KMAS9AA1 – Algebraic Topology

Exercise Sheet 4

1) Homological algebra

1. Let
$$0 \to A \to C \to F \to 0$$
be a short exact sequence and assume that F is a free R-module. Show that $C \cong A \oplus F$.
2. Find a counter example where F is non-free. This can be done over $R = \mathbb{Z}$ with $A = F = \mathbb{Z}/2\mathbb{Z}$. Check that the counter example is no longer a counter example if R is instead the ring $\mathbb{Z}/2\mathbb{Z}$.

2) Mayer–Vietoris

Let $R = \mathbb{Z}$.

1) Suppose that a topological space is written as a union of two open subspaces $X = U \cup V$ and consider the associated Mayer–Vietoris long exact sequence. Assuming that $U \cap V$ is path connected, use the explicit construction of the connecting morphism $\delta \colon H_1(X) \to H_0(U \cap V)$ to show that is the zero map.
2) Let (M, m) and (N, n) be two pointed spaces with open neighbourhoods deformation retracting to the respective points. Show that $H_d(M \vee N, *) = H_d(M, n) \oplus H_d(N, n)$.

3) Relative Homology

Let (X, A) be a topological pair.

1) Show that $H_0(X, A) = 0$ if and only if A intersects every path-connected component of X.
2) Let $Z_p(X, A) = \{\sigma \in C_p(X) \mid \partial\sigma \in C_{p-1}(A)\}$. Show that there is an isomorphism of modules
$$H_p(X, A) \cong \frac{Z_p(X, A)}{B_p(X) + C_p(A)}.$$

3) Provide an alternative proof of 1) using 2).
4) Show that $H_1(X, A) = 0$ if and only if the map $H_1(A) \to H_1(X)$ is surjective and every path-connected component of X contains at most one path-connected component of A.

4) Retract

1) Show that if X is a topological space and $A \subset X$ is a retract of X, then for all n, the map induced by inclusion $H_n(A) \to H_n(X)$ is injective.

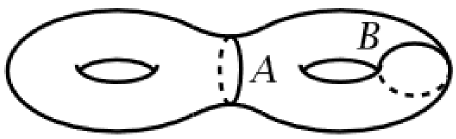
   Does this remain true if A is just a subspace of X?
2) Show that if A is a deformation retract of X, then $H_n(X, A) = 0$ for all n.

5) Surjective in Homology

Show that a surjective morphism $f: A \to B$ of chain complexes is not necessarily surjective in homology, but this is the case if $\ker f$ is acyclic.

6) Homological Calculations

1) Use the Mayer–Vietoris sequence to compute the homology of the sphere S^n.
2) Compute the homology of $S^n \times S^m$.
3) Show that the homology of a wedge of spheres $X = \bigvee_{i=1}^n S^k$ is $R^{\oplus n}$ in degree k and 0 in degrees different from $\{0, k\}$.
4) Compute the homology of Σ_2 (the closed orientable surface of genus 2).
5) Compute the homology $H(\Sigma_2, A)$ and $H(\Sigma_2, B)$, where A and B are the following circles[1]:



7) Excision Fails

Find $A \subset \mathbb{R}^n - 0$ such that

$$H_\bullet(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \neq H_\bullet(\mathbb{R}^n - A, (\mathbb{R}^n - \{0\}) - A).$$

[1]Drawing from [H, Exercise 2.1.17]

8) Cone and Suspension of a Topological Space

Let X be a topological space. The *suspension* of X is the topological space

$$SX = X \times [-1,1]/(x,-1) \sim (x',-1);\ (x,1) \sim (x',1)\ \forall x,y \in X.$$

The *cone* of X is the subspace of SX

$$CX = X \times [0,1]/(x,1) \sim (x',1).$$

1) Compute $H(CX)$.
2) Show that $H_{n+1}(SX) \simeq H_n(X)$ for $n \geq 1$ and that if X is path-connected, $H_1(SX) = 0$.

9) Homology of Manifolds with Points Removed

Let M be a topological manifold of dimension n, $* \in M$, and R a field. Compare $\dim H_d(M)$ with $H_d(M - *)$ for $d \neq n, n-1$.

10) Brouwer Fixed-Point Theorem

Show that the boundary of the disk ∂D^n is not a deformation retract of D^n. Deduce that every continuous map $D^n \to D^n$ has a fixed point.

Hint: Adapt the proof for the case $n = 2$.

11) Homology is not a Complete Invariant

(a) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology (for any ring R).

(b) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are not homotopy equivalent.