Filtered, cellular and CW algebras

- Idea: Repeat the topological construction of CW-complexes but with algebras over an operad.

A CW-complex is (successively) constructed by attaching cells:

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Now, if X is an algebra over an operad $\mathcal{O}$ we need to define a meaningful diagram

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & X \\ D^n & \end{tikzcd} \quad \text{in } Alg_{\mathcal{O}}$$

such that is compatible with filtrations

- Categorical setting

We will work with a "good" category S with some extra structure and satisfying some axioms.

$*$ S is closed monoidal

We have a tensor product, a unit $\mathbb{1} \in S$ and an internal hom $\mathcal{H}om_S(-,-) \in S$ satisfying

$$\mathrm{Hom}_S(X \otimes Y, Z) \cong \mathrm{Hom}_S(X, \mathcal{H}om_S(Y, Z))$$

$*$ Symmetry:

Depending on a parameter $k \in \{1, 2, \dots, \infty\}$ the monoidal category is

symmetric $(k > 2)$ / braided $(k=2)$ / non-sym $(k=1)$

This means that if $k \geq 2$ there is a braiding natural isomorphism $\beta_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ whose

inverse is $\beta_{Y,X}$ if $k > 2$.

We assume that k is fixed and depending on it there is an implicit notion of "symmetry".

* S is simplicially enriched

This is that S is a $sSet$-enriched category

In particular there is a simplicial set of morphism

$$\mathrm{Map}_S(X,Y) \in sSet \text{ for } X, Y \in S$$

* S is complete and cocomplete in the enriched sense ↖ all $sSet$-indexed colimits and limits exist

For us this implies two things:

** S is complete and cocomplete

** S has a copowering and a powering

The copowering is represented as

$$- \times - : sSet \times S \to S$$

The powering is represented as

$$(-)^{-} : sSet \times S \to S$$

and satisfies for $K, L \in sSet$ and $X \in S$

$$K \times (L \times X) \cong (K \times L) \times X,$$

$$(X^K)^L \cong X^{K \times L}$$

and they are adjoints (in the enriched sense)

$$\mathrm{Map}_S(K \times X, Y) \cong \mathrm{Map}_S(X, Y^K)$$

as simplicial sets

The compatibility of the structures gives us a functor

$$s: sSet \longrightarrow S$$
$$K \longmapsto K \times \mathbb{1}$$

strong monoidal

$$s(K \times L) \cong s(K) \otimes s(L)$$

If S is pointed, enriched over $sSet$ automatically

the terminal and initial object agree

$$\mathbb{t} = \mathbb{i}$$

implies enriched over $sSet_*$ pointed

Examples :

* sSet

$$\mathrm{Map}_{sSet}(X,Y)_n = \mathrm{Hom}_{sSet}(X\times\Delta^n, Y)$$

Cartesian product as both $\times$, $\otimes$

S: $sSet \xrightarrow{id} sSet$

* $sSet_*$ (similarly with the smash product)

* Top (Compactly generated weakly Hausdorff spaces)

$$\mathrm{Map}_{Top}(X,Y)_n = \mathrm{Hom}_{Top}(X\times\Delta^n, Y)$$

$K\times X = |K|\times X$, $X^K = \mathrm{Hom}_{Top}(|K|, X)$ compact-open topology

for $X, Y \in Top$, $K \in sSet$

$S = |-|: sSet \to Top$ (geometric realization)

$\otimes$ is the cartesian product

$\mathcal{H}om_{Top}(X,Y) = Hom_{Top}(X,Y)$ with the compact-open topology

* Top_* (similarly with the smash products)

* $sMod_{\Bbbk}$ (simplicial $\Bbbk$-modules, for $\Bbbk$ comm ring)

$X, Y \in sMod_{\Bbbk}$,

$$Map_{sMod_{\Bbbk}}(X,Y)_n = Hom_{Mod_{\Bbbk}}(X \otimes \Bbbk[\Delta^n], Y)$$

$K \in sSet$

levelwise

$$K \times X = \Bbbk[K] \otimes X$$

$X^K = Map_{sSet}(K, X)$ with the simplicial $\Bbbk$-module inherited from X

$\otimes_{\Bbbk}$ tensor product of $\Bbbk$-modules levelwise

$S : sSet \to sMod_{\Bbbk}$ free $\Bbbk$-module levelwise.
$X \mapsto \Bbbk[X]$

* Non-example: $Ch_{\Bbbk}$: chain complexes over $\Bbbk$

There is no strong monoidal functor
$S : sSet \longrightarrow Ch_{\Bbbk}$ so it is not a "good"
category.

