INTRODUCTION TO REPRESENTATIONS

OF LIE GROUPS AND LIE ALGEBRAS

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CONTENTS

NAMES

Wilhelm KILLING 1847 - 1923

Elie CARTAN 1869 - 1951

Hermann WEYL 1885 - 1955

Harold Scott MacDonald COXETER 1907 - 2003

Israel GELFAND 1913 - 2009

Eugene DYNKIN b. 1924

Joseph BERNSTEIN b. 1945

Chapter 1. Lie groups and Lie algebras. Main examples.

A. Lie groups

1.1. Haar measure. Topological groups. All our toplogies will be Hausdorff.

A topological group G is called *locally compact* if for every open neighbourhood $U \ni e$ there exist an open U' and a compact K such that $e \in U' \subset K \subset U$.

Let G be a locally compact group.

Theorem. (A. Haar and Von Neumann) *There exists a unique up to a multiplicative constant Borel measure μ_L on G which is left invariant and regular.*

Alfréd Haar (1885, Budapest, - 1933 Szeged) a Hungarian mathematician.

John von Neumann (1903, Budapest - 1955, Princeton) a famous Hungarian born American mathematician.

Left invariance: for each measurable $X \subset G$

$$\mu_l(X) = \mu_l(gX)$$

Regularity:

$$\mu(X) = \inf\{\mu(U)|U \supset X, U \text{ open } \} = \\ \sup\{\mu(K)|K \subset X, K \text{ compact } \}$$

The same for a right invariant measure, μ_R.

Define $\delta : \ G \longrightarrow \mathbb{R}_+^\times$ by

$$\int_G f(g^{-1}hg)d\mu_L(h) = \delta(g)\int_G f(h)d\mu_L(h).$$

Proposition. (i) *δ is a quasicharacter, i.e. a continuous homomorphism.*

(ii) *The measure $\delta(h)\mu_L(h)$ is right invariant.*

Corollary. *A compact group is unimodular, i.e. a left invariant Haar measure is right invariant.*

1.2. Local fields. Let F be a field (all fields will be commutative). *An absolute value* on F is a map

$$|.| : \ F \longrightarrow \mathbb{R}_{\geq 0}$$

such that (i) $|x| = 0$ iff $x = 0$; (ii) $|xy| = |x||y|$; (iii) $|x + y| \leq |x| + |y|$.

Example: all absolute values on $\mathbb{Q}$.

An absolute value defines a metrics, $d(x, y) = |x, y|$ on F and hence a topology.

We say that $|.|$ is non-trivial if there exists x with $|x| \neq 0, 1$. In that case the topology is not discrete.

A field with a non-trivial absolute value, complete wrt the corresponding metrics and locally compact is called *a local field.*

Examples: $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_q((t))$ and their finite extensions. That is all in fact.

1.2.1. The Haar measure on $\mathbb{Q}_p$.

Explicitly,

$$\mathbb{Q}_p = \{\sum_{i=n}^{\infty} a_i p^i,\ n \in \mathbb{Z},\ a_i \in \{0, 1, \ldots, p-1\}\} \supset$$

$$\mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i,\ n \in \mathbb{Z},\ a_i \in \{0, 1, \ldots, p-1\}\}.$$

By definition, $\mu(\mathbb{Z}_p) = 1$ it follows from $\mathbb{Q}_p$-invariance that $\mu(p^n\mathbb{Z}_p) = p^{-n}$ (explain this).

1.2.1.1. Exercice. (a) Show that $da/|a|_p$ is a Haar measure on the multiplicative group $\mathbb{Q}_p^\times$.

(b) Show that for a linear map $f:\ \mathbb{Q}_p^n \longrightarrow \mathbb{Q}_p^n$, $f_*\mu = |\det(f)|_p^{-n}\mu$.

1.3. Lie groups. Let F be a local field. Then a notion of an *analytic variety* over F is defined.

A **Lie group** over F is a group and an analytic variety G with both structures compatible.

