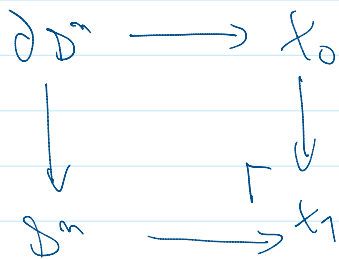


• CW Algebras

Idea: Mimick



for X_0 in $\text{Alg}_\sigma(\mathcal{C})$
 \uparrow operad

• Categories

We work with S a "good category"

\otimes_S closed monoidal

$$- \otimes_S - : S \times S \rightarrow S$$

$$\mathbb{1}_S$$

$$\text{Hom}_S(-, -) : S \times S \rightarrow S$$

$$k \in \{1, 2, \dots, \infty\}$$

$$k \geq 2 \Rightarrow S \text{ k-ary monoidal}$$

$$k = 2 \Rightarrow \text{braided}$$

$$k = 1 \Rightarrow \text{non-tern}$$

Examples

$$S = \text{TOP}$$

$$\otimes = \times \text{ cartesian prod}$$

$$\mathbb{1}_{\text{TOP}} = *$$

$$\text{Hom}(X, Y) = \text{Hom}_{\text{TOP}}(X, Y)$$

compact open top

and

...

$k \geq 1 \Rightarrow \text{non-sgn}$

and

$$\text{Map}_{\text{top}}(X, Y)_n = \text{Hom}_{\text{top}}(X \times \Delta^n, Y)$$

* S enriched over $s\text{Set}$

$$\text{Map}_S(X, Y) \in s\text{Set}$$

* S is bicomplete in the enriched sense

\uparrow
 $s\text{Set}$ -indexed limits/colimits exist

complete and cocomplete
powering and copowering

$$- \times - : s\text{Set} \times S \rightarrow S$$

$$k \in s\text{Set}$$

$$X \in \text{Top}$$

$$k \times X = |k| \times X$$

$$(-)^{\bar{}} : s\text{Set}^{\text{op}} \times S \rightarrow S$$

$$X^k = \text{Hom}_{\text{top}}(|k|, X)$$

n.t.

$$(k \times L) \times X = k \times (L \times X)$$

$$k, L \in s\text{Set}, X \in S$$

$$(X^k)^L = X^{k \times L}$$

$$* \mathfrak{s} : s\text{Set} \rightarrow S$$

$$k \mapsto k \times \mathbb{1}_S$$

• \mathfrak{s} strong monoidal

$$\mathfrak{s}(k) = |k|$$

$$\mathfrak{s}(k \times L) \cong \mathfrak{s}(k) \times \mathfrak{s}(L)$$

• If S pointed ($\emptyset = *$) and is "good" wrt $s\text{Set}$

\Rightarrow Also good wrt $s\text{Set}_*$

Examples: $* S = \text{top}_*$

$* s\text{Set}$

$* s\text{Mod}_k$ (simp mods over cRing)

\bigoplus_k levelwise

$X, Y \in s\text{Mod}_k$

$$\text{Map}_{s\text{Mod}_k} = \text{Hom}_{\text{Mod}_k}(X \otimes k[\Delta^n], Y)$$

$k \in s\text{Set}$

$$k \times X = k[k] \otimes X \quad \lambda(k) = k[k]$$

$$X^k = \text{Map}_{s\text{Set}}(k, X)$$

$* \text{Cb}_k$ not good even though they are eq to the previous example, via Dold-Kan. Issue: DK not monoidal

$* \text{Diagram categories } \mathcal{C} = \mathcal{S}^G$ \curvearrowright Sym monoidal

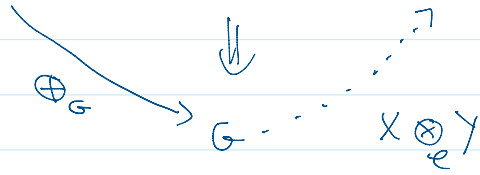
In practice $G = \text{discrete, point and maybe groupoid.}$

$\bullet S$ is good and G monoidal $\bigoplus_G \Rightarrow \mathcal{C} = \mathcal{S}^G$ is good.

\circ Tensor product: Day convolution

$$X, Y: G \rightarrow S$$

$$\begin{array}{ccc} G \times G & \xrightarrow{X \times Y} & S \times S \xrightarrow{\bigoplus_S} S \\ & \searrow \bigoplus & \downarrow \quad \nearrow \end{array}$$



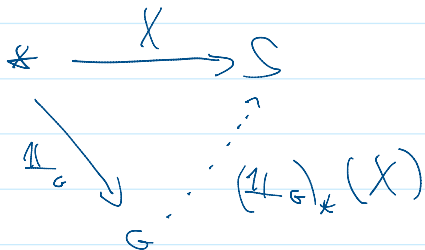
Example:

$$G = \mathbb{Z} \xrightarrow{\text{discrete}} \mathbb{k}\text{Mod}$$

$$(X \otimes Y)(n) = \bigoplus_{i+j=n} X(i) \otimes Y(j)$$

The unit of $e = S^G$ is $\text{hom}_G(\mathbb{1}_G, -) \otimes \mathbb{1}_S : G \rightarrow S$

$$\mathbb{1} : s\text{Set} \rightarrow S^G$$



$$\mathbb{1}_e : s\text{Set} \xrightarrow{\mathbb{1}} S \xrightarrow{(\mathbb{1}_G)_*} S^G = e$$

\mathcal{O} an operad in $e = S^G$ good

$$* \text{ unit } \mathbb{1}_e \rightarrow \mathcal{O}(1)$$

$$* \mathcal{O}(m) \otimes \mathcal{O}(k_1) \otimes \dots \otimes \mathcal{O}(k_m) \rightarrow \mathcal{O}(k_1 + \dots + k_m)$$

