Universitatea Tehnica Cluj-Napoca 2024-2025
Pregatire de Olimpiada de matematica universitara

Calcul diferential
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Exercise 1 *Determine all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *such that for every* $x, y \in \mathbb{R}$ *with* $x - y \in \mathbb{Q}$*, one has* $f(x) - f(y) \in \mathbb{Q}$.

Exercise 2 *We say that* $C \subset \mathbb{R}$ *is a closed set if its complementary* $\mathbb{R} \setminus C$ *is a union of open intervals. Let* $C \subset \mathbb{R}$ *be a non-empty, closed and bounded set and* $f : C \to C$ *be a nondecreasing function. Show that there is a point* $p \in C$ *such that* $f(p) = p$.

Exercise 3 *If* $I \subset \mathbb{R}$ *is an interval, we say that* $f : I \to \mathbb{R}$ *is a convex function if* $f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$ *for every* $x, y \in I$ *and every* $t \in [0,1]$*. If* f *is differentiable, then show that* f *is convex if and only if* f' *is nondecreasing.*

Exercise 4 *Let* $P_n(x)$ *be a polynomial of even degree* $n > 1$ *such that its dominant coefficient is positive and* $P_n(x) > P_n''(x), \forall x \in \mathbb{R}$*. Show that* $P_n(x) > 0, \forall x \in \mathbb{R}$.

Exercise 5 *Show that for any polynomial* $p(x)$ *of degree* $n > 1$ *and any point* Q*, the number of tangents to the graph of* $p(x)$ *passing at* Q *are not more than* n.

Exercise 6 *Let* $f : [0,1] \to \mathbb{R}$ *be a differentiable function with* $f(0) = 0$*. If there is some* $M \geqslant 0$ *such that* $|f'(x)| \leqslant M|f(x)|$ *for every* $x \in [0,1]$*, then* f *vanishes everywhere.*

Exercise 7 *Is there any continuous differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) > 0$ *and* $f'(x) = f(f(x))$ *for any* $x \in \mathbb{R}$*?*

Exercise 8 *Let* $f : (0,\infty) \longrightarrow \mathbb{R}$ *be a twice differentiable function on* $(0,\infty)$*. If* $|f(x)| \leqslant A$ *and* $|f''(x)| \leqslant B$ *for every* $x \in (0,\infty)$*, then* $|f'(x)| \leqslant 2\sqrt{AB}$ *for any* $x \in (0,\infty)$.

Exercise 9 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function such that* $f(0) = 2$, $f'(0) = -2$ *and* $f(1) = 1$*. Show that there is some* $\xi \in (0,1)$ *such that*

$$f(\xi)f'(\xi) + f''(\xi) = 0.$$

Exercise 10 *Let* $f : (-1,1) \to \mathbb{R}$ *be a continuous function such that* $f(0) = 0$ *and* f *is differentiable in* 0*. Show that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{1}{2} f'(0).$$

As an application, compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1\right).$$

Exercise 11 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a twice differentiable function. If* $f(x) \geqslant 0$ *and* $f''(x) \leqslant 0$ *for every* $x \in \mathbb{R}$*, then* f *is a constant function.*