But $Ch_{\Bbbk} \simeq sMod_{\Bbbk}$ by Dold-Kan theorem.

* Sp^{Σ} (Symmetric spectra)

Spectra $\{E_n\}_{n \geq 0}$ of pointed simplicial sets
with Σ_n-actions compatible with the maps

$$E_n \wedge S^1 \longrightarrow E_{n+1}$$

$(K \times E)_n = E_n \wedge K_+$ for $K \in Set$

$K \sqcup *$

$\mathbb{1} = \mathbb{S}^n = \{ S^n = (S^1)^{\wedge n} \}_{n \geq 0}$

$\otimes = \wedge$ smash product of symm. spectra

$$S : sSet \to Sp^{\Sigma}$$
$$K \mapsto \Sigma^{\infty} K_+ = \{ S^n \wedge K_+ \}_{n \geq 0}$$

* Diagram categories

$$\mathcal{C} = S^G = Fun(G, S)$$

G is normally discrete or a groupoid.

<u>Proposition</u>: S good and G (k-symmetric) monoidal

$\Rightarrow \mathcal{C} = S^G$ is good.

$\oplus_G$

The tensor product in $\mathcal{C}$ is given by the Day convolution: for $X, Y \in \mathcal{C} = S^G$

$$\begin{tikzcd}
G \times G \arrow[r, "X \times Y"] \arrow[dr, bend right, "\otimes_G"'] & S \times S \arrow[r, "\otimes_S"] \arrow[d, phantom, "\Downarrow"] & S \\
& G \arrow[ur, dashed, "X \otimes_{\mathcal{C}} Y"'] &
\end{tikzcd}$$

$\rightarrow$ It is a left Kan extension.

Example: $S = \mathrm{Vect}$ with $\otimes$ and $G = \mathbb{Z}$ discrete

$X, Y : \mathbb{Z} \longrightarrow \mathrm{Vect}$; then

$$X \otimes_{\mathcal{C}} Y : \mathbb{Z} \to \mathrm{Vect}, \quad (X \otimes Y)(k) = \bigoplus_{i+j=k} X(i) \otimes Y(j)$$

The unit of $\otimes_{\mathcal{C}}$ is

$$\hom_G(\mathbb{1}_G, -) \otimes \mathbb{1}_S : G \longrightarrow S$$

$S_{\mathscr{C}} : sSet \to S^G$ is defined as follows

Left Kan extension:
$$\begin{tikzcd} * \arrow[r, "X"] \arrow[d, "\mathbb{1}_G"'] \arrow[dr, phantom, "\Downarrow"] & S \\ G \arrow[ur, dashed, "(\mathbb{1}_G)_*(X)"'] & \end{tikzcd}$$

$$S_{\mathscr{C}} : sSet \xrightarrow{s} S = S^* \xrightarrow{(\mathbb{1}_G)_*} S^G$$

- Algebras over operads

G operad in $\mathscr{C}$ $(= S^G)$ a good category.

For $k \in \{1, 2, \ldots, \infty\}$ we write

$k > 2$ $\qquad G_n = \Sigma_n$ (symmetric group)

$k = 2$ $\qquad G_n = \beta_n$ (braid group)

$k = 1$ $\qquad G_n = \{1\}$ (trivial group)

Then $\mathcal{O}$ is a colection of objects $\mathcal{O}(n)$ with Σ_n actions, for $n \geq 0$ with morphisms

* Unit: $\quad 1_{\mathcal{O}} : \mathbb{1}_{\mathcal{C}} \to \mathcal{O}(1)$

* Composition

$$\mu_{\mathcal{O}}(n; k_1, \dots, k_n) : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \dots \otimes \mathcal{O}(k_n) \to$$

$$\to \mathcal{O}(k_1 + \dots + k_n)$$

which satisfies unit, associativity and equivariance axioms.

An algebra X over the operad $\mathcal{O}$ is an object $X \in \mathcal{C}$ together with morphism

$$\mathcal{O}(n) \otimes X^{\otimes n} \longrightarrow X$$

satisfying unit, associativity and equivariance axioms.

- <u>Filtered algebras</u>

We want a categorical approach of filtrations and gradings.

