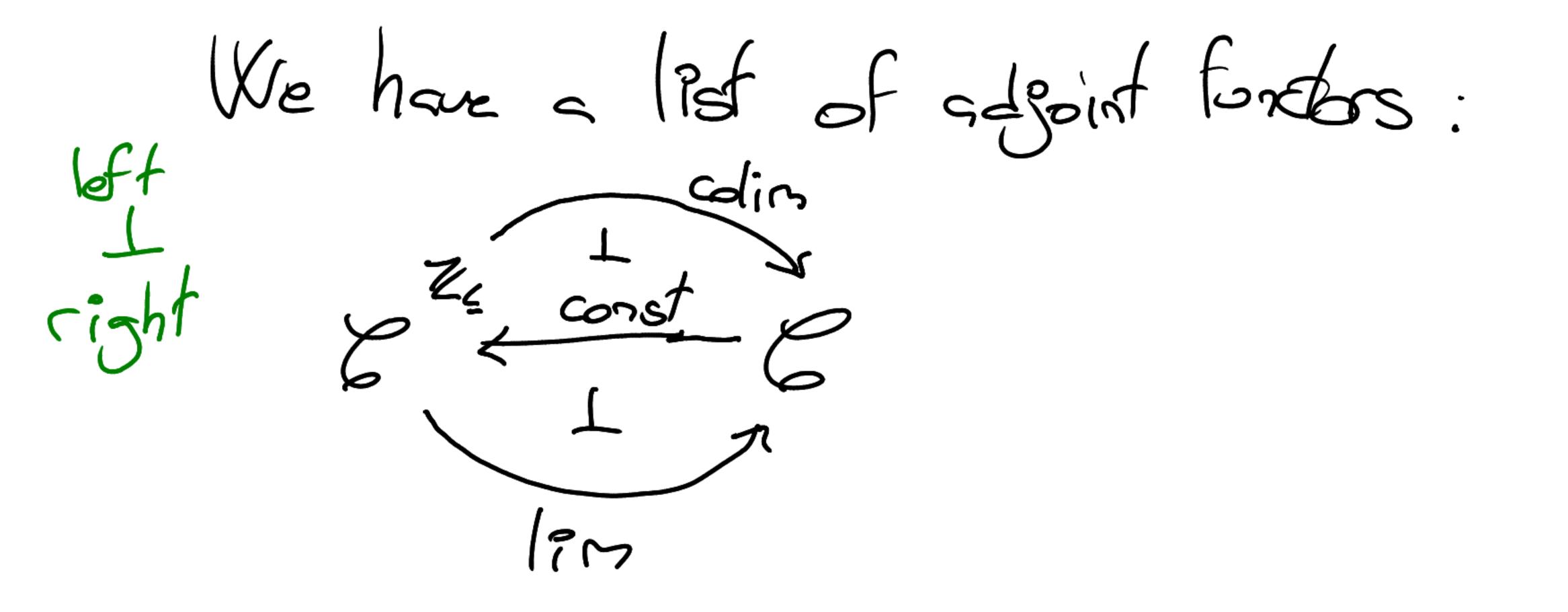
\* Zis the discrete steary with objects nEZ

+  $L_{\underline{z}}$  is the category associated to the poset  $(Z, \underline{z})$ 

 $-\frac{1}{2}$ 

Both ategories a symptric monoidal with the son

=> p<sup>z</sup> = and p<sup>z</sup> have the thy convolition.



Const:

colim: Q<sup>r</sup> - C

 $(--- \chi(0) \rightarrow \chi(0) \rightarrow ---) \rightarrow Golins \chi(2)$  $z \in \mathbb{Z}_{2}$ 

 $\lim_{n \to \infty} \mathcal{L}_{i} \xrightarrow{\mathcal{L}_{i}} \mathcal{L}_{i}$ (--- > X(o) - X(o) - --) ~ lin X(c) iezz

For a number a E ( identified with a torctor ) (\*) -> Zc V L ZE \* 1 ~ ~  $a^* = -6a : b^{\mathbb{Z}_2} \longrightarrow b$   $X \longrightarrow X(a)$ has a left adjoint given by a Kan extension