* Equivariant actions

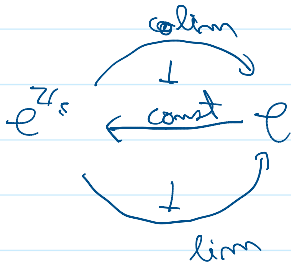
$$X \in \text{Alg}_{\mathbb{1}_G}(\mathcal{O})$$

$$X \in \text{Alg}_0(\mathcal{C})$$

• Filtration

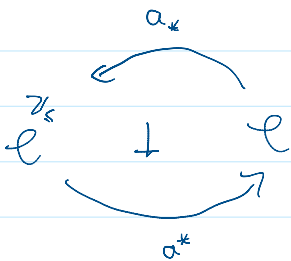
$$\cdot \mathbb{Z}_= = \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \mathbb{Z}_\leq = \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \quad \text{point}$$



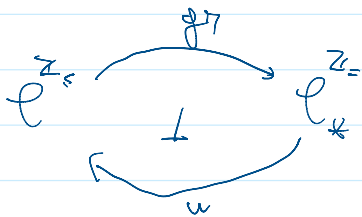
$$\text{Fix } \alpha \in \mathbb{Z}, \quad \alpha^* : \mathcal{C}^{\mathbb{Z}_\leq} \rightarrow \mathcal{C}$$

$$X \mapsto X(\alpha)$$



$$a_*(X) = (\dots \rightarrow \text{init} \xrightarrow{a_{-1}} \text{init} \xrightarrow{a} X \rightarrow X \rightarrow \dots)$$

We write $\mathcal{C}_* = \text{terminal} \downarrow \mathcal{C}$ "pointed category"



$$\text{gr}(X)(\alpha) = \text{colim} \left(\begin{array}{c} X(\alpha-1) \longrightarrow X(\alpha) \\ \downarrow \\ \text{term} \end{array} \right) \stackrel{=}{=} \frac{X(\alpha)}{X(\alpha-1)}$$

term \rightarrow pointed object

$$u(\dots, X(n), X(n+1), \dots) = \left(\dots \begin{array}{c} X(n) \quad X(n+1) \quad X(n+2) \dots \\ \searrow \quad \nearrow \quad \searrow \quad \nearrow \\ \text{turn} \quad \text{turn} \end{array} \right)$$

• The functor $\sigma_*; \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{Z}_2}$ is strong monoidal

Any operad \mathcal{O} in \mathcal{C} gives an operad in $\mathcal{C}^{\mathbb{Z}_2}$ (also denoted \mathcal{O})

$$\begin{array}{ccc} \mathcal{C} \xrightarrow{(-)_*} \mathcal{C}_* & \xrightarrow{\sigma_*} & \mathcal{C}_*^{\mathbb{Z}_2} \\ x \mapsto x \cup_{\text{turn}} & & \mathcal{O} \mapsto \text{operad in } \mathcal{C}_*^{\mathbb{Z}_2} \end{array} \quad \text{is strong monoidal}$$

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathbb{Z}_2}) & \begin{array}{c} \xrightarrow{g^{\mathbb{Z}_2}} \\ \xleftarrow{u} \end{array} & \text{Alg}_{\mathcal{O}}(\mathcal{C}_*^{\mathbb{Z}_2}) \\ \begin{array}{c} \uparrow f^{\mathbb{Z}_2} \\ \downarrow u^{\mathbb{Z}_2} \end{array} & & \begin{array}{c} \uparrow f^{\mathbb{Z}_2} \\ \downarrow u^{\mathbb{Z}_2} \end{array} \\ \mathcal{C}^{\mathbb{Z}_2} & \begin{array}{c} \xrightarrow{g^{\mathbb{Z}_2}} \\ \xleftarrow{u} \end{array} & \mathcal{C}_*^{\mathbb{Z}_2} \end{array}$$

commutes up to natural iso.

Cell Attachments

$\mathcal{C} = S^G$ good cat

Def: A cell attachment

Inputs: $*X \in \text{Alg}_{\mathcal{O}}(\mathcal{C})$

$*e; \partial D^n \rightarrow D^n$ a cofibration in sSet
whose realisation is homeo to $\partial D^n \rightarrow D^n$

* An object $g \in G$

* And a morphism

$$u^{\sigma}: \text{Alg}_{\sigma}(e) \xrightarrow{\text{forgetful}} e$$

$$e: \mathbb{1}(\partial D^m) \longrightarrow U^{\sigma}(X|g)$$

$$\quad \quad \quad \downarrow$$

$$\quad \quad \quad g^* U^{\sigma}(X)$$

This corresponds to $g_*(\mathbb{1}(\partial D^m)) \longrightarrow U^{\sigma}(X)$.

This corresponds to

$$\tilde{e}: F^{\sigma}(g_*(\mathbb{1}(\partial D^m))) \longrightarrow X$$

Notation $g_*(\mathbb{1}(\partial D^m)) = \partial D^{g,m}$ "disc of dim (g,m) "

Output:

$$\begin{array}{ccc} F^{\sigma}(\partial D^{g,m}) & \xrightarrow{\tilde{e}} & X \\ \downarrow & & \downarrow \\ F^{\sigma}(D^{g,m}) & \longrightarrow & Y \end{array} \quad \text{in } \text{Alg}_{\sigma}(e)$$

Y is obtained from X by attaching a cell of dim (g,m) along e

$$Y = X \cup_e D^m$$

Problem

$$\text{Alg}_{\sigma}(e)^{\sum_{g \in G} \mathbb{Z} e} \neq \text{Alg}_{\sigma}(e^{\sum_{g \in G} \mathbb{Z} e})$$

↑ want something here
↑ not here