A SHORT PROOF OF THE $\mathcal{C}^{1,1}$ REGULARITY FOR THE EIKONAL EQUATION

RADU IGNAT

Abstract. We give a short and self-contained proof of the interior $\mathcal{C}^{1,1}$ regularity of solutions $\varphi : \Omega \to \mathbb{R}$ to the eikonal equation $|\nabla\varphi| = 1$ in an open set $\Omega \subset \mathbb{R}^N$ in dimension $N \geq 1$ under the assumption that φ is pointwise differentiable in Ω.

1. Introduction

The aim of this note is to give a short and self-contained proof of the following result known in the theory of Hamilton-Jacobi equations:

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set in dimension $N \geq 1$ and $\varphi : \Omega \to \mathbb{R}$ be a pointwise differentiable solution to the eikonal equation $|\nabla\varphi| = 1$ in Ω. Then $\nabla\varphi$ is locally Lipschitz in Ω.*

The usual (standard) proof of this result is based on the following steps (see e.g. Lions [7], Cannarsa-Sinestrari [2]): first, one checks that φ (and $-\varphi$) is a viscosity solution to the eikonal equation (see [2, Definition 5.2.1]); second, one proves that φ is both semiconcave and semiconvex with linear modulus (see [2, Theorem 5.3.7]). Third, one proves that φ is C^1 (see [2, Theorem 3.3.7]) and finally, that φ is locally $C^{1,1}$ in Ω (see [2, Corollary 3.3.8]).

Our approach is based on the geometry of characteristics associated to the eikonal equation. More precisely, if $x_0 \in \Omega$, we say that $X := X_{x_0}$ is a characteristic of a solution φ passing through x_0 in some time interval $t \in [-T, T]$ if

$$\begin{cases} \dot{X}(t) = \nabla\varphi(X(t)) \text{ for } t \in [-T, T], \\ X(0) = x_0. \end{cases} \tag{1}$$

Then the beautiful proof of Caffarelli-Crandall [1, Lemma 2.2] shows in a short and self-contained manner that every point $x_0 \in \Omega$ has a characteristic X_{x_0} that is a straight line along which $\nabla\varphi$ is constant and φ is affine. Finally, we give a geometric argument on the structure of characteristics yielding the locally Lipschitz regularity of $\nabla\varphi$ in Ω.

The regularity result in Theorem 1 is optimal: such solution φ of the eikonal equation is not C^2 in general (see e.g. [5, Proposition 1]). We mention that a more general regularizing effect (i.e., $\nabla\varphi$ is locally Lipschitz away from vortex point singularities) is proved under a weaker assumption $\nabla\varphi \in W^{1/p,p}$ for $p \in [1, 3]$, see [5, 3]. Similar results are obtained in the context of the Aviles-Giga model which can be seen as a regularization of the eikonal equation (see [6, 4]).

2. Proof of the main result

The first step is to show that each point $x_0 \in \Omega$ has a characteristic $X := X_{x_0}$ that is a straight line in direction $\nabla\varphi(x_0)$. Moreover, $\nabla\varphi$ is constant while φ is affine along this characteristic. This fact yields $\varphi \in C^1(\Omega)$. In order to have a self-contained

Date: September 10, 2024.

proof of Theorem 1, we repeat here the very nice argument of Caffarelli-Crandall [1, Lemma 2.2] based on a maximum type principle for the eikonal equation.

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\varphi : \Omega \to \mathbb{R}$ be a pointwise differentiable solution of the eikonal equation $|\nabla\varphi| = 1$ in Ω. Then for every $x_0 \in \Omega$, $X(t) = x_0 + t\nabla\varphi(x_0)$ is a characteristic of* (1) *and*

$$\nabla\varphi(X(t)) = \nabla\varphi\,(x_0)\,,\; \varphi(X(t)) = \varphi(x_0) + t,\; \forall t \in [-T, T]$$

for some $T > 0$. As a consequence, $\varphi \in C^1(\Omega)$.

Proof. This proof follows the lines in [1, Lemma 2.2]. Let $R > 0$ be such that $\bar{B}_R(x_0) \subset \Omega$ and consider

$$M_r = \max_{\bar{B}_r(x_0)} \varphi, \quad m_r = \min_{\bar{B}_r(x_0)} \varphi, \quad \forall r \in [0, R].$$

Claim 3. *$M_r = \varphi(x_0) + r$ and $m_r = \varphi(x_0) - r$ for every $r \in [0, R]$.*