イギンーメント We write B = #16 the pointed category ot 6, whose objects are / # - X/in E (It has a initial and torminal object this th)  $gr: \mathcal{E}_{*} \to \mathcal{E}_{*}^{\mathbb{Z}_{2}}, X=(--,X(n-1)-X(n)-2--) \mapsto gr(X)$  $Gr(X)(n) = colim \begin{pmatrix} X(n-1) \rightarrow X(n) \\ \downarrow \\ \downarrow \\ \downarrow \end{pmatrix}$ 

Pointed by ۴  $X(n-1) \longrightarrow X(n)$  $t \xrightarrow{id} t \xrightarrow{f} f \xrightarrow{f} f$  $t \xrightarrow{f} f \xrightarrow{f} f(X)(n)$ Intorthely gr(K)(n) = K(n)/(n-1)

This is notation: X(n-1) -> X(n) needs noto

Le injective.

 $U: \mathcal{L}_{\ast}^{\mathbb{Z}_{=}} \to \mathcal{L}_{\ast}^{\mathbb{Z}_{=}}$  $(1) \longrightarrow (1) \times (1)$  $\left( --, \chi(0) \chi(0) \chi(0) \right) \mapsto \left( -- \chi(0) \right)$ 

We reed the basepoint to contract a morphism

tron X(n-1) to X(n) 1

· Opendes on filtered Jarded categories The functor of: E-, E is story model zero XH (--- il-il-iX-1X----) -( 0 1 So an opened G in G induces an opened in 6 24

Similarly with  $O_{*}o(-)_{+}: \mathcal{C} \to \mathcal{C}_{*} \to \mathcal{C}_{*}$  $\begin{array}{c} (-)_{+} \\ X \mapsto X_{+} = X \sqcup U \longrightarrow (--, \tilde{u}, \tilde{u}, \chi_{+}, \tilde{u}, \tilde{u}_{--}) \\ & & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & & \\ & & \\ & & \\$ which is strongly motidel. Notition we continue to call the opends in

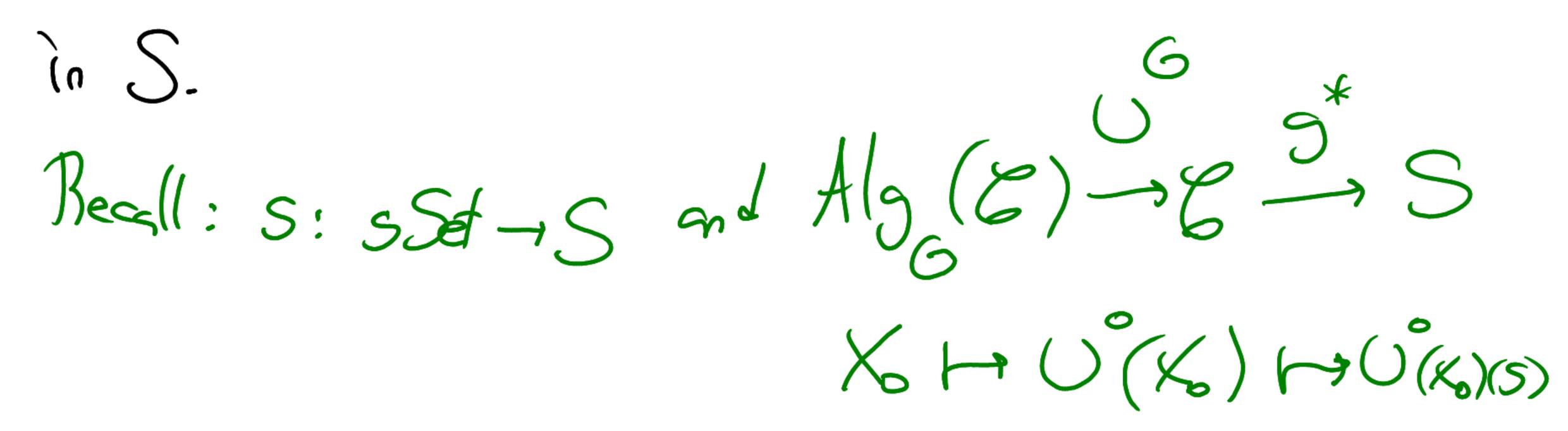
 $Z_{1}^{Z}$  and  $Z_{1}^{Z} = as$ 

There is a completive (upto natural isomorphism) dissen of the form  $Alg(\mathcal{C}^{\mathbb{Z}_{g}}) \xrightarrow{\mathcal{G}} Alg(\mathcal{C}^{\mathbb{Z}_{g}})$ FGJUG FZ FGTU  $z_{i} \xrightarrow{g_{i}}$ F'isthe free opened toretor and U torgets the O-algebra studure.