This means that the the multiplication and taking the inverse maps

$$m:\ G \times G \longrightarrow G,\ Inverse: G \longrightarrow G,\ Inverse(x) = x^{-1}$$

are morphisms of analytic varieties.

Examples. Classical groups. "Die Königin" (Her Majesty). Let V be a finite dimensional vector space over F. $GL(V)$.

$GL_n(F) = GL(F^n)$. All other Lie groups are its subgroups.

Suppose the F is equipped by a *symmetric* (resp. *antisymmetric*) bilinear form (x, y). Then the group $G = \{g \in GL(V)|\ (gx, gy) = (x, y)\}$ is called orthogonal $O(V)$ (resp. symplectic $Sp(V)$).

The classical series:

$A_n = SL_{n+1},\ n \geq 1.$

$B_n = SO(2n + 1),\ n \geq 2,$

$C_n = Sp(2n),\ n \geq 2,$

$D_n = SO(2n),\ n \geq 3,\ D_3 = A_3.$

Here "S" means "with det = 1".

In this course, if not specified otherwise, the base field $F = \mathbb{R}$. In exercises we will probably discuss a little the p-adic case which is important for the number theory.

Compact and non-compact Lie groups.

Exercice. (i) Let $F = \mathbb{R}, \mathbb{C}$, $\mathbb{Z}_p$ or $\mathbb{Q}_p$. Show that

$$dg = \frac{|\prod_{i,j=1}^n dg_{ij}|}{|\det(g)|^n}$$

is a left- and right invariant measure on $GL_n(F)$.

Cf. [A. Knightly, Ch. Li, Traces of Hecke operators], 7.6.

(ii) Consider the group

$$G = SL_2(F) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$$

Consider a three form

$$\omega = \frac{dbdcdd}{d} = \frac{dadcdd}{c} = -\frac{dadbdd}{b} = -\frac{dadbdc}{a} \in \Omega^3(G)$$

Show that

$$dadbdcdd = d(ab - cd)\omega.$$

Show that ω is left and right G-invariant. Deduce that $dg = (i/2)^3\omega\bar{\omega}$ is a Haar measure on $G(\mathbb{C})$.

Cf. [I.M.Gelfand, M.I.Graev, A.N.Vilenkin, Integral geometry and representation theory, Generalized functions, v. 5], Ch. IV, Appendix.

B. Lie algebras

1.4. Lie algebras. *Motivation*: let $X, Y \in \mathrm{End}(V)$. For small ϵ $1 + \epsilon X \in GL(V)$.

$$(1 + X\epsilon)(1 + Y\epsilon)(1 + X\epsilon)^{-1} = 1 + [X, Y]\epsilon + O(\epsilon^2)$$

where $[X, Y] = XY - YX$.

Definition. A Lie algebra over a field F is a vector space $\mathfrak{g}$ equipped with a bilinear pairing $[.,] :\ \mathfrak{g} \times \mathfrak{g} \longrightarrow F$ satisfying two axioms:

(i) skew symmetry:

$$[x, y] = -[y, x];$$

(ii) the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Example. A an associative ring; $[x, y] = xy - yx$. This Lie algebra will be denoted A^{Lie}.

In this course, unless specified otherwise, "a Lie algebra" will mean "a finite dimensional Lie algebra".

Example: $\mathfrak{gl}(V)$.

1.5. The Lie algebra of a Lie group. Exponential map. Let X be a smooth variety (over $\mathbb{R}$); $C(X)$: the algebra of smooth functions $X \longrightarrow \mathbb{R}$.

The Lie algebra of vector fields:

$$\mathcal{T}(X) = Der(C(X)),$$

Lie bracket = the commutator (check that a commutator of two derivations is a derivation).

Local form.

Each vector field $\tau \in \mathcal{T}(X)$ gives rise to a tangent vector $\tau(x)$ in the tangent space $T_x X$ at each point $x \in X$ (explain in the exercises).