<u>Definition</u>:

* $\mathbb{Z}_=$ is the discrete category with objects $n \in \mathbb{Z}$
* $\mathbb{Z}_\leq$ is the category associated to the poset $(\mathbb{Z}, \leq)$

$$\begin{tikzcd} \cdots & \underset{-1}{\bullet} \arrow[r] & \underset{0}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} & \cdots \end{tikzcd}$$

Both categories are symmetric monoidal with the sum

$\Rightarrow$ $\mathcal{C}^{\mathbb{Z}_=}$ and $\mathcal{C}^{\mathbb{Z}_\leq}$ have the Day convolution.

We have a list of adjoint functors:

left
$\perp$
right

$$\begin{tikzcd} \mathcal{C}^{\mathbb{Z}_\leq} \arrow[rr, bend left=50, "\mathrm{colim}"] \arrow[rr, bend right=50, "\lim"'] & & \mathcal{C} \arrow[ll, "\mathrm{const}" description] \end{tikzcd}$$

const: $\mathcal{C} \longrightarrow \mathcal{C}^{\mathbb{Z}_\le}$

$$X \longmapsto \left(\cdots \to \overset{0}{X} \xrightarrow{id} \overset{1}{X} \xrightarrow{id} \overset{2}{X} \to \cdots \right)$$

colim: $\mathcal{C}^{\mathbb{Z}_\le} \longrightarrow \mathcal{C}$

$$\left(\cdots \to X(0) \to X(1) \to \cdots \right) \longmapsto \operatorname*{colim}_{i \in \mathbb{Z}_\le} X(i)$$

lim: $\mathcal{C}^{\mathbb{Z}_\le} \longrightarrow \mathcal{C}$

$$\left(\cdots \to X(0) \to X(1) \to \cdots \right) \longmapsto \lim_{i \in \mathbb{Z}_\le} X(i)$$

For a number $a \in \mathbb{Z}$ (identified with a functor $\{*\} \longrightarrow \mathbb{Z}_\le$, $* \longmapsto a$)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "a_*"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbb{Z}_\le} \arrow[l, bend left, "a^*"] \end{tikzcd}$$

$a^* = - \circ a : \mathcal{C}^{\mathbb{Z}_\le} \longrightarrow \mathcal{C}$

$$X \longmapsto X(a)$$

has a left adjoint given by a Kan extension

$$\begin{tikzcd} \{*\} \arrow[r, "X"] \arrow[d, "a"'] & \mathcal{C} \\ \mathbb{Z}_{\le} \arrow[ur, dashed, "a_* X"'] & \end{tikzcd}$$

$$\begin{array}{ccccccccccccc} & & & & & a-1 & & a & & a+1 & & a+2 & \\ a_*X = (\cdots & \to & \emptyset & \to & \emptyset & \to & \emptyset & \to X & \xrightarrow{id} & X & \xrightarrow{id} & X & \to \cdots) \end{array}$$

We write $\mathcal{C}_* = \mathbb{1} \downarrow \mathcal{C}$ the pointed category of $\mathcal{C}$, whose objects are $\{\mathbb{1} \to X\}$ in $\mathcal{C}$

(It has a initial and terminal object $\mathbb{1} \xrightarrow{id} \mathbb{1}$)

$$\begin{tikzcd} \mathcal{C}^{\mathbb{Z}_{\le}} \arrow[r, bend left, "gr"] \arrow[r, phantom, "\perp"] & \mathcal{C}_*^{\mathbb{Z}_{=}} \arrow[l, bend left, "u"] \end{tikzcd}$$

$gr : \mathcal{C}^{\mathbb{Z}_{\le}} \to \mathcal{C}_*^{\mathbb{Z}_{=}}$, $X = (\cdots \to X(n-1) \to X(n) \to \cdots) \mapsto gr(X)$

$$gr(X)(n) = \operatorname{colim} \left(\begin{tikzcd} X(n-1) \arrow[r] \arrow[d] & X(n) \\ \mathbb{1} & \end{tikzcd} \right)$$

Pointed by

$$\begin{tikzcd}
 & X(n-1) \arrow[r] \arrow[d] & X(n) \arrow[d] \\
\mathbb{t} \arrow[r, "id"] & \mathbb{t} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & gr(X)(n)
\end{tikzcd}$$

Intuitively "$gr(X)(n) = X(n) / X(n-1)$"

This is notation: $X(n-1) \to X(n)$ needs not to be injective.

$$U : \mathcal{C}_*^{\mathbb{Z}_=} \longrightarrow \mathcal{C}^{\mathbb{Z}_\le}$$

$$(\dots, X(0), X(1), X(2), \dots) \longmapsto \left(\dots \begin{tikzcd}[column sep=small]
X(0) \arrow[dr] & & X(1) \arrow[dr] & & X(2) & \dots \\
 & \mathbb{t} \arrow[ur] & & \mathbb{t} \arrow[ur] & &
\end{tikzcd} \right)$$

We need the basepoint to construct a morphism from $X(n-1)$ to $X(n)$!

