MINIMALITY OF THE VORTEX SOLUTION FOR GINZBURG-LANDAU SYSTEMS

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ABSTRACT. We consider the Ginzburg-Landau system for N-dimensional maps defined in the unit ball for some parameter $\varepsilon > 0$. For a boundary data corresponding to a vortex of topological degree one, the aim is to prove the symmetry of the ground state of the system. We show this conjecture for every $\varepsilon > 0$ in any dimension $N \geq 7$, and then, we also prove it in dimension $N = 4, 5, 6$ provided that the admissible maps are gradient fields.

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The Ginzburg-Landau model. Let $B^N \subset \mathbb{R}^N$ be the unit ball, $N \geq 2$. For $u : B^N \to \mathbb{R}^N$, consider the Ginzburg-Landau functional for a parameter $\varepsilon > 0$:

$$G_\varepsilon(u) = \int_{B^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2}W(1 - |u|^2)\, dx,$$

where $W : (-\infty, 1] \to \mathbb{R}_+$ is C^1 convex, $W(0) = 0$, $W(t) > 0$ for $t \neq 0$. Typically, $W(t) = \frac{t^2}{2}$. As $\varepsilon \to 0$, the limit maps take values into the unit sphere $\mathbb{S}^{N-1}$, so the limit model is the $\mathbb{S}^{N-1}$-harmonic map problem (HMP). Thus, our results are expected to be closely related with those obtained for HMP.

We focus on critical points u of G_ε for fixed $\varepsilon > 0$:

$$-\Delta u = \frac{1}{\varepsilon^2}W'(1 - |u|^2)u \quad \text{in } B^N \tag{1}$$

under the boundary condition

$$u(x) = x \quad \text{on } \partial B^N = \mathbb{S}^{N-1}. \tag{2}$$

Such critical points u (e.g., minimizers) exist. In particular, by the maximum principle, $|u| \leq 1$ in B^N and then, the standard elliptic theory yields $u \in W^{2,p} \cap C^{1,\alpha}$ for every $p < \infty$ and $\alpha \in (0, 1)$. Moreover, the topological constraint in (2) implies that u has a zero point inside B^N that plays an important role in this theory. The main question concerns the uniqueness of solutions in (1) & (2).

The vortex solution. For every $\varepsilon > 0$, there exists a unique solution to (1) & (2) that is invariant under the special orthogonal group $SO(N)$, i.e., the group action $u \mapsto u^R(x) = R^{-1}u(Rx)$ for every $R \in SO(N)$ that keeps invariant the functional G_ε

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and the boundary condition (2). This is the so-called *vortex solution* (of topological degree 1) given by

$$u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}, \quad x \in B^N \setminus \{0\}.$$

The radial profile $f_\varepsilon : [0,1] \to \mathbb{R}$ is the unique solution to the singular ODE:

$$\begin{cases} -f_\varepsilon'' - \frac{N-1}{r}f_\varepsilon' + \frac{N-1}{r^2}f_\varepsilon = \frac{1}{\varepsilon^2}W'(1-f_\varepsilon^2)f_\varepsilon & \text{in } (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1, \end{cases} \tag{3}$$

where $r = |x|$ (see [3, 4, 7]). In particular, $1 > f_\varepsilon > 0$ and $f_\varepsilon' > 0$ in $(0,1)$. The aim is to study the minimality of the vortex solution:

Question 1: Is $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ the (unique) minimiser of G_ε under the boundary condition (2) for every $\varepsilon > 0$?

For large ε, i.e., $\varepsilon \geq \varepsilon_{conv}$, the functional G_ε is strictly convex yielding uniqueness in (1) & (2) (in particular, the positive answer to Question 1), see [1, 9]. For $\varepsilon < \varepsilon_{conv}$, there are only some partial results. In dimension $N = 2$, Bethuel-Brezis-Hélein [1] proved in the regime $\varepsilon \to 0$ that a minimizer u of G_ε under (2) has a unique topological zero converging to the origin, while Pacard-Rivière [17] proved that u_ε is the unique solution to (1) & (2) for very small $\varepsilon > 0$; we also mention the work of Mironescu [16] for the corresponding blow-up problem in the domain $\mathbb{R}^2$. In dimension $N \geq 3$, we quote the works of Millot-Pisante [14] and Pisante [18] for the blow-up problem in the domain $\mathbb{R}^N$. Finally, for the $\mathbb{S}^{N-1}$-harmonic map problem, $u_*(x) = \frac{x}{|x|}$ is the unique minimizing harmonic map in B^N under (2) if $N \geq 3$ (see Jäger-Kaul [11], Brezis-Coron-Lieb [2], Lin [13]).

