KMAS9AA1 – Algebraic Topology

Exercise Sheet 7

1. **Eckmann–Hilton argument** https://en.wikipedia.org/wiki/Eckmann%E2%80%93Hilton_argument

   The way I presented the group structure on higher homotopy groups, it seems that the first coordinate plays a privileged role, when compared to the other ones. In fact, with an argument not so different from the proof of commutativity, one can show that the product defined similarly but with other coordinates ends up giving the same result. Here, we present a purely algebraic proof of a much more general result.

   1) Let $\times$ and $\bullet$ be two unital binary operations on a set X. Suppose

   $$(a \times b) \bullet (c \times d) = (a \bullet c) \times (b \bullet d)$$

   for all $a, b, c, d \in X$. Then $\times$ and $\bullet$ are in fact the same operation, and are commutative and associative.

   2) Consider the usual product on higher homotopy groups

   $$(f \times g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \dots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

   and define as well

   $$(f \bullet g)(t_1, \dots, t_n) = \begin{cases} f(t_1, 2t_2, \dots, t_n) & t_1 \in [0, 1/2] \\ g(t_1, 2t_2 - 1, \dots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

   Show that these operations satisfy the conditions from the previous exercise.

2. **Homotopy equivalence**

   Adapt the proof that π_1 is invariant under homotopy equivalence to show the same for π_n. To be precise, show that if $f\colon X \to Y$ is a homotopy equivalence, $f\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.

3. Relative homotopy groups

1) Check that the relative homotopy groups $\pi_n(X, A, x_0)$ can have equivalently been defined as homotopy classes of maps $(I^n, \partial I_n, J^{n-1}) \to (X, A, x_0)$, where J^{n-1} is the union of all but one face of I^n, i.e. $J^{n-1} = \delta I^n - I^{n-1}$. What is the group structure?

   I^n and ∂I^n are homeomorphic to the disc and sphere respectively. For the rest, we use as always that a map $S^n \cong \partial I^n / J^{n-1} \to A$ is equivalent to a map $\partial I^n \to A$ sending J^{n-1} to a point.

   The group structure is defined exactly the same way. However, notice that there is a problem when $n = 1$. In fact, the definition we gave in class also does not define a group structure when $n = 1$, I invite you to try to see why.

2) Show that the end of the long exact sequence of relative homotopy groups is indeed exact:

$$\pi_1(X, x_0) \to \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \to \pi_0(X, x_0)$$

3) Let CX be the cone of $X \ni x_0$. Show that $\pi_n(CX, X, x_0) = \pi_{n-1}(X, x_0)$. Deduce that a relative π_2 need not be abelian.

   Use the long exact sequence and contractibility of the cone. Notice that from here we also see that a relative π_1 cannot have a canonical group structure.

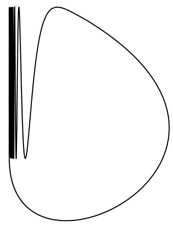
4. Whitehead theorem

1) Show that S^∞ is contractible using the Whitehead theorem. (It can be useful to use that a compact subspace of a CW complex is contained in a finite sub-complex [H, Prop A.1].)

   We show that $\pi_n(S^\infty) = 0$. Take a map $f\colon S^n to S^\infty$. By compactness, f factors through $f_k\colon S^n \to S^k$, where k is bigger than n. It follows that $\pi_n(f) = \pi_n(i) \circ \pi_n(f_k) = \pi_n(i) \circ 0 = 0$, where $i\colon S^k \hookrightarrow S^\infty$.

2) Consider the *Warsaw circle* W, which is given by the graph of $y = \sin(1/x)$ for $x \in (0, 1]$, then we add the vertical segment between $(0, 1)$ and $(0, -1)$ and finally we connect this segment to the point $(1, \sin(1))$ via some disjoint curve.

   See [H, Exercise 4.1.10]



Show that $\pi_n(W) = 0$, but W is not contractible.

5. **Hurewicz theorem for π_1** This is the proof strategy from [Bredon, IV Theorem 3.4].

   Recall that we have a map $A\colon \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$, $[\gamma]_\pi \mapsto [\gamma]_H$ which induces an isormophism between $H_1(X)$ and the abelianization of $\pi_1(X)$ when X is path-connected. Most of the proof of this theorem is quite doable as an exercise, as you will show next:

   1) Show that the constant loop is sent to 0 under A.
   2) Show that A is a well defined map. This can be done by collapsing one of the edges of the square which is the domain of the homotopy $h\colon I \times I \to X$ such that $\gamma \sim_h \gamma'$.
   3) Show that A is a homomorphism by finding an explicit simplex σ such that $\partial\sigma = \gamma + \gamma' - \gamma \star \gamma'$. Conclude that A factors through the abelianization $A_*\colon \pi_1/[\pi_1, \pi_1] \to A$.
   4) Show that A is surjective by finding an explicit pre-image of a cycle in $Z_1(X)$.

   In fact the conclusion of the proof does not need showing surjectivity beforehand (it is just easier in case you struggle with the following exercise). We can provide the explicit inverse as follows:

   5) For every point $x \in X$, let $\lambda_x\colon I \to X$ be a path from x_0 to x (take the constant one if $x = x_0$). Define $\psi\colon C_1(X) \to \pi_1/[\pi_1, \pi_1]$, mapping a 1-simplex $\sigma\colon I \to X$ to $\lambda_{\sigma(0)} \star \sigma \star \lambda_{\sigma(1)}^{-1}$.

      Check that ψ induces a well defined map $\psi_*\colon H_1(X) \to \pi_1/[\pi_1, \pi_1]$ and show that it is a left and right inverse to A_*.