- Operads on filtered/graded categories

The functor $0_* : \mathcal{C} \to \mathcal{C}^{\mathbb{Z}_\leq}$ is strongly monoidal

zero

$X \mapsto (\cdots \to \overset{0}{\mathbb{1}} \to \overset{0}{\mathbb{1}} \to X \to X \to \cdots)$
$ {}_{-1} \quad {}_{0} \quad {}_{1}$

So an operad $\mathcal{O}$ in $\mathcal{C}$ induces an operad in $\mathcal{C}^{\mathbb{Z}_\leq}$

Similarly with $0_* \circ (-)_+ : \mathcal{C} \to \mathcal{C}_* \to \mathcal{C}_*^{\mathbb{Z}_=}$

$$X \overset{(-)_+}{\longmapsto} X_+ = X \sqcup \mathbb{1} \overset{0_*}{\longmapsto} (\underset{}{\ldots}, \underset{-2}{\overset{0}{\mathbb{1}}}, \underset{-1}{\overset{0}{\mathbb{1}}}, \underset{0}{X_+}, \underset{1}{\overset{0}{\mathbb{1}}}, \underset{2}{\overset{0}{\mathbb{1}}}, \ldots)$$

$\mathbb{1}$

which is strongly monoidal.

Notation we continue to call the operads in $\mathcal{C}^{\mathbb{Z}_\leq}$ and $\mathcal{C}_*^{\mathbb{Z}_=}$ as $\mathcal{O}$.

There is a commutative (up to natural isomorphism) diagram of the form

$$\begin{tikzcd}
\mathrm{Alg}_G\left(\mathcal{C}^{\mathbb{Z}_{\le}}\right) \arrow[r, shift left, "gr"] \arrow[d, shift left, "U^G"] & \mathrm{Alg}_G\left(\mathcal{C}_*^{\mathbb{Z}_=}\right) \arrow[l, shift left, "o"] \arrow[d, shift left, "U^G"] \\
\mathcal{C}^{\mathbb{Z}_{\le}} \arrow[r, shift left, "gr"] \arrow[u, shift left, "F^G"] & \mathcal{C}_*^{\mathbb{Z}_=} \arrow[l, shift left, "u"] \arrow[u, shift left, "F^G"]
\end{tikzcd}$$

F^G is the 'free operad' functor and U^G forgets the G-algebra structure.

- Cell Attachments

$\mathcal{C} = S^G$ for S "good" category, G an operad in $\mathcal{C}$

Inputs: consider $X_0 \in \mathrm{Alg}_G(\mathcal{C})$

* $\partial D^d \hookrightarrow D^d$ a cofibration in sSet whose geometric realization is homeomorphic to $\partial \mathbb{D}^d \hookrightarrow \mathbb{D}^d$ in Top.

  topological disk.

* An object $g \in G$.

* A morphism $e : s(\partial D^d) \to U^0(X_0)(g)$ in S.

Recall: $s : \mathrm{sSet} \to S$ and $\mathrm{Alg}_G(\mathcal{C}) \xrightarrow{U^G} \mathcal{C} \xrightarrow{g^*} S$

$X_0 \mapsto U^0(X_0) \mapsto U^0(X_0)(g)$

Additional construction

Use the adjunction $\mathcal{C} = S^G \underset{g^*}{\overset{g_*}{\rightleftarrows}} S$ for each $g \in G$

$$X \longmapsto X(g)$$

to define

$$D^{g,d} = g_*(D^d)$$
$$\partial D^{g,d} = g_*(\partial D^d)$$

Notation: $s(\partial D^{g,d})$
$\quad\quad\quad\quad \overset{\shortparallel}{\partial D^{g,d}}$