Main results. Our first result gives a positive answer to Question 1 in dimension $N \geq 7$ (see [8, 9]):

Theorem 2 If $N \geq 7$, then $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ is the unique minimiser of G_ε under (2) for every $\varepsilon > 0$.

Sketch of the proof. The idea is to linearize the potential energy in G_ε. More precisely, the convexity of W yields for every $v \in H_0^1(B^N, \mathbb{R}^N)$:

$$G_\varepsilon(u_\varepsilon + v) - G_\varepsilon(u_\varepsilon) \geq \frac{1}{2}F_\varepsilon(v) \tag{4}$$

where $F_\varepsilon(v) = \int_{B^N} |\nabla v|^2 - \frac{1}{\varepsilon^2}W'(1-|u_\varepsilon|^2)|v|^2\,dx$. To conclude, we need to prove that for every $\varepsilon > 0$, $F_\varepsilon(v) = \displaystyle\int_{B^N} L_\varepsilon v \cdot v\,dx \geq 0$, $\forall v \in H_0^1(B^N, \mathbb{R}^N)$, where $L_\varepsilon = -\Delta - \frac{1}{\varepsilon^2}W'(1-f_\varepsilon^2)$. Let $\ell(\varepsilon) = \lambda_1(L_\varepsilon, B^N)$ be the first eigenvalue of L_ε in B^N under zero Dirichlet condition. The conclusion follows by:

Lemma 3 If $N \geq 7$, then $\ell(\varepsilon) \geq c_N = \frac{(N-2)^2}{4} - (N-1) > 0$, $\forall \varepsilon > 0$.

Sketch of the proof. For $v \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$, we use the Hardy decomposition $v = f_\varepsilon s$. Integration by parts combined with (3) imply

$$F_\varepsilon(v) = \int_{B^N} L_\varepsilon v \cdot v = \int_{B^N} (f_\varepsilon^2|\nabla s|^2 + s^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon) = \int_{B^N} f_\varepsilon^2\big(|\nabla s|^2 - \frac{N-1}{r^2}s^2\big).$$

The limit case $\varepsilon \to 0$ follows from the fact that $f_\varepsilon \to 1$ in $(0,1]$ combined with Hardy's inequality:

$$\int_{B^N} L_\varepsilon v \cdot v \to \int_{B^N} |\nabla s|^2 - \frac{N-1}{r^2} s^2 \geq \int_{B^N} \big(\frac{(N-2)^2}{4} - (N-1)\big)\frac{s^2}{r^2} \geq c_N \int_{B^N} s^2.$$

For the general case $\varepsilon > 0$ (fixed), one decomposes $s = \phi \tilde{s}$ with $\phi = r^{-\frac{N-2}{2}}$ and obtains $F_\varepsilon(v) \geq c_N \int_{B^N} \frac{v^2}{r^2}$ yielding the conclusion of Lemma 3 together with the uniqueness of the minimizer u_ε in Theorem 2. □

In dimension $N \in [2,6]$, the above argument does not yield the answer to Question 1. Indeed, the first eigenvalue $\ell(\varepsilon)$ of L_ε in B^N becomes negative for small $\varepsilon > 0$ if $2 \leq N \leq 6$. However, the above argument improves the range of ε where u_ε is the unique minimizer of G_ε under (2) (with respect to ε_{conv} above which G_ε is strictly convex), see [5, 10]:

Lemma 4 If $2 \leq N \leq 6$, then there is $\varepsilon_N \in (0, \varepsilon_{conv})$ such that $\ell(\varepsilon_N) = 0$ and $\ell(\varepsilon) < 0$ if $\varepsilon < \varepsilon_N$ (resp. $\ell(\varepsilon) > 0$ if $\varepsilon > \varepsilon_N$). In particular, if $\varepsilon > \varepsilon_N$, then the vortex solution u_ε is the unique minimizer of G_ε under (2).

The minimality of u_ε is still an open question if $\varepsilon < \varepsilon_N$ and $N \in [2,6]$. A partial result is the *local* minimality of u_ε for every $\varepsilon > 0$. This is known in dimension $N = 2$ thanks to the works of Mironescu [15] and Lieb-Loss [12], while in dimension $N \in [3,6]$, this is proved by Ignat-Nguyen [5]:

Theorem 5 If $3 \leq N \leq 6$, then $u_\varepsilon = f_\varepsilon(|x|)\frac{x}{|x|}$ is a *local* minimizer of G_ε under (2) for every $\varepsilon > 0$.