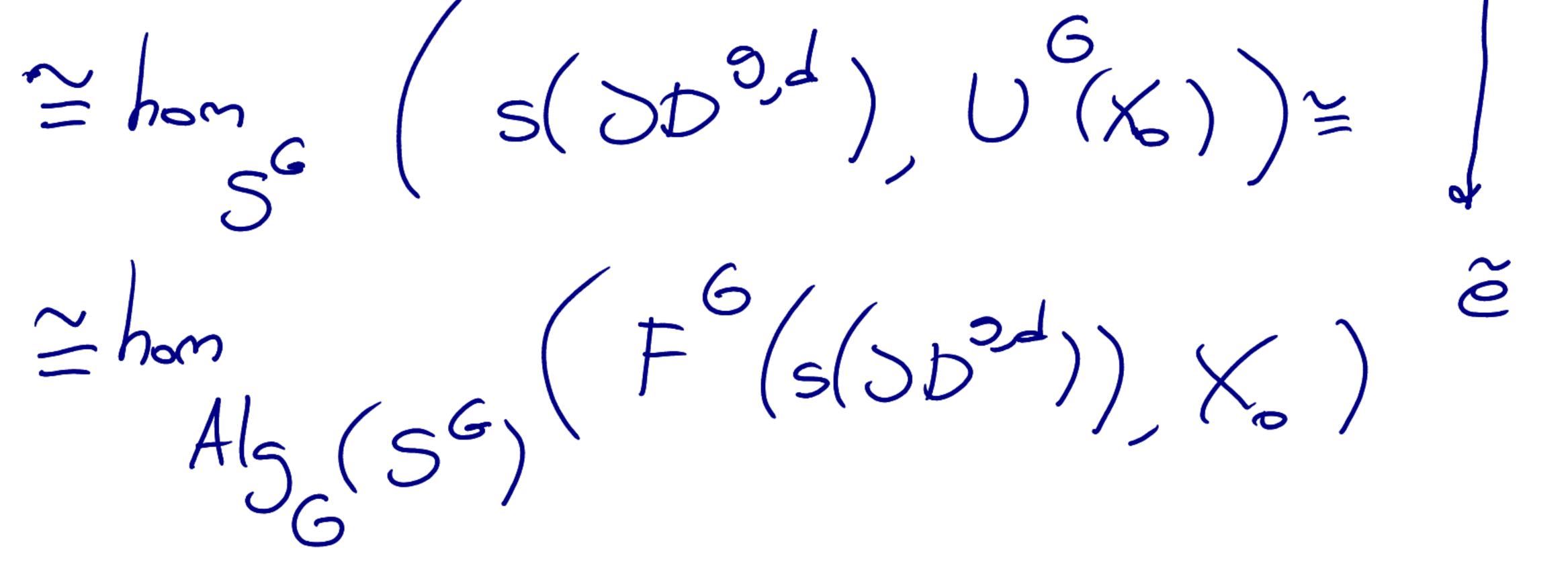
Cell Attachments G=SforSgood ategory, Garaperd

5

Inpots: consider XoEAlg(E) \* DD d d a cofibration in sSet where georetric realization is horse morphic to DD ~ D in Top. I Bodgical dish. \* Anobject geG.  $# A complish e: s(UDd) \rightarrow U(X_{5}, X_{5})$ 



Additional construction J+ Usethe adjunction 6=5+25 For each  $\begin{array}{c} \mathfrak{I} \\ \mathfrak$ GEG to define Notion: s(JD<sup>9,4</sup>)  $D^{3,d} = 3_*(D^4)$  $3D^{3,d} = 3_*(3D^4)$ Jobsi Also use adjunctions:  $hors\left(s(JD^{4}) \cup (X_{6})(g)\right) \cong$ 