Also use adjunctions:

$$\hom_S\left(s(\partial D^d), U^G(X_0)(g)\right) \cong$$
$$\cong \hom_{S^G}\left(s(\partial D^{g,d}), U^G(X_0)\right) \cong$$
$$\cong \hom_{Alg_G(S^G)}\left(F^G(s(\partial D^{g,d})), X_0\right)$$

$$e \longmapsto \tilde{e}$$

To define $\tilde{e} : F^T(s(\partial D^{g,d})) \to X_0$

Notation : $\tilde{e} = e$

Output: consider the pushout

$$\begin{tikzcd}
F^T(s(\partial D^{g,d})) \arrow[r, "\tilde{e}"] \arrow[d] & X_0 \arrow[d] \\
F^T(s(D^{g,d})) \arrow[r] & X_1 \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

We say that X_1 is obtained from X_0 by attaching a G- cell of dimension (g,d) along e.

Notation : $X_1 = X_0 \cup^G_e D^{g,n}$

What about filtrations? (Not necessary!)

Fact: there is a functor colim $: Alg_G(\mathcal{C}^{\mathbb{Z}_{\leq}}) \to Alg_G(\mathcal{C})$

such that

$$\begin{tikzcd} \mathrm{Alg}_G(\mathcal{C}^{\mathbb{Z}_\leq}) \arrow[r, "\mathrm{colim}"] \arrow[d, "U^G"'] & \mathrm{Alg}_G(\mathcal{C}) \arrow[d, "U^G"] \\ \mathcal{C}^{\mathbb{Z}_\leq} \arrow[r, "\mathrm{colim}"] & \mathcal{C} \end{tikzcd}$$

commutes.

If $X \in \mathrm{Alg}_G(\mathcal{C}^{\mathbb{Z}_\leq})$ is a filtered G-algebra we think of $\mathrm{colim}(X) \in \mathrm{Alg}_G(\mathcal{C})$ as the "underlying" G-algebra and an isomorphism

$$R \xrightarrow{\cong} \mathrm{colim}\, X \text{ in } \mathrm{Alg}_G(\mathcal{C})$$

induces a multiplicative filtration on R.

Consider $R_0 \in \mathrm{Alg}_G(\mathcal{C})$ and an attachment of a G-cell

$$\begin{tikzcd} F^G(\partial D^{g,d}) \arrow[r, "e"] \arrow[d] & R_0 \\ F^G(D^{g,d}) & \end{tikzcd}$$

This is a diagram in $Alg_G(\mathcal{C})$: we want a (non)-trivial diagram on $Alg_G(\mathcal{C}^{\mathbb{Z}_\le})$

$$\begin{tikzcd}
F^G(1_* \partial D^{g,d}) \arrow[r, "e"] \arrow[d] & 0_* R_0 \arrow[d] \\
F^G(1_* D^{g,d}) \arrow[r] & f R_1 \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

$$\begin{tikzcd}
-1 & & 0 & & 1 & & 2 \\
0 \arrow[rr] \arrow[r] \arrow[d] & 0 \arrow[d] & 0 \arrow[rr] \arrow[r] \arrow[d] & R_0 \arrow[d] & F^G(\partial D^{g,d}) \arrow[r] \arrow[d] & R_0 \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[ul, phantom, "\ulcorner"] & 0 \arrow[r] & R_0 \arrow[ul, phantom, "\ulcorner"] & F^G(D^{g,d}) \arrow[r] & (\text{non trivial}) \arrow[ul, phantom, "\ulcorner"] &
\end{tikzcd}$$

(The situation disaggregated)

$fR_1 \in Alg_G(\mathcal{C}^{\mathbb{Z}_\le})$ is the cell attachment filtration

(it is not concentrated in any degree)

- <u>Cellular algebras</u>

Cellular $\mathcal{O}$-algebras are constructed by iterated cell attachments starting at $\mathbb{I}$.

