KMAS9AA1 – Algebraic Topology

Exercise Sheet 3

1. Chain Complexes

1) Show that homotopy of morphisms of chain complexes is an equivalence relation. Also show that homotopy equivalence between two chain complexes (i.e., there exist morphisms in both directions such that their compositions are homotopic to the identity) is an equivalence relation. Show that $f \sim f'$ and $g \sim g'$, then if the composites are defined $f \circ f' \sim g \sim g'$.

2) Show that any short exact sequence is isomorphic to a short exact sequence of the form

$$0 \to B \to A \to A/B \to 0$$

where A is an R-module and B is a submodule of A.

If we have a SES $0 \to B \to A \xrightarrow{p} C \to 0$, we can define a map $f : C \to A/B$ by defining $f(c) = [a]$,where $p(a) = c$. We check that this is a well defined linear map. We can conclude by either using the 5 Lemma to argue that this is an isomorphism or showing that there is a well defined map $A/B \to C$ which is a both sided inverse to f.

2. Deformation Retraction

Let C and A be two chain complexes over R. A *deformation retraction* of C onto A is a triple (r, i, h) with $r\colon C \to A$ and $i\colon A \to C$ chain complex morphisms satisfying $r \circ i = \mathrm{id}_A$, and $h\colon C_\bullet \to C_{\bullet+1}$ is a homotopy between $i \circ r$ and id_C (i.e., $ir - \mathrm{id}_C = h\partial + \partial h$).

2) Show that over a field, any chain complex retracts by deformation onto its homology.

Let us write $C_i = \ker \partial_i \oplus I_i$, where I_i is an arbitrary choice of complement of $\ker \partial_i$ (which exists since we are working over a field). Let us further decompose $\ker \partial_i = \mathrm{Im}(\partial_{i+1}) \oplus H_i$, where H_i is a chosen complement. Notice that that $H_i \cong H_i(C)$. Under this decomposition, we can write for all degrees $C = H \oplus \mathrm{Im}(\partial) \oplus I$, where on the right hand side the only non-trivial part of the differential is $\partial I \to \mathrm{Im}(\partial)$ and this restriction is an isomorphism.

Now, one can check that there is a deformation rectraction given by projecting (resp. including) C to H (resp. H in C) and the homotopy can be taken to be the inverse of $\partial I \to \mathrm{Im}(\partial)$.

1) Show that a morphism of chain complexes that admits a left inverse (a "retract") is injective and induces an injective morphism in homology. Show that the converse is true over a field.

   For one implication just take the homology of $r \circ i = \mathrm{id}_A$.

   For the other implication, take $f\colon C \to D$. We do the decomposition for $C = H^C \oplus \mathrm{Im}(\partial^C) \oplus I^C$ of the previous exercise and then similarly for D, but in a way that is compatible with f, namely we pick H^D such that $H(f)(H^C) \subset H^D$. Now we can define $g\colon D \to C$ component by component. Using injectivity in homology we define g in H^D and the injectivity of f helps us defining g in the other components.

3) Show that this is not true in general. A counterexample can be found for $R = \mathbb{Z}$ with $C_0 = C_1 = \mathbb{Z}$ and $C_{i \neq 0,1} = 0$.

   Taking $C = [\mathbb{Z} \xrightarrow{2} \mathbb{Z}]$, there is no map from $H(C) = \mathbb{Z}/2\mathbb{Z}$ to C besides the zero map.

3. Euler Characteristic

Let C be a chain complex over a field such that for all i, C_i is finite-dimensional, and for $i \gg 0$ or $i \ll 0$, $C_i = 0$. The *Euler characteristic* of C is

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(C_i).$$

Show that the Euler characteristic depends only on the homology of C.

This follows from the rank nullity theorem: $\dim \ker \partial_i + \dim \mathrm{Im} \partial_i = \dim C_i$ plus the fact that $\dim H_i(C) = \dim \ker \partial_i - \dim \mathrm{Im} \partial_{i+1}$

4. Exact Sequence of Chain Complexes

1) Let

$$0 \to A \to B \to C \to 0$$

   be a short exact sequence of chain complexes. Show that if two of the three complexes are acyclic (i.e., $H_i = 0$ for all i), then the third complex is also acyclic.

   Take the associated long exact sequence and get $0 \to H_i \to 0$

2) Let A be a complex and B an acyclic subcomplex. Show that the quotient $A \to A/B$ is a quasi-isomorphism. [This can be done by hand or by a long exact sequence argument.]

   LES of $0 \to B \to A \to A/B \to 0$.

3) Let A be a complex and B a quasi-isomorphic subcomplex (i.e., the inclusion induces an isomorphism in homology). Show that the quotient A/B is acyclic. [This can be done by hand or by a long exact sequence argument.]

   LES of $0 \to B \to A \to A/B \to 0$.

5. Five Lemma

Consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Assume that the rows are exact at B, C, D, B', C', D', and that all vertical maps except the middle one are isomorphisms. Prove that the map $C \to C'$ is also an isomorphism.

`https://en.wikipedia.org/wiki/Five_lemma`

6. Connectedness

Let X be a topological space and $(X_\alpha)_{\alpha \in E}$ the family of its path-connected components.

Show that for all $n \in \mathbb{N}$

$$H_n(X) = \bigoplus_{\alpha \in E} H_n(X_\alpha).$$

This exercise often allows us to assume X is connected without loss of generality.

For complexes $H(\bigoplus_\alpha A_\alpha) = \bigoplus_\alpha H(A_\alpha)$ since both the kernel and the image decompose. It is clear that $C(X)$ decomposes as $\bigoplus_\alpha C(X_\alpha)$ as complexes.

7. Simplicial Identities

Let $\sigma\colon \Delta_p \to X$ be a p-simplex of X and $0 \le j < i \le p$. Show that $\partial_j \partial_i(\sigma) = \partial_{i-1}\partial_j(\sigma)$.