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Pregatire de Olimpiada de matematica universitara

Siruri si serii numerice
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Exercise 1 *Let $f : [1,\infty) \to \mathbb{R}$ be a nonincreasing function that is bounded from below. Show that the sequence $(a_n)_{n\geqslant 1}$ given by*

$$a_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx, \quad \forall n \geqslant 1,$$

is convergent. What do you deduce for $f(x) = \frac{1}{x}$, $\forall x \geqslant 1$?

Exercise 2 *Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$. Show that the set of limit points of $(a_n)_{n\geqslant 1}$ is a closed interval.*

Exercise 3 *Let (x_n) be a sequence of real numbers such that $\lim_{n\to\infty}(2x_{n+1} - x_n) = L$ for some $L \in \mathbb{R}$. Show that $\lim_{n\to\infty} x_n = L$.*

Exercise 4 *Let a and b be two positive numbers. Compute the limit of the sequence (x_n) defined by*

$$x_1 = \sqrt{a}, \quad x_{n+1} = \sqrt{a + bx_n}, \forall n \geqslant 1.$$

In particular, compute

$$\lim_{n\to\infty} \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}, \quad (n \text{ square roots }).$$

Exercise 5 *If the sequence $(a_n)_n$ is decreasing to zero and the series $\sum_{n=1}^{\infty} a_n$ converges, then show that*

$$\lim_{n\to\infty} na_n = 0.$$

Exercise 6 *We consider the sequence $(a_n)_n$ defined by*

$$a_1 = 1, \quad a_{n+1} = \ln(1 + a_n), \forall n \geqslant 1.$$

a) Show that $\lim_{n\to\infty} a_n = 0$.
b) Show that the series $\sum_{n=1}^{\infty} a_n$ diverges.
c) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Exercise 7 *Let $\sum_{n=1}^{\infty} a_n$ be a divergent series with positive terms and $(S_n)_n$ be the sequence of its partial sums $S_n = \sum_{k=1}^{n} a_k$. Prove that:*
a) The series $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ diverges.
b) The series $\sum_{n=1}^{\infty} \frac{a_n}{S_n^{1+\alpha}}$ converges for $\alpha > 0$.

Exercise 8 *Let $(\varepsilon_n)_n$ be a sequence with terms $\varepsilon_n \in \{-1, 1\}, \forall n \in \mathbb{N}$. Show that the series $\sum_{n=0}^{\infty} \frac{\varepsilon_n}{n!}$ converges to an irrational number.*

Exercise 9 *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive terms. Show that the series $\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \ldots a_n}$ is convergent and the following inequality holds true:*

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \ldots a_n} < e \sum_{n=1}^{\infty} a_n.$$