A map $f: X \to Y$ of $\mathcal{O}$-algebras is <u>cellular</u> if it is a transfinite composition of cell attachments. This means that there exists a diagram

$$\begin{tikzcd}
X_{-1} = X \arrow[r] \arrow[d, dashed, "f"'] & X_0 \arrow[r] \arrow[dl, "f_0"] & X_1 \arrow[r] \arrow[dll, "f_1"] & \cdots \\
Y & & &
\end{tikzcd}$$

indexed by some ordinal κ, such that

* $\operatorname{colim}_{i \in \kappa} f_i$ is an isomorphism
* for each successor ordinal $i \in \kappa$

$$\begin{tikzcd}
F^G\left(\coprod_{\alpha \in I_i} \partial D^{g_\alpha, d_\alpha}\right) \arrow[r, "\amalg h_\alpha"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_{i-1} \arrow[d] \\
F^G\left(\coprod_{\alpha \in I_i} D^{g_\alpha, d_\alpha}\right) \arrow[r] & X_i
\end{tikzcd}$$

is a pushout diagram for some maps $h_\alpha : \partial D^{g_\alpha, d_\alpha} \to X_{i-1}$

* For each limit ordinal $i \in \kappa$, $f_i = \operatorname{colim}_{i' < i} f_{i'} : X_i \to Y$

<u>Definition</u>: An $\mathcal{O}$-algebra Y is <u>cellular</u> if $\mathring{\mathbb{1}} \to Y$ is cellular.

• <u>What about filtrations</u>?

Cellular $\mathcal{O}$-algebras do not have a useful filtration. If we attach cells in increasing order of dimension we obtain a filtered object in $Alg_{\mathcal{O}}(\mathcal{C})$ i.e. an object in $Alg_{\mathcal{O}}(\mathcal{C})^{\mathbb{Z}_{\leq}} \neq$

$\neq Alg_{\mathcal{O}}(\mathcal{C}^{\mathbb{Z}_{\leq}})$

Definition: $\partial D^d \hookrightarrow D^d$ is always a cofibration in sSet whose geometric realization is homeomorphic to $\partial \mathbb{D}^d \to \mathbb{D}^d$ in Top. Define objects in $sSet^{\mathbb{Z}_\leq}$:

$$\begin{array}{ccccccccc} & & & & d-2 & d-1 & d & d+1 & \\ D^d[d] = (\cdots \to & \emptyset & \to & \emptyset & \to \emptyset & \to \partial D^d & \hookrightarrow D^d & \xrightarrow{id} D^d & \to \cdots) \end{array}$$

$$\partial D^d[d-1] = (\cdots \to \emptyset \to \emptyset \to \emptyset \to \partial D^d \xrightarrow{id} \partial D^d \xrightarrow{id} \partial D^d \to \cdots)$$

Recall that we have a strong monoidal functor

$$S_G : sSet \xrightarrow{S} \mathcal{S} \xrightarrow{(\mathbb{1}_G)_*} \mathcal{S}^G = \mathcal{C}$$

that induces a functor $sSet^{\mathbb{Z}_\leq} \longrightarrow \mathcal{C}^{\mathbb{Z}_\leq}$

So we consider $D^d[d]$ as living in $\mathcal{C}^{\mathbb{Z}_\leq}$

Notation: Given $X \in \mathcal{C}^{\mathbb{Z}_\leq} = (\mathcal{S}^G)^{\mathbb{Z}_\leq} = \mathcal{S}^{G \times \mathbb{Z}_\leq}$ we write

$X(g,n) \in \mathcal{S}$ for its value at $(g,n) \in G \times \mathbb{Z}_\leq$.

Definition: A <u>CW-algebra structure</u> on $Y \in Alg_G(\mathcal{C})$ is a relative CW-structure on $\mathbb{1} \to Y$.

<u>Definition</u>: A <u>relative CW-structure</u> on a morphism

$f: X \to Y$ in $\mathrm{Alg}_G(\mathbb{Z}^{\leq})$ is

* A diagram in $\mathrm{Alg}_G(\mathcal{C}^{\mathbb{Z}_{\leq}})$ (countable)

$$O_*(X) = sk_{-1}(f) \xrightarrow{f_0} sk_0(f) \xrightarrow{f_1} sk_1(f) \to \cdots$$

* For $d \geq 0$, a set I_d, objects $\{g_\alpha \in G \mid \alpha \in I_d\}$,

and morphisms

$$e_\alpha : \partial D^d_\alpha \longrightarrow sk_{d-1}(f)(g_\alpha, d-1) \text{ in } S$$

adjoint to

$$\tilde{e}_\alpha : \partial D_\alpha^{g_\alpha, d}[d-1] \to sk_{d-1}(f) \text{ in } S^{G \times \mathbb{Z}_{\leq}} = \mathcal{C}^{\mathbb{Z}_{\leq}}$$

such that there is a pushout diagram of the form

$$\begin{tikzcd}
F^G\left(\bigsqcup_{\alpha\in I_d} \partial D_\alpha^{g_\alpha,d}[d-1]\right) \arrow[r, "\sqcup\tilde{e}"] \arrow[d] & sk_{d-1}(f) \arrow[d, "f_d"] \\
F^G\left(\bigsqcup_{\alpha\in I_d} D_\alpha^{g_\alpha,d}[d]\right) \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & sk_d(f)
\end{tikzcd}$$

