KMAS9AA1 – Algebraic Topology

Exercise Sheet 6

1. Tor

1) Check carefully that the fundamental theorem of homological algebra implies that for any two resolutions of the R-module M, $F_\bullet \to M$ and $F'_\bullet \to M$, we have that $H_n(F_\bullet \otimes B)$ is canonically isomorphic to $H_n(F'_\bullet \otimes B)$.
2) Show that $\mathrm{Tor}_i^r(A \oplus B, C) = \mathrm{Tor}(A, C) \oplus \mathrm{Tor}(B, C)$.

   Recall that given an exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence of Tor functors given by tensoring ths short exact sequence with $F_\bullet$, a resolution of M. From now on, assume that R is a PID.
3) Take a free resolution $E_\bullet \to C$ such that $E_i = 0$ for $i \geq 2$. Consider the long exact sequence associated to

$$0 \to E_1 \otimes F_\bullet \to E_0 \otimes F_\bullet \to C \otimes F_\bullet \to 0$$

   to conclude that $\mathrm{Tor}_1(C, M) = \mathrm{Tor}_1(M, C)$.
4) Assume that $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$ are finitely generated. Show that for any prime p, $H_n(X; \mathbb{Z}/p\mathbb{Z})$ consists of:
   i. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}$ summand of $H_n(X; \mathbb{Z})$,
   ii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X; \mathbb{Z})$,
   iii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X; \mathbb{Z})$,
5) Use the universal coefficient theorem to show that if $H_*(X; \mathbb{Z})$ is finitely generated, so the Euler characteristic

$$\chi(X) = \sum_n (-1)^n \mathrm{rank} H_n(X; \mathbb{Z})$$

   is defined, then for any coefficient field $\mathbb{F}$ we have $\chi(X) = \sum_n (-1)^n \dim H_n(X; \mathbb{F})$.

2. Torsion-free I claimed in class that while $\mathbb{Q}$ is not free, it is torsion-free and therefore $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, C) = 0, \forall C \in R - \mathrm{Mod}$. Let us show this.

Let G be an abelian group.

1) Show that any element of $G \otimes \mathbb{Q}$ is of the form $g \otimes \frac{1}{n}$.
2) Show that if G is a torsion group, then $\mathbb{Q} \otimes G = 0$.
3) Show that if G is torsion free, then $g \otimes \frac{1}{n} = g' \otimes \frac{1}{n'}$ is equivalent to $gn' = ng'$.
4) Take a free resolution $F_1 \to F_0$ of G. Show that $F_1 \otimes \mathbb{Q} \to F_0 \otimes \mathbb{Q}$ is injective and conclude that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, G) = 0$.

3. Ext

1) Show if $A \to B \to C \to 0$ is exact, then $\mathrm{Hom}(A, N) \leftarrow \mathrm{Hom}(B, N) \leftarrow \mathrm{Hom}(C, N) \leftarrow 0$. [This is what is used to conclude that $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N)$!]
2) Show that $\mathrm{Ext}_R^i(A \oplus B, N) = \mathrm{Ext}_R^i(A, N) \oplus \mathrm{Ext}_R^i(B, N)$ and $\mathrm{Ext}_R^i(R^7, N) = 0$.
3) Show that $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, N) = N/nN$.
4) Show that $\mathrm{Ext}_R^1(M, -)$ is a covariant functor and that $\mathrm{Ext}^1(-, N)$ is a contravariant functor.
5) Show that $\mathrm{Ext}_{\mathbb{Z}/4\mathbb{Z}}^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

4. Cup product

1) Show that $H^\bullet(X \sqcup Y; R)$ and $H^\bullet(X; R) \oplus H^\bullet(Y; R)$ are isomorphic as graded commutative R-algebras. Deduce a similar statement for the wedge product (assuming that the basepoints are deformation retracts of open neighbourhoods).
2) Let X be a CW complex with one 0-cell, one 5-cell, one 7-cell and one 10-cell. What is the cohomology ring structure of X with coefficients in $\mathbb{Q}$?

   The cohomology of the torus with coefficients in $\mathbb{F}_2$ is spanned by **degree 0:** 1, **degree 1:** α, β and **degree 2:** γ.
3) Use the same strategy that we used in class for $\mathbb{RP}^2$ to show that $\alpha \cup \beta = \gamma$.
4) Show that $\alpha \cup \alpha = 0$.