KMAS9AA1 – Algebraic Topology

Exercise Sheet 1

1. Quotient topology, spheres, and discs

Let X be a topological space and $A \subset X$. We denote by X/A the quotient of X by the equivalence relation

$$x \sim y \iff x = y \text{ or } x, y \in A$$

equipped with the quotient topology.

1. Let $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional disc, and $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subset D^n$ the $(n-1)$-dimensional sphere. Show that D^n/S^{n-1} is homeomorphic to S^n.

   This answer can come with a varied amount of details depending on how many formulas we want to write down (also other definitions of the sphere are reasonable..). A minimalistic acceptable argument would be to claim that S^n minus a point is homeomorphic to R^n and so is the interior of D^n. By the universal property of the quotient, this homeomorphism extends to a continuous bijection $D^n/S^{n-1} \to S^n$. Since both the source and target are compact Hausdorff, this is a homeomorphism.

   Rule of thumb: An argument with low levels of rigour is acceptable if it is relatively clear that it could be given complete rigour if the writer has enough patience.

2. Given a topological space X, the cone of X is the quotient topological space $C(X) := X \times [0,1]/X \times \{0\}$. Show that $C(S^{n-1})$ is homeomorphic to D^n.

   Using the presentation above, can construct the map $S^{n-1} \times [0,1] \to D^n$ sending (x,t) to tx. Check that this map passes to the quotient and is a homeomorphism.

3. Show that the cone of a non-empty topological space is contractible. An explicit deformation retract into the vertex of the cone can be constructed $h_t(x,s) = (x, ts)$.

2. Path-connected components

1. Reprove that, up to isomorphism, the fundamental group $\pi_1(X, x)$ only depends on the path-connected component of $x \in X$. More precisely, if γ is a path between x and $y \in X$, then

$$\begin{aligned} \phi_\gamma \colon \pi_1(X, x) &\to \pi_1(X, y) \\ [\alpha] &\mapsto [\overline{\gamma} \cdot \alpha \cdot \gamma] \end{aligned}$$

is a group isomorphism.

2. If δ is another path joining x to y, then the isomorphisms ϕ_γ and ϕ_δ are conjugate. Deduce that if $\pi_1(X, x)$ is abelian, then this isomorphism is canonical.

3. Homotopy

1. Show that the homotopy equivalence relation is indeed an equivalence relation. Convince yourself that the same is not true for deformation retracts.
2. Show that $f \sim f'$ and $g \sim g'$ imply $f \circ g \sim f' \circ g'$.
3. Show with explicit formulas that any convex subset of $\mathbb{R}^n$ is contractible.
   Pick an arbitrary point $x \in X$. The homotopy $h_t(y) = x + t(y - x)$, sending y linearly to x is continuous and is fully contained in X due to convexity.

4. Fundamental Group

1. **Simple connectedness**: Let X be a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is trivial for any $x \in X$.
   b. There exists $x_0 \in X$ such that $\pi_1(X, x_0)$ is trivial.
   c. Any map $f \colon S^1 \to X$ extends to a map $D^2 \to X$. f is given equivalently by a map $I \to X$ sending the endpoints to the same point. If f is a contractible loop, there is a homotopy starting from $I \times I$... The key observation is that $I \times I \cong D^2$.
   d. There exist $x_0, x_1 \in X$ such that all paths from x_0 to x_1 are homotopic.
   e. For any $x_0, x_1 \in X$, all paths from x_0 to x_1 are homotopic.
2. **An abelian π_1**: Let x, y be two points in a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is abelian.
   b. For any paths α, β from x to y, the induced homomorphisms given by Exercise 2 from $\pi_1(X, x)$ to $\pi_1(X, y)$ are the same.

5. **Borsuk–Ulam Theorem** : *For any $f\colon S^n \to \mathbb{R}^n$, there exists a pair of antipodal points x and $-x$ in S^n such that $f(x) = f(-x)$.*

   Slogan: "There exist two antipodal points on Earth that have exactly the same temperature and pressure."

   1. Prove the theorem in dimension $n = 1$ using elementary methods.
   2. To prove the case $n = 2$, argue by contradiction and consider the function $g\colon S^2 \to S^1$, $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. Use g to construct a non-trivial loop in $\pi_1(S^1)$ and deduce a contradiction.
      [H, Thm 1.10]
   3. Suppose that $S^2 = A_1 \cup A_2 \cup A_3$ with A_i closed subsets. Show that at least one of these three sets contains a pair of antipodal points. You may use the functions $S^2 \to \mathbb{R}$,
      $$\text{distance}_i(x) = \inf_{y \in A_i} |x - y|.$$
      [H, Cor 1.11]

6. **Van Kampen**

   1) The *connected sum* $M\#N$ of two connected manifolds M and N of the same dimension n is obtained by removing a small neighborhood of a point formed by an open disc from each, and gluing the resulting manifolds along the two spheres S^{n-1} that appear. For example, $\Sigma_g\#\Sigma_{g'} = \Sigma_{g+g'}$, where Σ_g is the oriented surface of genus g.
      Assuming $n > 2$, compute $\pi_1(M\#N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.
      This is essentially the same strategy we used for $M \vee N$. On a different note, it is instructive to compute some examples when $n = 2$.
   2) Let $X \subset \mathbb{R}^n$ be a union of convex open sets $X_1, \ldots, X_n$ such that any triple intersection is non-empty $X_i \cap X_j \cap X_k \neq \emptyset$.
      Prove that X is simply connected.
      [H, Exercise 1.2-2]
   3) Prove that the complement of a finite number of points in $\mathbb{R}^n$, for $n \geq 3$, is simply connected.
      [H, Exercise 1.2-3]
   4) Compute the fundamental group of the complement of the unit circle in $\mathbb{R}^3$.
      [H, Example 1.23]
   5) Compute the fundamental group of the quotient of the disjoint union of two tori $S^1 \times S^1$ obtained by identifying the circle $S^1 \times \{x_0\}$ of one torus with the corresponding circle $S^1 \times \{x_0\}$ of the other torus.

Similar strategy as for 1), but now there is an extra S^1 around. All these nice spaces have the property that we want to pick a closed set (which we can't) to apply van Kampen, but there is an open neighbourhood deformation retracting into it.