Proof. For $r \in [0, R]$, we pick some maximum point $x_r^+ \in \bar{B}_r(x_0)$ such that $\varphi(x_r^+) = M_r$. First, we show that $r \in [0, R] \mapsto M_r$ is a nondecreasing 1-Lipschitz function. Indeed, for $R \geq r > \tilde{r}$, as $|x_r^+ - x_0| \leq r$, we can find a vector $e \in \mathbb{R}^N$ such that $|e| \leq r - \tilde{r}$ and $|x_r^+ + e - x_0| \leq \tilde{r}$, i.e., $x_r^+ + e \in \bar{B}_{\tilde{r}}(x_0)$; this yields

$$0 \leq M_r - M_{\tilde{r}} \leq \varphi(x_r^+) - \varphi(x_r^+ + e) \leq |e| \leq r - \tilde{r}$$

because φ is 1-Lipschitz. Second, we prove that $\frac{dM_r}{dr} = 1$ a.e. in $(0, R)$ because for $r \in (0, R)$ and for small $h > 0$, as $x_r^+ + h\nabla\varphi(x_r^+) \in \bar{B}_{r+h}(x_0)$, we have

$$\liminf_{h\to 0} \frac{M_{r+h} - M_r}{h} \geq \liminf_{h\to 0} \frac{\varphi(x_r^+ + h\nabla\varphi(x_r^+)) - \varphi(x_r^+)}{h} = |\nabla\varphi(x_r^+)|^2 = 1.$$

As $M_0 = \varphi(x_0)$, we conclude $M_r = \varphi(x_0) + r$. Up to changing φ in $-\varphi$, one also gets $m_r = \varphi(x_0) - r$ for $r \in [0, R]$. □

To conclude the proof of Lemma 2, pick some minimum point $x_R^- \in \bar{B}_R(x_0)$ such that $\varphi(x_R^-) = m_R = \varphi(x_0) - R$ (by Claim 3). As φ is 1-Lipschitz, we have, again by Claim 3:

$$2R = \varphi(x_R^+) - \varphi(x_R^-) \leq |x_R^+ - x_R^-| \leq 2R,$$

which means that $[x_R^+, x_R^-]$ is a diameter in $\bar{B}_R(x_0)$. Note that x_R^+ (resp. x_R^-) is the unique maximum (resp. minimum) of φ in $\bar{B}_R(x_0)$ because if $\tilde{x}_R^+$ is another maximum, then it has to be antipodal to x_R^-, that is, $\tilde{x}_R^+ = x_R^+$ (the same for the uniqueness of x_R^-). In particular, $e_* = \frac{x_R^+ - x_0}{R} \in \mathbb{S}^{N-1}$. Define $g : [-R, R] \to \mathbb{R}$ by $g(r) = \varphi(x_0 + re_*) - \varphi(x_0)$. Then g is 1-Lipschitz and $g(\pm R) = \varphi(x_R^\pm) - \varphi(x_0) = \pm R$ (by Claim 3). So $g(r) = r$ for every $r \in (-R, R)$ yielding $1 = g'(r) = e_* \cdot \nabla\varphi(x_0 + re_*)$ for every r. Thus, $\nabla\varphi(x_0 + re_*) = e_*$ for every $r \in [-R, R]$, in particular, $e_* = \nabla\varphi(x_0)$, i.e., $X(r) = x_0 + r\nabla\varphi(x_0)$ is a characteristic of (1) and

$$\nabla\varphi(X(r)) = \nabla\varphi\,(x_0)\,, \varphi(x_0 + r\nabla\varphi(x_0)) = \varphi(x_0) + r, \quad \forall r \in [-R, R].$$

In particular, the (unique) maximum and minimum of φ in $\bar{B}_R(x_0)$ are achieved at the points $x_R^\pm = x_0 \pm R\nabla\varphi(x_0)$.

It remains to prove that $\nabla\varphi$ is continuous in Ω. Indeed, let $x_n \to x_0$ in Ω and $\bar{B}_R(x_n) \subset \Omega$ for large n. Up to a subsequence, we may assume that $\nabla\varphi(x_n) \to e \in \mathbb{S}^{N-1}$. By above, we know that $\varphi(x_n + R\nabla\varphi(x_n)) = \varphi(x_n) + R$. Passing to the limit, we obtain $\varphi(x_0 + Re) = \varphi(x_0) + R$, meaning that $x_0 + Re$ is the maximum of φ in $\bar{B}_R(x_0)$. By uniqueness of the maximum point x_R^+, we conclude that $e = \nabla\varphi(x_0)$.

The uniqueness of the limit e for such subsequences yield the convergence of the whole sequence $(\nabla\varphi(x_n))_n$ to $\nabla\varphi(x_0)$. □

Proof of Theorem 1. Let B be a ball, $\bar{B}\subset\Omega$ and we consider $d\in(0,\frac{\mathrm{dist}(B,\partial\Omega)}{5})$. We will prove the following:

Claim 4. *There exists a universal constant $C>0$ such that*

$$|\nabla\varphi(x)-\nabla\varphi(y)|\leqslant\frac{C}{d}|x-y|,\quad \forall x,y\in B\ \textit{with}\ |x-y|<\frac{d}{10}.$$