To define  $\tilde{e}: f(s(\delta D^{3,d})) \to \chi$ Notation: e=e Output: consider the pushent F(s() D<sup>g,d</sup>)) - K  $F^{T}(s(D^{2,d})) \xrightarrow{F} X_{i}$ 

We say that X, is obtained from Ko by Attaching a G-cell of dimension (3,d) along e. Notation: X1=X006 Dgsn

What about fillestions? (Not recessive)

Fact: there is a functor colim:  $Alg(E^{z_{e}}) \rightarrow Alg(E)$ 

 $Ab_{G}(\mathcal{E}^{Z_{i}}) \xrightarrow{colim} Ab_{G}(\mathcal{E})$ Such that UCU Ze colim Ze colim comstes. IF XEAlg(6<sup>2</sup>) is a filtered G-algebra we think of colim(X) EX15 (E) as the underlying G-algebra nd a isomophism  $R \xrightarrow{\simeq} colon X in Ab_{G}(E)$ induces a milliplicative filtration on R. Consider Ro EAlg (E) and an attachment of a  $F(JD^{g,d}) \xrightarrow{e} \mathcal{R}_{o}$ FG(1) F(1)

This is a diagram in Alg (E): we want a (non)-trivid diagram on Alg (EZe) G(EZe)  $F(I_* DD^{g,d}) \xrightarrow{e} O_* R_{-}$  $G(I_{1}, D^{3, d}) \longrightarrow f \mathcal{B}_{1}$ 



fB, EAlg (6") is the cell attachment filtration (it is not concentrated in any degree)

· Cellute algebras Cellutr G-algebras are constructed by iterated collattachments starting at 11. Ampf: X-17 of G-algebras is cellular ifitis a transfinite composition of cellettadmonts. This mans that there exists a diagram

 $X = X \longrightarrow X_{o} \longrightarrow X_{o}$ FibE indexed by some ordinal K, such that the colimpter is a isomorphism

at lor each successor ordinal i ek FG(LG DD<sup>D</sup>, d, <u>Uh</u>, <u>X</u>e-(  $F = \begin{pmatrix} J \\ J \\ F \\ \downarrow \\ \neg z, da \end{pmatrix} \xrightarrow{F} \chi_{i}^{2}$ 

is a pushout diagram for some reps hz: UD 2, da



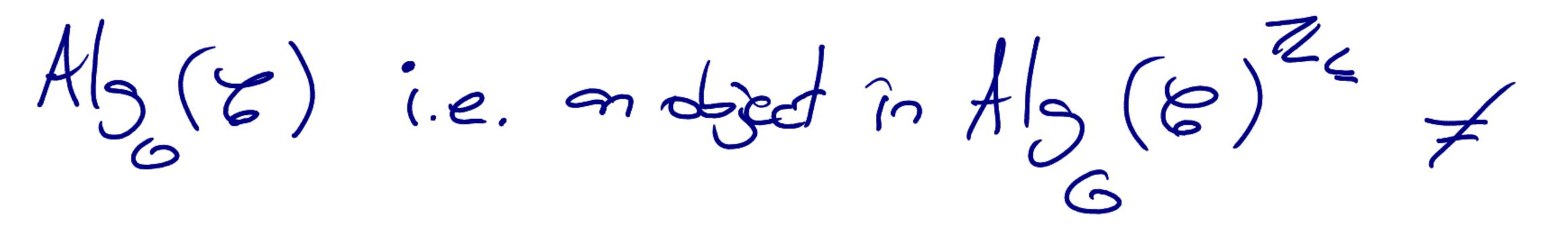
\* For each limit ordinal iek f= climfa: X.-14 i/i

