KMAS9AA1 – Algebraic Topology

Exercise Sheet 4

1) Homological algebra

1. Let
$$0 \to A \to C \to F \to 0$$
be a short exact sequence and assume that F is a free R-module. Show that $C \cong A \oplus F$.

   Pick a section of $C \to F$.

2. Find a counter example where F is non-free. This can be done over $R = \mathbb{Z}$ with $A = F = \mathbb{Z}/2\mathbb{Z}$. Check that the counter example is no longer a counter example if R is instead the ring $\mathbb{Z}/2\mathbb{Z}$.

   $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Notice that the sequence does not split seeing everything as $\mathbb{Z}$-modules (there is no non-trivial map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$).

   The second part of the question is deliberately vague, to make you think of how to fix the problem. While $\mathbb{Z}/2\mathbb{Z}$ is a free $\mathbb{Z}/2\mathbb{Z}$-module, the same sequence would split in the world of $\mathbb{Z}/2\mathbb{Z}$, but there is no $\mathbb{Z}/2\mathbb{Z}$-module structure on $\mathbb{Z}$.

   Here is another example: As we saw in class, there are two possible ways to fill out the question mark in the category of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z}/2\mathbb{Z} \to ? \to \mathbb{Z}/2\mathbb{Z} \to 0$$

   On the other hand, by the preceding question, over $\mathbb{Z}/2\mathbb{Z}$ there is only one way to fill the question mark, which is with $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed, $\mathbb{Z}/4\mathbb{Z}$ is not a $\mathbb{Z}/2\mathbb{Z}$-module

2) Mayer–Vietoris

Let $R = \mathbb{Z}$.

1) Suppose that a topological space is written as a union of two open subspaces $X = U \cup V$ and consider the associated Mayer–Vietoris long exact sequence. Assuming that $U \cap V$ is path connected, use

the explicit construction of the connecting morphism $\delta\colon H_1(X) \to H_0(U\cap V)$ to show that is the zero map.

This can be easily shown with the exactness of the LES, but let us use the explicit formula of the connecting morphism. The result actually holds over a general ring R.

Let us consider $[c] \in H_1(X)$, $c = \sum_{i=1}^k r_i\sigma_i$. First, we claim that c is a sum of elements of the form $r(\sum_j \gamma_j)$, where γ_j are 1-simplices forming a loop, i.e. $\gamma_j(1) = \gamma_{j+1}(0)$. Indeed, we start by building $r(\sum_j \gamma_j)$ by setting $r = r_1$ and $\gamma_1 = \sigma_1$. If $r\gamma_1$ is already closed, we're done by induction, as k is lowered. Otherwise, if $r\gamma_1$ is not closed (i.e. if $\gamma_1(1) \neq \gamma_1(0)$), since c itself is closed, these boundaries must cancel out in ∂c. Therefore there must be an element γ_2 among the other σ_i such that $\gamma_2(0) = \gamma_1(1)$. We add it to our recursive construction to get $r(\gamma_1+\gamma_2)$ (notice that the coefficient of γ_2 in c might not be r, but this does not matter). Since there is only a finite number of σ_i, this procedure must finish and it must finish with the last γ_{final} closing the loop, i.e. $\gamma_{\text{final}}(1) = \gamma_1(0)$, otherwise c would not be a cycle.

We now proceed by removing $r(\sum_j \gamma_j)$ from c, which gives us a sum of at most $k-1$ elements, so recursively we write c as a sum of elements of the form $r(\sum_j \gamma_j)$.

To prove the result, it therefore suffices to show that the image of such a chain $\sum_j \gamma_j$ such that $\gamma_j(1) = \gamma_{j+1(0)}$ under the connecting morphism δ is zero. By further decomposing the loop (using the compactness of I), we can assume that the image of each γ_j is fully contained in either U or V.

Let us write $c = \underbrace{\sum_{u\in\mathcal{U}} \gamma_u}_{c_U} + \underbrace{\sum_{v\in\mathcal{V}} \gamma_v}_{c_V} \in C^{\{U,V\}}(X) = C(U)+C(V)$, where $\mathcal{U}$ and $\mathcal{V}$ are chosen such that $\gamma_u(I) \subset U$ and $\gamma_v(I)\subset V$. Notice that there is a choice due to 1-simplices contained in the intersection. We can choose to send them all to the $\mathcal{U}$ summand.

Following the construction of the connecting morphism, pick as a pre-image of c in $C(U)\oplus C(V)$ the chains (c_U, c_V). We take its boundary to get a collection of 0-simplices $(\partial c_U, \partial c_V) \in C_0(U)\oplus C_0(V)$. Notice that ∂c_U only contains points living in $U\cap V$, as the points living in $U - U\cap V$ must cancel out in ∂c and they can't cancel out with the terms in c_V. Similarly, ∂c_V only contains points living in $U\cap V$ and indeed they are the exact same points, but with the opposite signs.

Following the construction of the connecting morphism we now take ∂c_U as the pre-image in $C_0(U\cap V)$. We now need to show that $[\partial c_U] = 0 \in H_0(U\cap V)$. But all points showing up in ∂c_U appear in pairs and each pair contributes with a $+$ point and a $-$ point. Since

$U \cap V$ is path connected, all points represent the same homology class, so they all cancel out.

