KMAS9AA1 – Algebraic Topology

Exercise Sheet 2

1. Products, coproducts, pullbacks and pushouts

Let $\mathcal{C}$ be a category.

1) Recall the notions in the title of this exercise and show that if they exist they are unique up to unique isomorphism.
2) Show that the category of fields does not have coproducts.
   There are no maps between fields of different characteristic so a coproduct of two such fields cannot exist.
3) Show that in the category of unital commutative rings, the coproduct of R and S is given by $R \otimes S$ with the maps $R \to R \otimes S, r \mapsto r \otimes 1$ and $S \to R \otimes S, s \mapsto 1 \otimes s$.
4) We say that $I \in \mathcal{C}$ is an *initial* object if for any object $X \in \mathcal{C}$, there is a unique morphism $I \to X$. Similarly, $T \in \mathcal{C}$ is *terminal* if there is a unique morphism from any $X \to T$.
   a. Show that if such objects exist, they are unique.
   b. Show that if an initial object exists, any coproduct can be written as a pushout. Similarly, if a terminal object exists, a product is a pullback.
      The pushout of $X \leftarrow I \to Y$ is the same as $X \sqcup Y$. Similarly, the pullback of $X \to T \leftarrow Y$ is the product $X \times Y$.
   c. Determine the initial and terminal object in the categories $\mathbf{Top}, \mathbf{Top}_*, \mathbf{Groups}, R-\mathbf{Mod}$.
      $(\emptyset, *), (*, *), (\{e\}, \{e\}), (0, 0)$
5) In **Set**, show that the pullback of $f\colon X \to Z$ and $g\colon Y \to Z$ is given by the set of pairs $X \times_Z Y = \{(x, y) | x \in X, y \in Y, f(x) = g(y)\}$.

2. CW Complexes

1) Show that a single cell attachment is a pushout along the attachment map.

2) Let X, Y be CW complexes containing finitely many cells. Show that $X \times Y$ is a CW complex with d-cells given by products of k and $d-k$ cells.

   We use that $D^n \times D^m$ is homeomorphic to D^{n+m} and under this identification the boundary is given by $\partial D^n \times D^m \cup D^n \times \partial D^m$. Then, the "inclusion" of the cell in the product $X \times Y$ is given by the product of the two maps $\phi_1 \times \phi_2 : D^n \times D^m \to X \times Y$. See [H, Theorem A.6] for a solution with no finiteness assumptions.

3) Let X, Y be CW complexes, A a subcomplex of X and $f : A \to Y$ a cellular map. Show that the pushout of f along the inclusion $A \hookrightarrow X$ is a CW complex with the cells of Y and of $X - A$ as cells. You can assume for convenience that all CW complexes have finitely many (and therefore it is enough to consider the case of a single cell attachment)

   See Theorem 4.14 of `https://www.mat.univie.ac.at/~kriegl/Skripten/2011WS.pdf` for a solution with no finiteness assumptions

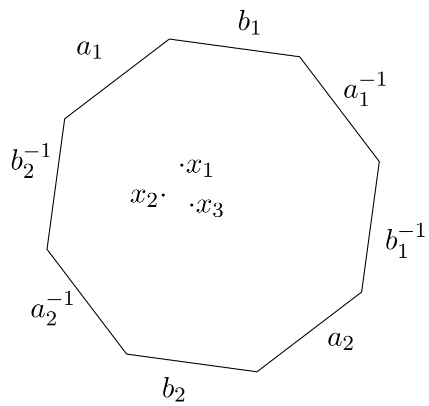
4) Use the cellular structure on $\mathbb{RP}^2$ to show that its fundamental group is a cyclic group with two elements.

3. Fundamental Group of a Punctured Surface

In this exercise, we will compute the fundamental group of oriented surfaces with boundary. Up to homotopy, we can assume that each connected boundary component is a point, reducing the problem to computing the fundamental group of $\Sigma_{g,n} := \Sigma_g - \{x_1, \ldots, x_n\}$, where Σ_g is the oriented surface of genus g ($g \geq 0, n \geq 1$).