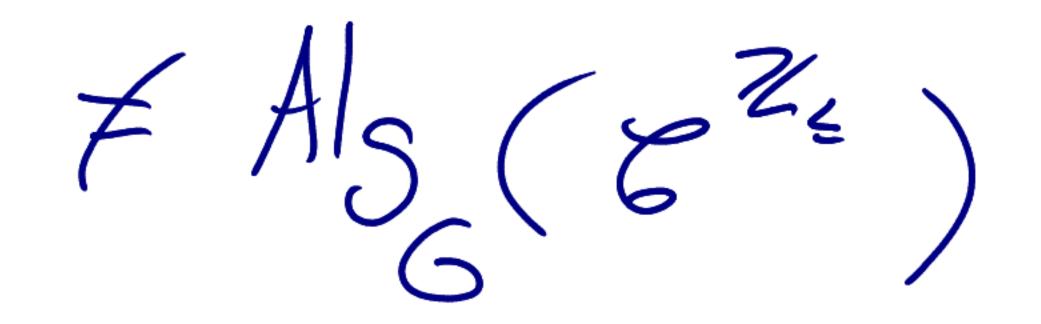
Définition: In Galgebra Mis <u>celluler</u> if

 $U \rightarrow \gamma \tilde{c}s$  allow.

· What about filtrations?

Cellular G-algebres do nothère à usefol filtation. If we attach cells in increasing order of dimension we dotain a fillered object in





Définition: DD des Dd is always a cofibration in sSet whose geométric realization is boreomorphic to DD - Dd inTop. Dofine dejects in sSet ". 

Becall that we have a story monoridal functor  $S_G: SSet \xrightarrow{S} S\xrightarrow{(M_G)} S \xrightarrow{=} S$ that induces a loober SSet Zig -> 6 Zig So we consider  $D^{d}(d)$  as living in  $\mathcal{B}^{\mathcal{R}_{d}}$ Notation: Given XE E<sup>z</sup> = (S<sup>G</sup>) = S<sup>G</sup> we write  $X(g,n) \in S$  for its value at  $(g,n) \in G \times \mathbb{Z}_{\ell}$ . Définition: A CVU-algebra structure on YEAG (8) is a relative du stratue on 11-37.

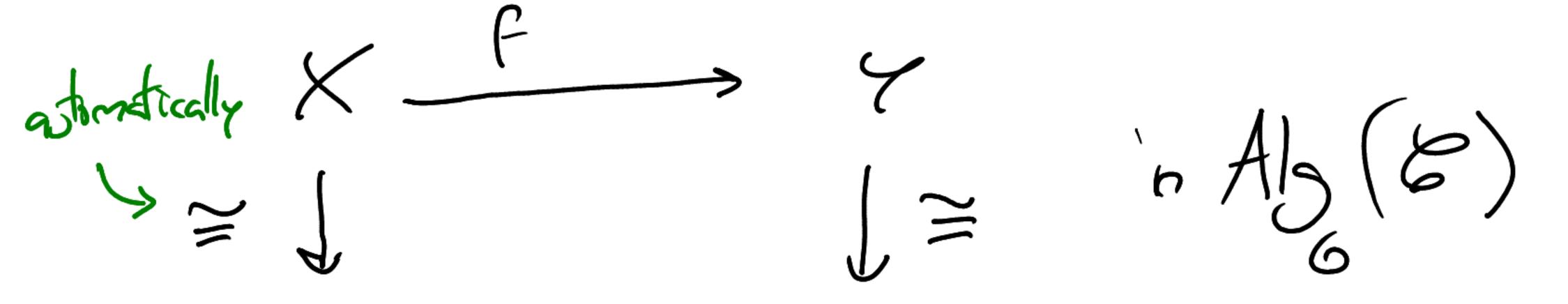
Définition: A relative Clu - structure on a morprism  $f: X \rightarrow Y$  in  $H_6(\mathbb{Z}^{\leq})$  is \* A diagram in Alg( $E^{z_e}$ )  $O_{*}(X) = Sl_e(F) \xrightarrow{f_e} Sl_e(F) \xrightarrow{f_e} Sl_e(F) \xrightarrow{f_e} Sl_e(F) \xrightarrow{f_e} Sl_e(F)$ \* For dro, a set II, objects / 5, EG / dE II?

and coophisms

 $e_{\chi}: J_{\chi} \longrightarrow s_{d}(F)(g_{\chi}, d-1)$  in S adjoint to  $\widetilde{e}_{\alpha}: \operatorname{JD}_{\alpha}^{\operatorname{Ju},d}\left[\operatorname{d-I}\right] \to \operatorname{Sh}_{\alpha}(F) \text{ in } S = \widetilde{e}^{\operatorname{Re}_{\alpha}}$ 

