# KMAS9AA1 – Algebraic Topology

# Exercise Sheet 2

**1. Products, coproducts, pullbacks and pushouts**

Let $\mathcal{C}$ be a category.

1) Recall the notions in the title of this exercise and show that if they exist they are unique up to unique isomorphism.
2) Show that the category of fields does not have coproducts.
3) Show that in the category of unital commutative rings, the coproduct of $R$ and $S$ is given by $R \otimes S$ with the maps $R \to R \otimes S, r \mapsto r \otimes 1$ and $S \to R \otimes S, s \mapsto 1 \otimes s$.
4) We say that $I \in \mathcal{C}$ is an *initial* object if for any object $X \in \mathcal{C}$, there is a unique morphism $I \to X$. Similarly, $T \in \mathcal{C}$ is *terminal* if there is a unique morphism from any $X \to T$.
   a. Show that if such objects exist, they are unique.
   b. Show that if an initial object exists, any coproduct can be written as a pushout. Similarly, if a terminal object exists, a product is a pullback.
   c. Determine the initial and terminal object in the categories $\mathbf{Top}, \mathbf{Top}_*, \mathbf{Groups}, R-\mathbf{Mod}$.
5) In **Set**, show that the pullback of $f\colon X \to Z$ and $g\colon Y \to Z$ is given by the set of pairs $X \times_Z Y = \{(x, y) | x \in X, y \in Y, f(x) = g(y)\}$.

**2. CW Complexes**

1) Show that a single cell attachment is a pushout along the attachment map.
2) Let $X, Y$ be CW complexes containing finitely many cells. Show that $X \times Y$ is a CW complex with $d$-cells given by products of $k$ and $d-k$ cells.
3) Let $X, Y$ be CW complexes, $A$ a subcomplex of $X$ and $f : A \to Y$ a cellular map. Show that the pushout of $f$ along the inclusion

$A \hookrightarrow X$ is a CW complex with the cells of $Y$ and of $X - A$ as cells. You can assume for convenience that all CW complexes have finitely many (and therefore it is enough to consider the case of a single cell attachment)

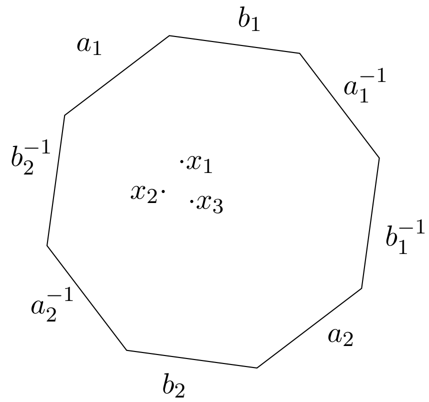
4) Use the cellular structure on $\mathbb{RP}^2$ to show that its fundamental group is a cyclic group with two elements.

**3. Fundamental Group of a Punctured Surface**

In this exercise, we will compute the fundamental group of oriented surfaces with boundary. Up to homotopy, we can assume that each connected boundary component is a point, reducing the problem to computing the fundamental group of $\Sigma_{g,n} := \Sigma_g - \{x_1, \dots, x_n\}$, where $\Sigma_g$ is the oriented surface of genus $g$ ($g \geq 0, n \geq 1$).

We will use the fact that $\Sigma_g$ admits a CW-complex structure with $2g$ 1-cells induced by the quotient of a polygon with $4g$ edges.

$\Sigma_{2,3} =$

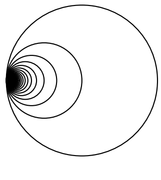



1) Compute $\pi_1(\Sigma_{g,1})$. You may construct a deformation retraction of $\Sigma_{g,1}$ onto a wedge product of circles.
2) Compute $\pi_1(\Sigma_{0,n})$. Similarly, you may construct a deformation retraction of $\Sigma_{0,n}$ onto a wedge product of circles.
3) Use the previous results and the van Kampen theorem to compute the fundamental group of $\Sigma_{g,n}$.

**4. Hawaiian Earrings**

The *Hawaiian rings* is the subspace $X$ of $\mathbb{R}^2$ obtained by the union of a sequence of circles $C_n$ where $C_n$ is the circle centered at $(1/n, 0)$ with radius $1/n$.

$X =$



1) Justify, by elementary methods, that $X$ is not homeomorphic to an infinite wedge product of circles.
2) Prove that $X$ is not *locally simply connected*: the point $(0,0) \in X$ does not have a simply connected neighborhood.
3) Construct a surjective homomorphism $\pi_1(X) \to \prod_{n=1}^{\infty} \mathbb{Z}$.
Deduce that the fundamental group of $X$ is not the same as the infinite wedge product of circles.

**5. Chain Complexes**

1) Show that the kernel, image, and cokernel of a chain complex morphism $f: C \to D$ are chain complexes. Show that if $H(\ker f) = 0 = H(\mathrm{coker} f)$, then $f$ induces an isomorphism in homology.
2) Show that the direct sum of chain complexes is a chain complex, and compare $H(A \oplus B)$ with $H(A) \oplus H(B)$.
3) Show that the dual of a chain complex $C^\vee = \{\mathrm{Hom}_R(C_i, R)\}_{i\in\mathbb{Z}}$ is a chain complex. Prove that over $\mathbb{Z}$, the dual of the homology of $C$ is not isomorphic to the homology of the dual of $C$.
4) Compute the homology of the complex

$$0 \to R \overset{\times 2}{\to} R \to 0$$

for $R = \mathbb{Z}$ and $R$ a field (pay attention to the characteristic). [Important to retain from this: The ring we are working with changes a lot the result. Hatcher only works with $\mathbb{Z}$ which does not allow us to see such differences later on.]

**6. Leftovers from class**

1) We saw in class that the fundamental group of a finite graph is a free group on finitely many generators. How many generators? The answer should depend only (linearly!) on the number of edges and vertices.
2) Use van Kampen to compute the fundamental group of a finite graph.
3) Memorize the presentation of the dihedral group.