Mario

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- CW Algebras

Idea : Mimick

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & X_0 \arrow[d] \\ D^n \arrow[r] & X_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

for X_0 in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ ← operad

- Categories

We work with S a "good category"

* S closed monoidal

$$- \otimes_S - : S \times S \to S$$

$$\mathbb{1}_S$$

$$\mathrm{Hom}_S(-,-) : S \times S \to S$$

$k \in \{1, 2, \dots \infty\}$

$k \geq 2 \Rightarrow S$ sym monoidal

$k = 2 \Rightarrow$ braided

$k = 1 \Rightarrow$ non-sym

Examples

- $S = \mathrm{TOP}$

$\otimes = \times$ cartesian prod

$\mathbb{1}_{Top} = *$

$\mathrm{Hom}(X,Y) = \mathrm{Hom}_{Top}(X,Y)$ compact open top

and

$K = 1 \Rightarrow$ non-sym

and

$\mathrm{Map}_{\mathrm{Top}}(X, Y)_n = \mathrm{Hom}_{\mathrm{Top}}(X \times \Delta^n, Y)$

* $\mathcal{S}$ enriched over sSet

$\mathrm{Map}_{\mathcal{S}}(X, Y) \in \mathrm{sSet}$

* $\mathcal{S}$ is bicomplete in the enriched sense

sSets-indexed limits/colimits exist

complete and cocomplet

powering and copowering

$- \times - : \mathrm{sSet} \times \mathcal{S} \longrightarrow \mathcal{S}$

$K \in \mathrm{sSet}$

$X \in \mathrm{Top}$

$K \times X = |K| \times X$

$(-)^{-} : \mathrm{sSet}^{op} \times \mathcal{S} \longrightarrow \mathcal{S}$

$X^K = \mathrm{Hom}_{\mathrm{Top}}(|K|, X)$

s.t.

$(K \times L) \times X = K \times (L \times X)$

$K, L \in \mathrm{sSet}, \; X \in \mathcal{S}$

$(X^K)^L = X^{K \times L}$

* $s : \mathrm{sSet} \longrightarrow \mathcal{S}$

$K \longmapsto K \times 1\!\!1_{\mathcal{S}}$

· s strong monoidal

$s(K) = |K|$

$s(K \times L) \cong s(K) \times s(L)$

· If $\mathcal{S}$ pointed $(\emptyset = *)$ and is "good" wrt sSet

$\Rightarrow$ Also good wrt $sSet_*$

Examples: * $S = Top_*$

* $sSet$

* $sMod_{\mathbb{K}}$ (simp mods over cRing)

$\otimes_{\mathbb{K}}$ levelwise

$X, Y \in sMod_{\mathbb{K}}$

$$Map_{sMod_{\mathbb{K}}} = Hom_{Mod_{\mathbb{K}}}(X \otimes \mathbb{K}[\Delta^n], Y)$$

$K \in sSet$

$$K \times X = \mathbb{K}[K] \otimes X \qquad s(K) = \mathbb{K}[K]$$

$$X^K = Map_{sSet}(K, X)$$

* $Ch_{\mathbb{K}}$ not good even though they are eq to the previous example, via Dold-Kan. Issue: DK not monoidal

* Diagram categories $\mathcal{C} = S^G$ sym monoidal

In practice G = discrete, poset and maybe groupoid.

• S is good and G monoidal $\oplus_G$ $\Rightarrow$ $\mathcal{C} = S^G$ is good.

• Tensor product: Day convolution

$X, Y : G \to S$ $\qquad G \times G \xrightarrow{X \times Y} S \times S \xrightarrow{\otimes_S} S$

$\Downarrow$

$\oplus$

$$\begin{tikzcd} \arrow[dr, "\oplus_G"'] & \Downarrow & \\ & G \arrow[r, dotted] & X \otimes_{\mathcal{C}} Y \end{tikzcd}$$

<u>Example</u> :

$$\underset{\text{discrete}}{G = \mathbb{Z}} \longrightarrow \Bbbk\,\mathrm{Mod}$$

$$(X \otimes Y)(n) = \bigoplus_{i+j=n} X(i) \otimes Y(j)$$

The unit of $\mathcal{C} = S^G$ is $\hom_G(\mathbb{1}_G, -) \otimes \mathbb{1}_S : G \longrightarrow S$

$$\delta : sSet \longrightarrow S^G$$

$$\begin{tikzcd} * \arrow[r, "X"] \arrow[d, "\mathbb{1}_G"'] & S \\ G \arrow[ur, dotted, "(\mathbb{1}_G)_*(X)"'] & \end{tikzcd}$$

$$\delta_{\mathcal{C}} : sSet \xrightarrow{\delta} S \xrightarrow{(\mathbb{1}_G)_*} S^G = \mathcal{C}$$

$\mathcal{O}$ an operad in $\mathcal{C} = S^G$ good

* Unit $\mathbb{1}_{\mathcal{C}} \longrightarrow \mathcal{O}(1)$
* $\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \dots \otimes \mathcal{O}(k_n) \longrightarrow \mathcal{O}(k_1 + \dots + k_n)$
* Equivariant actions

$$X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

$$X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

- Filtration
  - $\mathbb{Z}_= = \quad \bullet \quad \bullet \quad \bullet \quad \bullet$
  - $\mathbb{Z}_\le = \quad \bullet \to \bullet \to \bullet \to \qquad$ poset