We will use the fact that Σ_g admits a CW-complex structure with $2g$ 1-cells induced by the quotient of a polygon with $4g$ edges.

$\Sigma_{2,3} =$



1) Compute $\pi_1(\Sigma_{g,1})$. You may construct a deformation retraction of $\Sigma_{g,1}$ onto a wedge product of circles.

$\Sigma_{g,1}$ deformation retracts to the 1-skeleton of Σ_g, which is a wedge product of $2g$ circles. It follows that $\pi_1(\Sigma_{g,1})$ is the free group on $2g$ generators $(a_1, ..., b_g)$.

2) Compute $\pi_1(\Sigma_{0,n})$. Similarly, you may construct a deformation retraction of $\Sigma_{0,n}$ onto a wedge product of circles.

   $\Sigma_{0,1}$ is a sphere with 1 point removed and therefore is homeomorphic to $\mathbb{R}^2$. Similarly, by picking a any point (say x_n) $\Sigma_{0,n}$ is homeomorphic to $\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}$. Via a homeomorphism, we can assume $x_i = (i, 0) \in \mathbb{R}^2$.

   Let $X \subset \mathbb{R}^2$ be the union of the $n-1$ circles with center x_i and radius $1/2$ (these circles are tangent at $(i + \frac{1}{2}, 0)$ for $i = 1, \ldots n-2$). A good drawing makes it obvious that $\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}$ deformation retracts to X, but its good to convince yourself that you can write down the explicit formulas!

   Finally, by contracting the lower hemisphere of every circle (which is a homotopy equivalence, since it is a contractible sub-CW-complex), we get a wedge of $n-1$ circles. The fundamental group is therefore free on $n-1$ generators.

3) Use the previous results and the van Kampen theorem to compute the fundamental group of $\Sigma_{g,n}$.

   Pick an open disc fully contained in the interior of the 2-cell and containing all the x_i's as U and pick V to be a slight open enlargement of the complement.

   V is homotopy equivalent to $\Sigma_{g,1}$, U is homeomorphic to $\mathbb{R}^2 - \{x_1, \ldots, x_n\}$ and $U \cap V \sim S^1$.

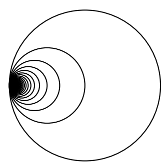
   The inclusion of $U \cap V \sim S^1$ in V sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} a_2 ... b_g^{-1}$, while the inclusion of $U \cap V \sim S^1$ in U sends the generator to $c_1 c_2 \ldots c_n$, where c_i is the generator of the fundamental group of the ith circle as per the previous question.

   Finally, van Kampen tells us that

$$\Sigma_{g,n} = \langle a_1, \ldots a_g, b_1, \ldots, b_g, c_1, \ldots c_n | a_1 b_1 a_1^{-1} b_1^{-1} a_2 ... b_g^{-1} = c_1, \ldots c_n \rangle.$$

4. Hawaiian Earrings

The *Hawaiian rings* is the subspace X of $\mathbb{R}^2$ obtained by the union of a sequence of circles C_n where C_n is the circle centered at $(1/n, 0)$ with radius $1/n$.

$X =$ 

1) Justify, by elementary methods, that X is not homeomorphic to an infinite wedge product of circles.

Let $n \in \mathbb{N}$, and $\widetilde{C}_n$ be the n-th circle, which we identify with $\mathbb{R}/\mathbb{Z}$, in the wedge product $\bigvee_{i \in \mathbb{N}}(0, \widetilde{C}_i)$ (where 0 is the class of 0 in $\mathbb{R}/\mathbb{Z}$).

If we had a homeomorphism $\varphi : X \to \bigvee_{i \in \mathbb{N}}(0, \widetilde{C}_i)$, then it must send $(0,0) \in X$ to the base point of the wedge product (since in each space these are the only points that when removed disconnect the space). By injectivity of φ, there exists $k \in \mathbb{N}$ such that $\varphi(C_n) \subseteq \widetilde{C}_k$ and by surjectivity $\widetilde{C}_k \subseteq \varphi(C_n)$. Thus, $\widetilde{C}_k = \varphi(C_n)$. By reordering the circles in the wedge product, we can assume that $\widetilde{C}_k = \varphi(C_n)$

Setting $x_i = \frac{1}{2} \in \widetilde{C}_i$, it is easy to see that the set $\{x_i\}_{i \in \mathbb{N}}$ has no accumulation point. On the other hand, since $\phi^{-1}(x_i) \in C_i$, the sequence $(\phi^{-1}(x_i))_{i \in \mathbb{N}}$ converges to $(0,0)$, which contradicts that ϕ is a homeomorphism.