Sketch of the proof. The aim is to prove that for every $\varepsilon > 0$, $G_\varepsilon(u_\varepsilon + v) - G_\varepsilon(u_\varepsilon) \geq C\|v\|_{H^1}^2$ if $\|v\|_{H^1} \leq \delta$ for some $\delta = \delta(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ small. For that, we analyse the second variation of G_ε at u_ε in direction $v \in H_0^1(B^N, \mathbb{R}^N)$:

$$Q_\varepsilon(v) = \frac{d^2}{dt^2}\Big|_{t=0} G_\varepsilon(u_\varepsilon + tv) = F_\varepsilon(v) + \frac{2}{\varepsilon^2}\int_{B^N} W''(1 - f_\varepsilon^2) f_\varepsilon^2 (v \cdot \frac{x}{|x|})^2 \, dx.$$

This is done by writing $v(x) = s(x)\frac{x}{|x|} + \tilde{v}(x)$ for some scalar function s and a tangent vector field $\tilde{v}(x) \cdot x = 0$ and then use the Hodge decomposition in the tangent space for every $x \in B^N \setminus \{0\}$: $\tilde{v}(r,\cdot) = v^\circ(r,\cdot) + \not\nabla\psi(r,\cdot)$ on $\mathbb{S}^{N-1}$ where $\not\nabla \cdot v^\circ(r,\cdot) = 0$ in $\mathbb{S}^{N-1}$ and ψ is a scalar function. (Here, $\not\nabla$ is the covariant derivative.) The spectral decomposition of $s(r,\cdot)$ and $\psi(r,\cdot)$ in $L^2(\mathbb{S}^{N-1})$ yields a decomposition of $v - v^\circ$ in modes v_k and furthermore, the following decomposition of the second variation

$$Q_\varepsilon(v) = Q_\varepsilon(v^\circ) + \sum_{k\geq 0} Q_\varepsilon(v_k).$$

Using Hardy decompositions for v° and each v_k, we obtain $Q_\varepsilon(v) \geq C(\varepsilon)\|v\|_{H^1}^2$ for every $v \in H_0^1(B^N, \mathbb{R}^N)$ and $\varepsilon > 0$. An extra argument yields local minimality of u_ε. □

The Aviles-Giga model. Note that the vortex solution is a gradient field, i.e., $u_\varepsilon = \nabla\phi_\varepsilon$ for some radial function $\phi_\varepsilon : B^N \to \mathbb{R}$ determined by $\phi_\varepsilon' = f_\varepsilon$ in $(0,1)$. Therefore, in dimension $N \in [2,6]$, it is natural to study the minimality of u_ε restricted to the class of gradient fields.

Question 2: Is u_ε the (unique) minimizer of G_ε for every $\varepsilon > 0$ over gradient fields

$$\mathcal{V} = \{u = \nabla\phi \,:\, \phi \in H^2(B^N, \mathbb{R}),\ \nabla\phi = Id \ \text{ on } \partial B^N\}?$$

This is the so-called Aviles-Giga model corresponding to the functional

$$G_\varepsilon(\nabla\phi) = \int_{B^N} \frac{1}{2}|\nabla^2\phi|^2 + \frac{1}{2\varepsilon^2}W(1 - |\nabla\phi|^2)\, dx.$$

We are able to improve Theorem 2 to the dimensions $N = 4, 5, 6$ in this restricted class $\mathcal{V}$, see Ignat-Nahon-Nguyen [6].

Theorem 6 If $N \geq 4$, then u_ε is the unique global minimizer of G_ε over $\mathcal{V}$ for every $\varepsilon > 0$.