* For $sk(f) = \operatorname{colim}_d sk_d(f)$ a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\
\operatorname{colim}(sk_{-1}(f)) \arrow[r] & \operatorname{colim}(sk(f))
\end{tikzcd} \quad \text{in } Alg_G(\mathcal{C})$$

automatically

Theorem: $gr(sk(f))$ in $Alg_G(\mathcal{C}_*^{\mathbb{Z}_=})$ is isomorphic to

$$O_*(X) \vee^G F^G\left(\bigvee_{d\geq 0} \bigvee_{\alpha\in I_d} d_*(S_\alpha^{g_\alpha,d})\right)$$

$\vee^G$ denotes the coproduct in $\mathrm{Alg}_G(\mathcal{C}_*^{\mathbb{Z}_=})$

$\vee$ is the coproduct in $\mathcal{C}_*^{\mathbb{Z}_=}$

$$\mathbb{S}^{g,d}_= = D^{g,d} \Big/ \partial D^{g,d} = \mathrm{colim}\left(\begin{tikzcd} \partial D^{g,d} \arrow[r] \arrow[d] & D^{g,d} \\ * & \end{tikzcd}\right) \in \mathcal{C}_*$$

in $\mathcal{C}$

Intuitively if we make a "quotient" by things of degree $k-1$, we send $\partial D^{g,d}[d-1]$ to a point and we have a free operad generated by an "sphere" $\mathbb{S}^{g,d}$ at degree d.

We have the expression

$$gr(sk_d(f)) \cong gr(sk_{d-1}(f)) \vee F^G\left(\bigvee_{\alpha\in I_k} d_*(\mathbb{S}^{g_\alpha,d}_\alpha)\right)$$

• Model categories for operads and algebras

We assume that S has a (cofibrantly generated) model category structure.

The projective model structure: consider an adjunction

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "U"] \end{tikzcd}$$

$\mathcal{D}$ with a model category. We declare a morphism f in $\mathcal{E}$ to be a

* fibration if $U(f)$ is a fibration
* weak equivalence if $U(f)$ is a weak equivalence

This is a model category on $\mathcal{E}$ if some conditions holds (always in our case): this is the projective model category on $\mathcal{E}$ transfered along $F \dashv U$.

Consider now $\mathcal{C} = S^G$ and the functor

$$U : S^G \longrightarrow \prod_{ob(G)} S$$

$$X \mapsto (X(g))_{g \in G}$$

with left adjoint

$$F((X_g)_{g\in G}) = \coprod_{g \in G} hom_G(g, -) \times X_g$$

$\Rightarrow$ $S^G = \mathcal{C}$ has the projective model category

<u>Definition</u>: We define groupoids FB_k depending on $k \in \{1, 2, \dots \infty\}$ with $ob(FB_k) = \{0, 1, 2, \dots\}$

and

$$\underset{FB_k}{hom}(n, n) = G_n = \begin{cases} \Sigma_n \text{ (symmetric group) if } k \geq 2 \\ B_n \text{ (Braided group) if } k = 2 \\ \{id\} \text{ (trivial group) if } k = 1 \end{cases}$$

We consider the category of <u>k-symmetric sequences</u> in $\mathscr{C}$, $FB_k(\mathscr{C}) = \mathscr{C}^{FB_k}$. Then an operad is a unital monoid in $FB_k(\mathscr{C})$ with respect to the composition product.

Then $FB_k(\mathscr{C})$ has a projective model category. In particular, $X \in FB_k(\mathscr{C})$ is cofibrant if and only if $X(n)$ is cofibrant in $\mathscr{C}^{G_n}$ for each $n \geq 0$

The projective model structure in $Alg_G(\mathscr{C})$ is transfered along

$$\mathscr{C} \underset{U^G}{\overset{F^G}{\rightleftarrows}} Alg_G(\mathscr{C})$$

Moreover, if the underlying k-symmetric sequence of G is cofibrant $\Rightarrow U^G$ preserves cofibrations and trivial cofibrations between cofibrant objects.