KMAS9AA1 – Algebraic Topology

Exercise Sheet 6

1. Tor See [H,Prop 3A.5]. For the first two questions the base ring is irrelevant.

1) Check carefully that the fundamental theorem of homological algebra implies that for any two resolutions of the R-module M, $F_\bullet \to M$ and $F'_\bullet \to M$, we have that $H_n(F_\bullet \otimes B)$ is canonically isomorphic to $H_n(F'_\bullet \otimes B)$.
2) Show that $\mathrm{Tor}_i^r(A \oplus B, C) = \mathrm{Tor}(A, C) \oplus \mathrm{Tor}(B, C)$.

   Recall that given an exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence of Tor functors given by tensoring ths short exact sequence with $F_\bullet$, a resolution of M. From now on, assume that R is a PID.
3) Take a free resolution $E_\bullet \to C$ such that $E_i = 0$ for $i \geq 2$. Consider the long exact sequence associated to

$$0 \to E_1 \otimes F_\bullet \to E_0 \otimes F_\bullet \to C \otimes F_\bullet \to 0$$

   to conclude that $\mathrm{Tor}_1(C, M) = \mathrm{Tor}_1(M, C)$.
4) Assume that $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated. Show that for any prime p, $H_n(X;\mathbb{Z}/p\mathbb{Z})$ consists of:
   i. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   ii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   iii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$,
5) See [H,3A.1.1] Use the universal coefficient theorem to show that if $H_\bullet(X;\mathbb{Z})$ is finitely generated, so the Euler characteristic

$$\chi(X) = \sum_n (-1)^n \mathrm{rank} H_n(X;\mathbb{Z})$$

   is defined, then for any coefficient field $\mathbb{F}$ we have $\chi(X) = \sum_n (-1)^n \dim H_n(X;\mathbb{F})$.

2. Torsion-free I claimed in class that while $\mathbb{Q}$ is not free, it is torsion-free and therefore $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, C) = 0, \forall C \in R-\mathrm{Mod}$. Let us show this.

Let G be an abelian group.

1) Show that any element of $G \otimes \mathbb{Q}$ is of the form $g \otimes \frac{1}{n}$.
Any element of the tensor product is a finite sum of pure tensors, so it suffices to show that any sum of two pure tensors can be written a single pure tensor. Indeed

$$g\otimes\frac{a}{b}+g'\otimes\frac{a'}{b'} = ag\otimes\frac{1}{b}+a'g'\otimes\frac{1}{b'} = ab'g\otimes\frac{1}{bb'}+a'bg'\otimes\frac{1}{bb'} = (ab'g+a'bg')\otimes\frac{1}{bb'}.$$

2) Show that if G is a torsion group, then $\mathbb{Q} \otimes G = 0$.
Suppose $mg = 0$, for $m \in \mathbb{Z}$ and $g \in G$. Then

$$b \otimes g = m\frac{b}{m} \otimes g = \frac{b}{m} \otimes mg = 0.$$

3) Show that if G is torsion free, then $g \otimes \frac{1}{n} = g' \otimes \frac{1}{n'}$ is equivalent to $gn' = ng'$.
$g \otimes \frac{1}{n} = g' \otimes \frac{1}{n'} \Rightarrow (n'g - g'n) \otimes \frac{1}{nn'} = 0$. We're done if we show that $x \otimes 1/k = 0 \Rightarrow x = 0$.
Let us more generally give a complete characterisation of $G \otimes \mathbb{Q}$. Let us define the *rationalization* of G to be E, the set of formal symbols g/n, where $g \in G$ and $n \in \mathbb{Z} - 0$. We define the equivalence relation $g/n = g'/n'$ if $n'g = ng'$ (this is only an equivalence relation since G is torsion free!). There is an obvious addition that can be defined making E into an abelian group. There is a bilinear map $G \times \mathbb{Q} \to E$, sending $(g, a/b)$ to ag/b, which thus induces a map $f\colon G \otimes \mathbb{Q} \to E$. We already see from this that if $x \neq 0$, then $f(x \otimes 1/k) \neq 0$, which concludes the proof. But furthermore, we can see that f is an isomorphism, by defining the inverse $f^{-1}(g/n) = g \otimes \frac{1}{n}$ and checking that it is indeed well defined and an inverse.

4) Take a free resolution $F_1 \to F_0$ of G. Show that $F_1 \otimes \mathbb{Q} \to F_0 \otimes \mathbb{Q}$ is injective and conclude that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, G) = 0$.
The question as stated is mildly incorrect[1]. We need to take a free resolution such that the map $F_1 \to F_0$ is injective. We know this exists since we can take F_0 to be generated by G as a set, and F_1 to

[1] In fact, no. Even if we weren't over a PID, if there is a two step resolution, then we know that $\partial_1\colon F_1 \to F_0$ must be injective. This is the case since $0 = H_1(F_\bullet) = \ker\partial_1/\mathrm{Im}\partial_2 = \ker\partial_1$. But being over a PID guarantees that a two step resolution always exists.

be the kernel of the morphism $F_0 \to G$. Then, by the question above $F_1 \otimes \mathbb{Q} \to F_0 \otimes \mathbb{Q}$ has trivial kernel. But in fact this kernel is by definition $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, G)$, using $F_\bullet$ as a resolution.

3. Ext

1) Show if $A \to B \to C \to 0$ is exact, then $\mathrm{Hom}(A, N) \leftarrow \mathrm{Hom}(B, N) \leftarrow \mathrm{Hom}(C, N) \leftarrow 0$ is exact.
   [This is what is used to conclude that $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$!]
2) Show that $\mathrm{Ext}^i_R(A \oplus B, N) = \mathrm{Ext}^i_R(A, N) \oplus \mathrm{Ext}^i_R(B, N)$ and $\mathrm{Ext}^i_R(R^7, N) = 0$.
   Not a typo. It's really just that R^7 is free.
3) Show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) = N/nN$.
4) Show that $\mathrm{Ext}^1_R(M, -)$ is a covariant functor and that $\mathrm{Ext}^1(-, N)$ is a contravariant functor.
5) Show that $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all n.
   There is a $(\mathbb{Z}/4\mathbb{Z})$-free resolution of $\mathbb{Z}/2\mathbb{Z}$ given by

$$... \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}$$

   which we can use to compute the Ext functors. Notice that in particular we deduce that there is no finite free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$, otherwise $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n >> 0$.

4. Cup product

1) Show that $H^\bullet(X \sqcup Y; R)$ and $H^\bullet(X; R) \oplus H^\bullet(Y; R)$ are isomorphic as graded commutative R-algebras. Deduce a similar statement for the wedge product (assuming that the basepoints are deformation retracts of open neighbourhoods).
2) Let X be a CW complex with one 0-cell, one 5-cell, one 7-cell and one 10-cell. What is the cohomology ring structure of X with coefficients in $\mathbb{Q}$?

   The cohomology of the torus with coefficients in $\mathbb{F}_2$ is spanned by **degree 0:** 1, **degree 1:** α, β and **degree 2:** γ.
3) Use the same strategy that we used in class for $\mathbb{RP}^2$ to show that $\alpha \cup \beta = \gamma$.
4) Show that $\alpha \cup \alpha = 0$.