2) Let (M, m) and (N, n) be two pointed spaces with open neighbourhoods deformation retracting to the respective points. Show that $H_d(M \vee N, *) = H_d(M, n) \oplus H_d(N, n)$.

   Use the Mayer–Vietoris sequence, as for the similar result for π_1.

3) Relative Homology

Let (X, A) be a topological pair.

1) Show that $H_0(X, A) = 0$ if and only if A intersects every path-connected component of X.
2) Let $Z_p(X, A) = \{\sigma \in C_p(X) \mid \partial\sigma \in C_{p-1}(A)\}$. Show that there is an isomorphism of modules

$$H_p(X, A) \cong \frac{Z_p(X, A)}{B_p(X) + C_p(A)}.$$

3) Provide an alternative proof of 1) using 2).
4) Show that $H_1(X, A) = 0$ if and only if the map $H_1(A) \to H_1(X)$ is surjective and every path-connected component of X contains at most one path-connected component of A.

4) Retract

1) Show that if X is a topological space and $A \subset X$ is a retract of X, then for all n, the map induced by inclusion $H_n(A) \to H_n(X)$ is injective.

   Does this remain true if A is just a subspace of X?

   For the first part just use that $A \hookrightarrow\to X \to A$ is the identity and take the homology functor.

   For the second one, consider $S^1 \subset \mathbb{R}^2$.
2) Show that if A is a deformation retract of X, then $H_n(X, A) = 0$ for all n.

   We can observe that a deformation retract gives a homotopy from (X, A) to (A, A) and conclude using that $H_n(A, A) = 0$. Alternatively, the long exact sequence of a pair also works.

5) Surjective in Homology

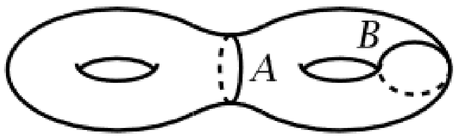
Show that a surjective morphism $f: A \to B$ of chain complexes is not necessarily surjective in homology, but this is the case if $\ker f$ is acyclic.

The projection $[\mathbb{Z} \to \mathbb{Z}] \to \mathbb{Z}$ is surjective but its homology is $0 \to \mathbb{Z}$.

The other statement follows from the long exact sequence of $0 \to \ker f \to A \to B \to 0$.

6) Homological Calculations

1) Use the Mayer–Vietoris sequence to compute the homology of the sphere S^n.
2) Compute the homology of $S^n \times S^m$.
   Use Mayer–Vietoris removing one point from each factor.
3) Show that the homology of a wedge of spheres $X = \bigvee_{i=1}^{n} S^k$ is $R^{\oplus n}$ in degree k and 0 in degrees different from $\{0, k\}$.
4) Compute the homology of Σ_2 (the closed orientable surface of genus 2).
   Can use Mayer–Vietoris splitting Σ_2 into two opens that are each homotopy equivalent to the torus minus one point, which in turn is homotopy equivalent to a wedge of two circles.
   Alternatively, cellular homology works.
5) Compute the homology $H(\Sigma_2, A)$ and $H(\Sigma_2, B)$, where A and B are the following circles[1]:



[H, Exercise 2.1.17]

7) Excision Fails

Find $A \subset \mathbb{R}^n - 0$ such that

$$H_\bullet(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \neq H_\bullet(\mathbb{R}^n - A, (\mathbb{R}^n - \{0\}) - A).$$

Take $A = \mathbb{R}^n - \{0\}$

8) Cone and Suspension of a Topological Space

Let X be a topological space. The *suspension* of X is the topological space

$$SX = X \times [-1, 1]/(x, -1) \sim (x', -1);\ (x, 1) \sim (x', 1)\ \forall x, y \in X.$$

The *cone* of X is the subspace of SX

$$CX = X \times [0, 1]/(x, 1) \sim (x', 1).$$

[1]Drawing from [H, Exercise 2.1.17]

1) Compute $H(CX)$.

   The cone is contractible.

2) Show that $H_{n+1}(SX) \simeq H_n(X)$ for $n \geq 1$ and that if X is path-connected, $H_1(SX) = 0$.

   Mayer–Vietoris splitting the suspension into two cones.

9) Homology of Manifolds with Points Removed

Let M be a topological manifold of dimension n, $* \in M$, and R a field. Compare $\dim H_d(M)$ with $H_d(M - *)$ for $d \neq n, n-1$.

Mayer–Vietoris on M, with opens $M - *$ and a small disc around the point $D^n \ni *$. Notice that $M - D^n \sim M - *$.

10) Brouwer Fixed-Point Theorem

Show that the boundary of the disk ∂D^n is not a deformation retract of D^n. Deduce that every continuous map $D^n \to D^n$ has a fixed point.

Hint: Adapt the proof for the case $n = 2$.

Hint.

11) Homology is not a Complete Invariant

(a) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology (for any ring R).

(b) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are not homotopy equivalent.

   Non-isomorphic fundamental groups.