Sketch of the first proof. As before, for every $\nabla\psi \in H_0^1(B^N, \mathbb{R}^N)$, we have $G_\varepsilon(u_\varepsilon + \nabla\psi) - G_\varepsilon(u_\varepsilon) \geq \frac{1}{2}F_\varepsilon(\nabla\psi)$. As $\nabla\psi = 0$ on ∂B^N, we have

$$F_\varepsilon(\nabla\psi) = \int_{B^N} (\Delta\psi)^2 - \frac{1}{\varepsilon^2}W'(1 - f_\varepsilon^2)|\nabla\psi|^2\, dx.$$

In the limit case $\varepsilon \to 0$, we expect that $F_\varepsilon(\nabla\psi) \to \int_{B^N}(\Delta\psi)^2 - \frac{N-1}{r^2}|\nabla\psi|^2$ and the conclusion would follow by the Hardy inequality in $\mathcal{V}$:

$$\int_{B^N} (\Delta\psi)^2 \geq K_N \int_{B^N} \frac{|\nabla\psi|^2}{r^2} \quad \text{ with } K_N = \begin{cases} N^2/4 & \text{if } N \geq 5 \\ N-1 & \text{if } N = 4 \\ 25/36 & \text{if } N = 3 \end{cases}.$$

For the general case $\varepsilon > 0$, we use the spherical harmonic decomposition for ψ and based again on some Hardy decompositions, we get $F_\varepsilon(\nabla\psi) \geq 0$ provided that $N \geq 4$.

Sketch of the second proof if $N \geq 5$: This second proof is based on the following symmetrization of gradient fields. More precisely, for the stream function $\phi \in H^1(B^N, \mathbb{R})$, we associate the radial function $\phi_* = \phi_*(r)$ defined by

$$\phi_*'(r) = \Big(\fint_{\mathbb{S}^{N-1}} |\nabla\phi(r\theta)|^2 d\sigma(\theta) \Big)^{1/2} \geq 0, \quad r \in (0,1).$$

As W is convex, Jensen's inequality yields

$$\int_{B^N} W(1 - |\nabla\phi|^2)\, dx \geq \int_{B^N} W(1 - |\nabla\phi_*|^2)\, dx.$$

Moreover, if $\nabla\phi = Id$ on ∂B^N and $N \geq 5$ then

$$\int_{B^N} |\nabla^2\phi|^2\, dx \geq \int_{B^N} |\nabla^2\phi_*|^2\, dx$$

with equality if and only if ϕ is radial. Thus, for every $N \geq 5$ and any $\varepsilon > 0$, $G_\varepsilon(\nabla\phi) \geq G_\varepsilon(\nabla\phi_*) \geq G_\varepsilon(u_\varepsilon = \nabla\phi_\varepsilon)$. □

$\mathbb{R}^{N+1}$-valued vortex solutions. We can solve completely Question 1 when we add one target dimension, i.e., the admissible maps are $U = (u, U_{N+1}) : B^N \to \mathbb{R}^{N+1}$ satisfying the boundary condition

$$U(x) = (x, 0) \in \mathbb{S}^{N-1} \times \{0\} \text{ on } \partial B^N. \tag{5}$$

We prove that for every $\varepsilon > 0$, minimizers of G_ε under (5) are vortex type solutions that are either *non-escaping* (i.e., their $(N+1)$-component vanishes in B^N), or they are *escaping*, i.e., their $(N+1)$-component is positive (or negative) in B^N, see Ignat-Rus [10].

Theorem 7 Every minimizer of G_ε under (5) is symmetric of vortex type and the following dichotomy holds in dimension $2 \le N \le 6$:

a) if $\varepsilon \ge \varepsilon_N$, then the *non-escaping* vortex solution $\bar{U}_\varepsilon = (f_\varepsilon(|x|)\frac{x}{|x|}, 0)$ is the unique minimizer of G_ε under (5).

b) if $\varepsilon < \varepsilon_N$, then the two *escaping* vortex solutions $(\tilde{f}_\varepsilon(|x|)\frac{x}{|x|}, \pm g_\varepsilon(|x|))$ with $g_\varepsilon > 0$ are the only minimizers of G_ε under (5). In this case, the non-escaping solution $\bar{U}_\varepsilon$ is unstable.

The idea of the proof is the following: point a) is implied by the proof of Theorem 2. For point b), if an escaping critical point $U = (u, U_{N+1})$ of G_ε exists under (5), then it is a minimizer and the set of minimizers is given by $\{(u, \pm U_{N+1})\}$ (this phenomenon is explained in [9]). Restricting to the class of symmetric vortex type maps, Lemma 4 implies that the non-escaping vortex solution $\bar{U}_\varepsilon$ is unstable if $\varepsilon < \varepsilon_N$ and therefore, an escaping symmetric vortex solution exists, which determines the set of minimizers. Of course, by the proof of Theorem 2, the non-escaping vortex solution $\bar{U}_\varepsilon$ is the unique minimizer of G_ε under (5) in dimension $N \ge 7$.

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