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{Z}_\le} \arrow[r, bend left=60, "\mathrm{colim}", "\perp"'] \arrow[r, bend right=60, "\lim"', "\perp"] & \mathcal{C} \arrow[l, "\mathrm{const}"']
\end{tikzcd}$$

Fix $a \in \mathbb{Z}$, $\quad a^* : \mathcal{C}^{\mathbb{Z}_\le} \longrightarrow \mathcal{C}$

$$X \longmapsto X(a)$$

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{Z}_\le} \arrow[r, bend right=50, "a^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right=50, "a_*"']
\end{tikzcd}
\qquad
a_*(X) = \left(\cdots \to \mathrm{init} \to \underset{a-1}{\mathrm{init}} \to \underset{a}{X} \to X \to \cdots \right)$$

We write $\mathcal{C}_* = \text{terminal} \downarrow \mathcal{C}$ "pointed category"

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{Z}_\le} \arrow[r, bend left=50, "\mathrm{gr}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_*^{\mathbb{Z}_=} \arrow[l, bend left=50, "u"]
\end{tikzcd}$$

$$\mathrm{gr}(X)(n) = \mathrm{colim}\left(\begin{tikzcd} X(n-1) \arrow[r] \arrow[d] & X(n) \\ \mathrm{term} & \end{tikzcd} \right) \text{ "="} \; X(n)/X(n-1)$$

term $\quad$ pointed object

$$u(\dots, X(n), X(n+1), \dots) = \left(\begin{tikzcd}[column sep=small] \cdots & X(n) \arrow[dr] & & X(n+1) \arrow[dr] & & X(n+2) & \cdots \\ & & \text{term} \arrow[ur] & & \text{term} \arrow[ur] & & \end{tikzcd}\right)$$

• The functor $0_* : \mathcal{C} \longrightarrow \mathcal{C}^{\mathbb{Z}_\leq}$ is strong monoidal

Any operad $\mathcal{O}$ in $\mathcal{C}$ gives an operad in $\mathcal{C}^{\mathbb{Z}_\leq}$ (also denoted $\mathcal{O}$)

$$\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_* \xrightarrow{0_*} \mathcal{C}_*^{\mathbb{Z}_=}$$

$$X \longmapsto X \sqcup \text{term}$$

is strong monoidal

$\mathcal{O} \rightsquigarrow$ operad in $\mathcal{C}_*^{\mathbb{Z}_=}$

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathbb{Z}_\leq}) \arrow[r, shift left, "gr"] \arrow[d, shift left, "u^{\mathcal{O}}"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_*^{\mathbb{Z}_=}) \arrow[l, shift left, "u"] \arrow[d, shift left, "u^{\mathcal{O}}"] \\ \mathcal{C}^{\mathbb{Z}_\leq} \arrow[u, shift left, "\mathcal{F}^{\mathcal{O}}"] \arrow[r, shift left, "gr"] & \mathcal{C}_*^{\mathbb{Z}_=} \arrow[l, shift left, "u"] \arrow[u, shift left, "\mathcal{F}^{\mathcal{O}}"] \end{tikzcd}$$

commutes up to natural iso.

Cell Attachments

$\mathcal{C} = \mathcal{S}^G$ good cat

Def: A cell attachment

Inputs: * $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$

* $e_i : \partial D^n \longrightarrow D^n$ a cofibration in sSet whose realization is homeo to $\partial \mathbb{D}^n \longrightarrow \mathbb{D}^n$

* An object $g \in G$

* And a morphism $\qquad U^\theta : \mathrm{Alg}_\theta(\mathcal{C}) \xrightarrow{\text{forgetful}} \mathcal{C}$

$$e : \Delta(\partial D^n) \longrightarrow \begin{array}{c} U^\theta(X)(g) \\ \shortparallel \\ g^* U^\theta(X) \end{array}$$

This corresponds to $\quad g_*(\Delta(\partial D^n)) \longrightarrow U^G(X).$

This corresponds to

$$\tilde{e} : F^\theta(g_*(\Delta(\partial D^n))) \longrightarrow X$$

Notation $\quad g_*(\Delta(\partial D^n)) = \partial D^{gn} \quad$ "disc of dim (g,n)"

Output:

$$\begin{tikzcd} F^\theta(\partial D^{gn}) \arrow[r, "\tilde{e}"] \arrow[d] & X \arrow[d] \\ F^\theta(D^{gn}) \arrow[r] & Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \quad \text{in } \mathrm{Alg}_\theta(\mathcal{C})$$

Y is obtained from X by attaching a cell of dim (g,n) along e

$Y = X \cup_e D^n$

Problem

$$\mathrm{Alg}_\theta(\mathcal{C})^{\mathbb{Z}_\xi} \neq \mathrm{Alg}_\theta(\underbrace{\mathcal{C}^{\mathbb{Z}_\xi}}_{})$$

$$\mathcal{C}^{\mathbb{Z}_\xi} = S^{G \times \mathbb{Z}_\xi}$$

want something here

Not here