Mario 2

Tuesday 5 November 2024 14:01

Last time: S, $\mathcal{C} = S^G$ ← monoidal, $sSet \xrightarrow{s} S \xrightarrow{(1_G)_*} S^G = \mathcal{C}$

monoidal by convolution

$\mathcal{O}$ operad in $\mathcal{C}$

$\partial \mathbb{D}^d \hookrightarrow \mathbb{D}^d$ cofibration in sSet

its geom realization is homeomorphic to $\partial \mathbb{D}^d \hookrightarrow \mathbb{D}^d$

$\partial \mathcal{D}^d = s(\partial \mathbb{D}^d) \longrightarrow s(\mathbb{D}^d) = \mathcal{D}^d$

$g \in G$
$$\begin{tikzcd} S^G \arrow[r, bend right, "g^*"'] \arrow[r, phantom, "\perp"] & S \arrow[l, bend right, "g_*"'] \end{tikzcd}$$

$X \longmapsto X(g)$

- Cellular algebra

"iterated attachment of cells"

<u>Definition</u>: $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a <u>cellular algebra</u> if $i \to X$ is a <u>cellular map</u>

<u>Def</u>: $f: X \longrightarrow Y$ in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a cellular map if it is a transfinite composition of cell attachments. This means

$$\begin{tikzcd} X = X_{-1} \arrow[r] \arrow[dr, bend right, "f"'] & X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[dl, "f_1"] & \cdots \\ & Y & & \end{tikzcd}$$

indexed over cardinal κ

* $\operatorname{colim}_{i \in K} f_i$ is an isomorphism

* $i \in K$ succession

$F^{O}\left(\bigsqcup_{\alpha \in I_i} \partial D^{g_\alpha, d}\right)$

$$\begin{tikzcd}
F^{O}\left(\bigsqcup_{\alpha \in I_i} \partial D^{g_\alpha, d_\alpha}\right) \arrow[r, "\sqcup h_\alpha"] \arrow[d] & X_{i-1} \arrow[d] \\
F^{O}\left(\bigsqcup_{\alpha \in I_i} D^{g_\alpha, d_\alpha}\right) \arrow[r] \arrow[ur, phantom, "\text{pushout}\ \ulcorner"] & X^{i}
\end{tikzcd}$$

$h_\alpha : \partial D^{d_\alpha} \longrightarrow U^{O}\left(X_{i-1}(g_\alpha)\right)$ in S

$\Updownarrow$

$h_\alpha : F^{O}\left(\partial D^{d_\alpha, g_\alpha}\right) \longrightarrow X_{i-1}$ in $\mathrm{Alg}_{O}(\mathcal{C})$

* $i \in K$ limit

$\operatorname{colim}_{i' < i} f_{i'} = f_i : X_i \longrightarrow Y$

$\rightarrow$ we want an object in $\mathrm{Alg}_{O}(\mathcal{C}^{K_\varepsilon}) \neq \mathrm{Alg}_{O}(\mathcal{C}^{K_\varepsilon})$

__Def:__ $\partial D^{d} \hookrightarrow D^{d}$ in sSet

$$d_* \mathbb{D}^d = \mathbb{D}^d[d] = (\cdots \longrightarrow \emptyset \longrightarrow \partial\mathbb{D}^d \hookrightarrow \mathbb{D}^d \xrightarrow{id} \mathbb{D}^d \to \cdots)$$

$d-1 \qquad d \qquad d+1$

$$\partial\mathbb{D}^d[d-1] = (\cdots \longrightarrow \emptyset \longrightarrow \partial\mathbb{D}^d \hookrightarrow \partial\mathbb{D}^d \longrightarrow \partial\mathbb{D}^d \to \cdots)$$

objects in $sSet^{\mathbb{Z}_\le}$

$sSet \to S \to S^G = \mathcal{C}$

we consider that $\mathbb{D}^d[d]$, $\partial\mathbb{D}^d[d-1] \in \mathcal{C}^{\mathbb{Z}_\le}$

$X \in \mathcal{C}^{\mathbb{Z}_\le} \simeq \mathcal{C}^{G\times\mathbb{Z}_\le}$ $\qquad X(g,d) \in S$

Def: $X \in Alg_{\mathcal{O}}(\mathcal{C})$ is a CW-algebra if $i \to X$ is a relative CW-structure