Proof. Let X_x and X_y be the characteristics passing through x and y constructed in Lemma 2 (that are lines in direction $\nabla\varphi(x)$ and $\nabla\varphi(y)$). If X_x and X_y coincide inside B (in particular, $\nabla\varphi(x)=\pm\nabla\varphi(y)$ by Lemma 2), then Lemma 2 implies that $\nabla\varphi(x)=\nabla\varphi(y)$ and the claim is trivial in that case. Otherwise, $\nabla\varphi(x)\neq\pm\nabla\varphi(y)$ and X_x and X_y cannot intersect inside Ω (as $\nabla\varphi$ is continuous in Ω by Lemma 2). Let $|x-y|<\frac{d}{10}$ and x' be the projection of y on X_x. Clearly,

$$\ell=|x'-y|\leq|x-y|<\frac{d}{10},$$

$\mathrm{dist}(x',\partial\Omega)>5d-\ell$ and $\nabla\varphi(x')=\nabla\varphi(x)$. Up to changing φ in $-\varphi$, we may assume that $\varphi(x')\leq\varphi(y)$ and up to an additive constant, we also may assume that $0=\varphi(x')\leq\varphi(y)=a$. As $|\nabla\varphi|=1$ on the segment $[x'y]$, we deduce that $0\leq a=\varphi(y)-\varphi(x')\leq|x'-y|=\ell$. Let z be the point on the characteristic X_y reached at time $t=d$ in direction $\nabla\varphi(y)$, i.e., $|z-y|=d$, $\nabla\varphi(y)=\frac{z-y}{d}$ and $\varphi(z)=a+d$; in particular, $z\in\Omega$. Let w be the point on the characteristic $X_{x'}$ reached at time $t=-d$ in direction $\nabla\varphi(x')$, i.e., $|x'-w|=d$, $\nabla\varphi(x')=\frac{x'-w}{d}$ and $\varphi(w)=-d$; in particular, $w\in\Omega$. We deduce that

$$|z-w|=\int_{[zw]}|\nabla\varphi|\,d\mathcal{H}^1\geq\left|\int_{[zw]}\nabla\varphi\cdot\frac{z-w}{|z-w|}\,d\mathcal{H}^1\right|=|\varphi(z)-\varphi(w)|=2d+a\geq 2d.$$

By Pitagora's formula in the triangle $x'yw$ we have $|y-w|^2=\ell^2+d^2$. Denoting α the angle between $\vec{wy}$ and $\nabla\varphi(y)$, the cosine formula in the triangle wyz yields

$$\begin{aligned}-\cos\alpha&=\frac{|y-w|^2+|y-z|^2-|z-w|^2}{2|y-z|\cdot|y-w|}\\&=\frac{2d^2+\ell^2-|z-w|^2}{2d\sqrt{d^2+\ell^2}}\leq-\frac{2d^2-\ell^2}{2d\sqrt{d^2+\ell^2}}=-1+O(\frac{\ell^2}{d^2}).\end{aligned}$$

As $\alpha\in(0,\frac{\pi}{2})$, it yields $\sin^2\alpha=1-\cos^2\alpha\leq 2(1-\cos\alpha)=O(\frac{\ell^2}{d^2})$. So, $\sin\alpha=O(\frac{\ell}{d})$. Denoting $\beta\in(0,\frac{\pi}{2})$ the angle between $\vec{wy}$ and $\nabla\varphi(x)$, we compute in the triangle $x'yw$:

$$\sin\beta\leq\tan\beta=\frac{|y-x'|}{|x'-w|}=\frac{\ell}{d}.$$

In particular, $0\leq\alpha+\beta\leq\frac{\pi}{2}$ if $\ell<\frac{d}{10}$. Finally, denoting γ the angle between $\nabla\varphi(x)$ and $\nabla\varphi(y)$, the triangle inequality yields

$$\sin\gamma\leq\sin(\alpha+\beta)\leq\sin\alpha+\sin\beta=O(\frac{\ell}{d}).$$

We conclude

$$|\nabla\varphi(x)-\nabla\varphi(y)|=2\sin\frac{\gamma}{2}=O(\sin\gamma)=O(\frac{\ell}{d})\leq\frac{C}{d}|x-y|$$

for some universal $C > 0$. □

The conclusion of Theorem 1 follows. □

Acknowledgement. The author thanks Xavier Lamy for very useful discussions. The author is partially supported by the ANR projects ANR-21-CE40-0004 and ANR-22-CE40-0006-01.

References

[1] L.A. Caffarelli, M. G. Crandall, *Distance functions and almost global solutions of eikonal equations*, Comm. Partial Differential Equations **35** (2010), 391–414.
[2] P. Cannarsa, C Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control*. Progress in Nonlinear Differential Equations and Their Applications, Vol. 58. Berlin: Birkhäuser, 2004.
[3] C. De Lellis, R. Ignat, *A regularizing property of the 2D-eikonal equation*, Comm. Partial Differential Equations **40** (2015), 1543-1557.
[4] F. Ghiraldin, X. Lamy, *Optimal Besov differentiability for entropy solutions of the eikonal equation*, Comm. Pure Appl. Math. **73** (2020), 317-349.
[5] R. Ignat, *Two-dimensional unit-length vector fields of vanishing divergence*, J. Funct. Anal. **262** (2012), 3465-3494.
[6] P.-E. Jabin, F. Otto, B. Perthame, *Line-energy Ginzburg-Landau models: zero-energy states*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), 187-202.
[7] P.-L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations*. Research Notes in Mathematics, Vol. 69. Boston: Pitman, 1982.

Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France.
Email address: Radu.Ignat@math.univ-toulouse.fr