Such that there is a pushout diagram of the form

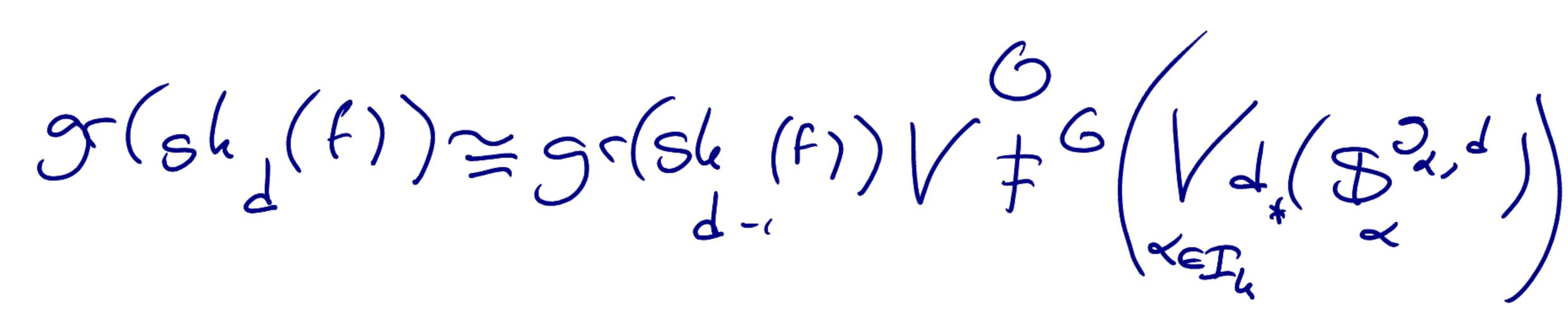
\* For Sh(F) = colim Sh(F) a connective diagram



 $\operatorname{colim}(\operatorname{sh}(F)) \rightarrow \operatorname{colim}(\operatorname{sh}(F))$ 

Theorem: gr(sk(F)) in Alg(6, ) is isomorphic to of(X) v & E (V V d'(S d', d)) dzo deta

V denotes the coproduct in Ab (& =) V is the coproduct in G = Intuitively it we make a quotient by thing's of degree k-1, we sand JD<sup>3,d</sup> [d-] to apoint ad vehue a free apoil georited by an "Sphe" & I degree d. We have the expression



· Model categories for opends and algobres We assume that Shas a (cofibrantly generated) model category structure. The projective model structure: consider on adjuction DL D with a model category. We declare a rarphism Fine to be a \* fibration if ((F) is a fibration

\* kege equivalence if U(F) is a weak equivalence

This is a model category on E it some conditions

holds (always in ar case): this is the projective

model ategory on & transferred along F-1U.

Consider nou &= S and the functor

 $U: S^{G} \to M S$ 

 $X \mapsto (X(g))_{g \in G}$ 

with left algoint

 $F((X_3)) = \iint_{G} hom(g, -) \times X_g$ geG =) S= & has the projective model ategory Définition: l'édéfine groupoiles FB, dependingon 

and  $I_{n}$  (symptric group) if k=2  $horn(n, n)=G_{n}=\begin{cases} B_{n} (Braided group) & if k=2 \\ id ? (Hrivid group) & if k=1 \end{cases}$ 

We orside the category of <u>h-symptric squares</u> in  $\mathcal{B}$ ,  $fB_{\ell}(\mathcal{B}) = \mathcal{B}^{fB_{\ell}}$ . Then an opened is north movid in FB (E) with respect to the onposition product. Then FB(E) has a projective model category. In particula, XEFB (6) is cotionant it and als it X(n) is ofibrat in 6 Gn for each nzo The projective model structure in Alg (E) is transferred along  $F^{G}$  $E \xrightarrow{F^{G}} Ab_{G}(E)$  $G^{G}$ Moreover, if the underlying k-symmetric sequence of Gis cofibrat = ) Opresences cofibrations and

trivial cofibrations between cofibrant objects.