KMAS9AA1 – Algebraic Topology

Exercise Sheet 5

1. Degree of a map $S^n \to S^n$ See [Hatcher, Beginning of Section 2.2]

Given a continuous map $f\colon S^n \to S^n$, we consider the induced map $H_n(f)\colon \mathbb{Z} \to \mathbb{Z}$. The *degree* of f, $\deg f$ is defined to be $H_n(f)(1)$. In other words, $H_n(f)$ is multiplication by $\deg f$. In this exercise we will prove some properties of the degree of a map.

1) Show that if f is not surjective, then $\deg f = 0$.
2) Given $f'\colon S^n \to S^n$, show that $\deg f \circ f' = \deg f \cdot \deg f'$. Conclude that if f is a homotopy equivalence, then $\deg f = \pm 1$.
3) Let $r^1\colon S^1 \to S^1$ be the reflection along the vertical axis, i.e. $r^1(x_0, x_1) = (-x_0, x_1)$. Show that $\deg r^1 = -1$.
4) Let $r\colon S^n \to S^n$ be a reflection along some hyperplane. Show that $\deg r = -1$.

   Hint: By change of coordinates we can suppose $r = r^n(x_0, \dots, x_n) = (-x_0, \dots, x_n)$. One can use Mayer–Vietoris to show that $\deg r^n = \deg r^{n-1}$.
5) Show that the degree of the antipodal map $x \mapsto -x$ is $(-1)^{n+1}$.
6) Suppose that f has no fixed points. Construct a homotopy between f and the antipodal map and conclude that $\deg f = (-1)^{n+1}$.

   Hint: A formula such as $(1-t)f(x) - tx$ almost does the trick, but this does not land in the S^n...

2. Actions on spheres

1) Let n be an even number. Suppose that a group G acts freely on the sphere S^n. Use the previous exercise to deduce that either $G = \{e\}$ or $G = \mathbb{Z}/2\mathbb{Z}$.

   See [Hatcher, 2.29]
2) Find one infinite group that acts freely on all spheres of odd dimension.

Identify $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, |(z_1, \dots, z_n)| = 1\}$. The circle acts freely by rotating every coordinate: $e^{i\pi\theta} \cdot (z_1, \dots, z_n) = (e^{i\pi\theta} z_1, \dots, e^{i\pi\theta} z_n)$.

3. Homology of projective space

Recall that $\mathbb{RP}^n$ has a cellular structure with one cell in each dimension up to n and its k skeleton is $\mathbb{RP}^k$.

1) Show that there exists a commutative diagram as follows

$$
\begin{array}{ccccc}
H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) & \xrightarrow{\delta} & H_{n-1}(\mathbb{RP}^{n-1}) & \longrightarrow & H_{n-1}(\mathbb{RP}^{n-1}, \mathbb{RP}^{n-2}) \\
\cong\uparrow & & \uparrow & \searrow & \downarrow\cong \\
H_n(D^n, S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(S^{n-1}) & \xrightarrow{f_*} & H_{n-1}(D^{n-1}/S^{n-2}, *)
\end{array}
$$

2) Check that f_* is the map induced by the composite of the commuting diagram

$$
\begin{array}{ccccc}
S^{n-1} & \xrightarrow{\text{quotient}} & \mathbb{RP}^{n-1} & \xrightarrow{\text{pinch } \mathbb{RP}^2} & D^{n-1}/S^{n-2} \cong S^{n-1} \\
& \searrow & & \nearrow & \\
& & S^{n-1}/S^{n-2} \cong S^{n-1} \vee S^{n-1} & &
\end{array}
$$

where one of the maps from the wedge is the identity and the other is the antipodal map.

3) Conclude that the cellular complex of $\mathbb{RP}^n$ is

$$
\begin{array}{ccccccc}
\text{deg} & -1 & 0 & 1 & \cdots & n & n+1 \cdots \\
 & 0 \longleftarrow & R \xleftarrow[0]{} & R \xleftarrow[\times 2]{} & \cdots \xleftarrow[2 \text{ or } 0]{} & R \longleftarrow & 0 \quad \cdots
\end{array}
$$

4) Compute explicitly (for both parities of n) and check that the homology gives very different results for

   a. $R = \mathbb{Z}$.

   b. A field of characteristic 2 (or in fact any ring in which $2 = 0$).

   c. A field of characteristic $\neq 2$ (or in fact any ring in which the endomorphism $\times 2$ is invertible.)

   d. $R = \mathbb{Z}/8\mathbb{Z}$.

   See https://topospaces.subwiki.org/wiki/Homology_of_real_projective_space