KMAS9AA1 – Algebraic Topology

Exercise Sheet 1

1. Quotient topology, spheres, and discs

Let X be a topological space and $A \subset X$. We denote by X/A the quotient of X by the equivalence relation

$$x \sim y \iff x = y \text{ or } x, y \in A$$

equipped with the quotient topology.

1. Let $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional disc, and $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subset D^n$ the $(n-1)$-dimensional sphere. Show that D^n/S^{n-1} is homeomorphic to S^n.
2. Given a topological space X, the cone of X is the quotient topological space $C(X) := X \times [0,1]/X \times \{0\}$. Show that $C(S^{n-1})$ is homeomorphic to D^n.
3. Show that the cone of a non-empty topological space is contractible.

2. Path-connected components

1. Reprove that, up to isomorphism, the fundamental group $\pi_1(X, x)$ only depends on the path-connected component of $x \in X$. More precisely, if γ is a path between x and $y \in X$, then

$$\begin{aligned} \phi_\gamma \colon \pi_1(X, x) &\to \pi_1(X, y) \\ [\alpha] &\mapsto [\overline{\gamma} \cdot \alpha \cdot \gamma] \end{aligned}$$

   is a group isomorphism.
2. If δ is another path joining x to y, then the isomorphisms ϕ_γ and ϕ_δ are conjugate. Deduce that if $\pi_1(X, x)$ is abelian, then this isomorphism is canonical.

3. Homotopy

1. Show that the homotopy equivalence relation is indeed an equivalence relation. Convince yourself that the same is not true for deformation retracts.
2. Show that $f \sim f'$ and $g \sim g'$ imply $f \circ g \sim f' \circ g'$.
3. Show with explicit formulas that any convex subset of $\mathbb{R}^n$ is contractible.

4. Fundamental Group

1. **Simple connectedness**: Let X be a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is trivial for any $x \in X$.
   b. There exists $x_0 \in X$ such that $\pi_1(X, x_0)$ is trivial.
   c. Any map $f \colon S^1 \to X$ extends to a map $D^2 \to X$.
   d. There exist $x_0, x_1 \in X$ such that all paths from x_0 to x_1 are homotopic.
   e. For any $x_0, x_1 \in X$, all paths from x_0 to x_1 are homotopic.
2. **An abelian π_1**: Let x, y be two points in a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is abelian.
   b. For any paths α, β from x to y, the induced homomorphisms given by Exercise 2 from $\pi_1(X, x)$ to $\pi_1(X, y)$ are the same.

5. Borsuk–Ulam Theorem

Borsuk–Ulam Theorem *: For any $f \colon S^n \to \mathbb{R}^n$, there exists a pair of antipodal points x and $-x$ in S^n such that $f(x) = f(-x)$.*

Slogan: “There exist two antipodal points on Earth that have exactly the same temperature and pressure.”

1. Prove the theorem in dimension $n = 1$ using elementary methods.
2. To prove the case $n = 2$, argue by contradiction and consider the function $g \colon S^2 \to S^1$, $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. Use g to construct a non-trivial loop in $\pi_1(S^1)$ and deduce a contradiction.
3. Suppose that $S^2 = A_1 \cup A_2 \cup A_3$ with A_i closed subsets. Show that at least one of these three sets contains a pair of antipodal points. You may use the functions $S^2 \to \mathbb{R}$,

$$\mathrm{distance}_i(x) = \inf_{y \in A_i} |x - y|.$$

6. Van Kampen

1) The *connected sum* $M\#N$ of two connected manifolds M and N of the same dimension n is obtained by removing a small neighborhood of a point formed by an open disc from each, and gluing the resulting manifolds along the two spheres S^{n-1} that appear. For example, $\Sigma_g\#\Sigma_{g'} = \Sigma_{g+g'}$, where Σ_g is the oriented surface of genus g.

   Assuming $n > 2$, compute $\pi_1(M\#N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.

2) Let $X \subset \mathbb{R}^n$ be a union of convex sets $X_1, \ldots, X_n$ such that any triple intersection is non-empty $X_i \cap X_j \cap X_k \neq \emptyset$.

   Prove that X is simply connected.

3) Prove that the complement of a finite number of points in $\mathbb{R}^n$, for $n \geq 3$, is simply connected.

4) Compute the fundamental group of the complement of the unit circle in $\mathbb{R}^3$.

5) Compute the fundamental group of the quotient of the disjoint union of two tori $S^1 \times S^1$ obtained by identifying the circle $S^1 \times \{x_0\}$ of one torus with the corresponding circle $S^1 \times \{x_0\}$ of the other torus.