2) Prove that X is not *locally simply connected*: the point $(0,0) \in X$ does not have a simply connected neighborhood.

For any i there is a map $f_i \colon S^1 \to X$ mapping S^1 to C_i in the obvious way. It is enough to show that $\pi_1(f_i)$ is injective. This follows by defining a left inverse to f_i and then taking the fundamental groups. This is given by the continuous map $r_i \colon X \to S^1$ sending C_i to S_1 and every other point in X to 0.

3) Construct a surjective homomorphism $\pi_1(X) \to \prod_{n=1}^{\infty} \mathbb{Z}$.

Deduce that the fundamental group of X is not the same as the infinite wedge product of circles.

Taking the product of all $\pi_1(r_i)$ we get a homomorphism $r \colon \pi_1(X) \to \prod_{\mathbb{N}} \mathbb{Z}$.

To show that this is surjective, take a sequence $(a_1, a_2, \dots) \in \prod_{\mathbb{N}} \mathbb{Z}$. We define a loop $\gamma \colon I \to X$ by mapping $[0, \frac{1}{2}]$ to a loop wrapping around C_1 a_1 times, then mapping $[\frac{1}{2}, \frac{1}{3}]$ wrapping around C_2 a_2 times, etc. This is obviously continuous for $t < 1$ and continuity on $t = 1$ is given by the fact that every neighbourhood of $(0,0)$ containing all circles after some point. It is now clear that $\gamma \mapsto (a_1, a_2, ...)$

5. Chain Complexes

1) Show that the kernel, image, and cokernel of a chain complex morphism $f \colon C \to D$ are chain complexes. Show that if $H(\ker f) = 0 = H(\operatorname{coker} f)$, then f induces an isomorphism in homology.

2) Show that the direct sum of chain complexes is a chain complex, and compare $H(A \oplus B)$ with $H(A) \oplus H(B)$.

We use that $\ker(f \oplus g) = \ker f \oplus \ker g$ and the same for the image.

3) Show that the dual of a chain complex $C^\vee = \{\mathrm{Hom}_R(C_i, R)\}_{i\in\mathbb{Z}}$ is a chain complex. Prove that over $\mathbb{Z}$, the dual of the homology of C is not isomorphic to the homology of the dual of C.

The complex given by the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has homology (isomorphic to) $[\mathbb{Z} \mid 0]$, whereas its dual has trivial differential and therefore its homology is itself.

4) Compute the homology of the complex

$$0 \to R \stackrel{\times 2}{\to} R \to 0$$

for $R = \mathbb{Z}$ and R a field (pay attention to the characteristic). [Important to retain from this: The ring we are working with changes a lot the result. Hatcher only works with $\mathbb{Z}$ which does not allow us to see such differences later on.]

Its homology is always $[\ker f \mid \mathrm{coker} f]$, but for a field of characteristic $\neq 2$ both are 0, whereas for $\mathbb{F}_2$ the homology is $[\mathbb{F}_2 \mid \mathbb{F}_2]$. Finally, for $\mathbb{Z}$, the homology is $[0 \mid \mathbb{Z}/2\mathbb{Z}]$

6. Leftovers from class

1) We saw in class that the fundamental group of a finite graph is a free group on finitely many generators. How many generators? The answer should depend only (linearly!) on the number of edges and vertices.

Contracting an edge decreases the number of edges by 1 and the number of vertices by 1, so the quantity $e - v$ is preserved. At the end we are left with a single vertex and n edges, so $n - 1 = e - v$. Each of these edges represents a circle, so the fundamental group is free on $n = e - v + 1$ generators.

2) Memorize the presentation of the dihedral group.