Def: $f : X \to Y$ in $Alg_{\mathcal{O}}(\mathcal{C})$ is a (countable) diagram in $Alg_{\mathcal{O}}(\mathcal{C}^{\mathbb{Z}_\le})$

$$O_*(X) = sk_{-1} \xrightarrow{f_0} sk_0 \xrightarrow{f_1} sk_1 \to \cdots$$

* For each $d \geq 0$, we have I_d set $\{ g_\alpha \in G \mid \alpha \in I_d \}$

and morphisms $e_\alpha : \partial D^d_\alpha \longrightarrow U^{\mathcal{O}}\, sk_{d-1}(g_\alpha, d-1)$ in S

adjoint to

$$\begin{tikzcd}
F^{\mathcal{O}}(\partial \mathbb{D}^{g_\alpha, d}[d-1]) \arrow[r, "\sqcup e_\alpha"] \arrow[d] & sk_{d-1} \in Alg_{\mathcal{O}}(\mathcal{C}^{\mathbb{Z}_\le}) \arrow[d, "f_d"] \\
F^{\mathcal{O}}(\sqcup \mathbb{D}^{g_\alpha, d}[d]) \arrow[r] & sk_d
\end{tikzcd}$$

$$\mathbb{F}^{\sigma}\Big(\coprod_{\alpha\in I_d} D^{g_\alpha,d}[d]\Big) \longrightarrow Sk_d$$

in $Alg_{\sigma}(\mathcal{C}^{\mathbb{Z}_{\le}})$

$\ast$ $Sk = \underset{i}{\mathrm{colim}}\, Sk_i \in Alg_{\sigma}(\mathcal{C}^{\mathbb{Z}_{\le}})$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"] \\
\mathrm{colim}(Sk_{-1}) \arrow[r] \arrow[d] & \mathrm{colim}(Sk) \\
\mathrm{colim}(Sk_0) \arrow[r] & \mathrm{colim}(Sk_1) \arrow[u]
\end{tikzcd}$$

in $Alg_{\sigma}(\mathcal{C})$

<u>Theorem</u>: let $X \to Y$ cw map

$gr(Sk)$ in $Alg_{\sigma}(\mathcal{C}^{\mathbb{Z}_{=}}_{*})$

$$O_*(X) \vee^{\sigma} \mathbb{F}^{\sigma}\Big(\bigvee_{d\geq 0}\ \bigvee_{\alpha\in I_d} d_*\big(S^{g_\alpha,d}_{\alpha}\big)\Big)$$

coproduct in $Alg_{\sigma}(\mathcal{C}^{\mathbb{Z}_{=}}_{*})$

$$S^{g_\alpha,d} = \underset{\text{in } \mathcal{C}}{\mathrm{colim}} \left(\begin{tikzcd} \partial D^{g_\alpha,d} \arrow[r] \arrow[d] & D^{g_\alpha,d} \\ \ast & \end{tikzcd}\right) \in \mathcal{C}_*$$

• <u>Model categories</u>

- <u>Model categories</u>

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "U"] \end{tikzcd}$$

model cat

Projective model structure

- f morphism in $\mathcal{E}$

$\to$ fib if $U(f)$ fib

$\to$ w.e. " " w.e.

$\Rightarrow$ (sometimes) $\mathcal{E}$ becomes a model cat

- S (cofibrantly gen) model cat

$$\mathcal{C} = \begin{tikzcd} S^G \arrow[r, bend right, "U"'] & \prod\limits_{ob(G)} S \arrow[l, bend right, "F"'] \end{tikzcd}$$

$\Rightarrow \mathcal{C}$ projective model cat

$$X \longmapsto (X(g))_{g \in G}$$

$$F((X_g)_{g\in G}) = \coprod_{g \in G} \hom_G(g,-) \times X_g$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F^{\mathcal{O}}"] \arrow[r, phantom, "\perp"] & Alg_{\mathcal{O}}(\mathcal{C}) \arrow[l, bend left, "U^{\mathcal{O}}"] \end{tikzcd}$$

$\Rightarrow$ projective model str on $Alg_{\mathcal{O}}(\mathcal{C})$

<u>Prop:</u>

Prop: If $\mathcal{O}$ is cofibrant (the underlying Sym seq is cofibrant)

$\Rightarrow$ $\mathcal{U}^{\mathcal{O}}$ preserves cofibrations between cofibrant objects.