KMAS9AA1 – Algebraic Topology

Exercise Sheet 3

1. Chain Complexes

1) Show that homotopy of morphisms of chain complexes is an equivalence relation. Also show that homotopy equivalence between two chain complexes (i.e., there exist morphisms in both directions such that their compositions are homotopic to the identity) is an equivalence relation. Show that $f \sim f'$ and $g \sim g'$, then if the composites are defined $f \circ f' \sim g \sim g'$.
2) Show that any short exact sequence is isomorphic to a short exact sequence of the form

$$0 \to A \to B \to A/B \to 0$$

where A is an R-module and B is a submodule of A.

2. Deformation Retraction

Let C and A be two chain complexes over R. A *deformation retraction* of C onto A is a triple (r, i, h) with $r \colon C \to A$ and $i \colon A \to C$ chain complex morphisms satisfying $r \circ i = \mathrm{id}_A$, and $h \colon C_\bullet \to C_{\bullet+1}$ is a homotopy between $i \circ r$ and id_C (i.e., $ir - \mathrm{id}_C = h\partial + \partial h$).

1) Show that a morphism of chain complexes that admits a left inverse (a "retract") is injective and induces an injective morphism in homology. Show that the converse is true over a field.
2) Show that over a field, any chain complex retracts by deformation onto its homology.
3) Show that this is not true in general. A counterexample can be found for $R = \mathbb{Z}$ with $C_0 = C_1 = \mathbb{Z}$ and $C_{i \neq 0,1} = 0$.

3. Euler Characteristic

Let C be a chain complex over a field such that for all i, C_i is finite-dimensional, and for $i \gg 0$ or $i \ll 0$, $C_i = 0$. The *Euler characteristic* of C is

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(C_i).$$

Show that the Euler characteristic depends only on the homology of C.

4. Exact Sequence of Chain Complexes

1) Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of chain complexes. Show that if two of the three complexes are acyclic (i.e., $H_i = 0$ for all i), then the third complex is also acyclic.

2) Let A be a complex and B an acyclic subcomplex. Show that the quotient $A \to A/B$ is a quasi-isomorphism. [This can be done by hand or by a long exact sequence argument.]

3) Let A be a complex and B a quasi-isomorphic subcomplex (i.e., the inclusion induces an isomorphism in homology). Show that the quotient A/B is acyclic. [This can be done by hand or by a long exact sequence argument.]

5. Five Lemma

Consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Assume that the rows are exact at B, C, D, B', C', D', and that all vertical maps except the middle one are isomorphisms. Prove that the map $C \to C'$ is also an isomorphism.

6. Connectedness

Let X be a topological space and $(X_\alpha)_{\alpha \in E}$ the family of its path-connected components.

Show that for all $n \in \mathbb{N}$

$$H_n(X) = \bigoplus_{\alpha \in E} H_n(X_\alpha).$$

This exercise often allows us to assume X is connected without loss of generality.

7. **Simplicial Identities**

   Let $\sigma\colon \Delta_p \to X$ be a p-simplex of X and $0 \leq j < i \leq p$. Show that $\partial_j \partial_i(\sigma) = \partial_{i-1}\partial_j(\sigma)$.