Let G be a Lie group. For each $g \in G$ let

$$L_g : \ G \xrightarrow{\sim} G, \ L_g(h) = gh,$$

whence

$$L_{g*} : \ \mathcal{T}(G) \xrightarrow{\sim} \mathcal{T}(G)$$

A vector field $\tau \in \mathcal{T}(G)$ is called left invariant if for each $g \in G$, $L_{g*}(\tau) = \tau$.

By definition, $Lie(G) \subset \mathcal{T}(G)$ is the Lie subalgebra of left invariant vector fields (one has to verify that it is closed wrt to the commutator).

As a vector space,

$$\mathfrak{g} = Lie(G) = T_e G$$

For $g \in G$ define

$$\mathrm{Ad}_g : \ G \longrightarrow G, \ \mathrm{Ad}_g(h) = ghg^{-1}.$$

Let $X \in \mathfrak{g}$. There exists a unique curve $g(t) = \exp(tX) \in G$ such that the tangent vector to $g(t)$ at $t = t_0$ is equal to $X(g(t_0))$.

Thus we get a map

$$\exp:\ \mathfrak{g} \longrightarrow G,\ \exp(X) = e^X := g(1).$$

1.6. The classical Lie algebras.

Let V be a finite-dimensional vector space with a non-degenerate bilinear form $(.,.) : V \times V \longrightarrow F$. Consider the following subspace

$$\{g \in \mathfrak{gl}(V) |\ \forall x, y \in V\ (gx, y) + (x, gy) = 0\} \subset \mathfrak{gl}(V).$$

Exercise: show that it is a Lie subalgebra.

When the form is symmetric (resp. antisymmetric), this Lie subalgebra is denoted by $\mathfrak{o}(V)$ (resp. $\mathfrak{sp}(V)$).

$A_n = \mathfrak{sl}_{n+1},\ n \geq 1.$

$B_n = \mathfrak{so}(2n+1),\ n \geq 2,$

$C_n = \mathfrak{sp}(2n),\ n \geq 2,\ B_2 = C_2.$

$D_n = \mathfrak{so}(2n),\ n \geq 3,\ D_3 = A_3.$

Here "$\mathfrak{s}$" means "with tr = 0".

1.6.1. Language of categories.

Categories, functors, natural transformations.

Exercise: Ioneda's lemma.

Adjoint functors.

1.7. Enveloping algebras.

Abstract definition. Let $\mathfrak{g}$ be a Lie algebra over a field F. Its envelopping algebra $U\mathfrak{g}$ is an associative algebra $U\mathfrak{g}$ over F together with a map of Lie algebras

$$i_U : \mathfrak{g} \longrightarrow U\mathfrak{g}^{Lie}$$

having the following universal property:

for any associative F-algebra A and a map of Lie algebras $i_A : \mathfrak{g} \longrightarrow A^{Lie}$ there exists a unique morphism of associative algebras $f : U\mathfrak{g} \longrightarrow A$ such that $i_A = f \circ i_U$.

A concrete definition.

Let V be a vector space. Its *tensor algebra*:

$$T^{\cdot}V = \oplus_{n=0}^{\infty} V^{\otimes n} = T^n \oplus_{n=0}^{\infty} T^n V.$$

Its *symmetric algebra*:

$$S^{\cdot}V = T^{\cdot}V/I$$

where I is the two-sided (homogeneous) ideal generated by all elements $xy - yx,\ x, y \in V$.

The enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$:

$$U\mathfrak{g} = T^{\cdot}\mathfrak{g}/I$$

where I is the two-sided ideal in $T^{\cdot}\mathfrak{g}$ generated by all elements $xy - yx - [x, y]$.

Canonical filtration.

Let

$$F_nU\mathfrak{g} = \mathrm{Im}(T^{\geq n}\mathfrak{g} \longrightarrow U\mathfrak{g}) \subset U\mathfrak{g},\ n \geq 0.$$

The associated graded

$$\mathrm{gr}\, U\mathfrak{g} := \sum_{n=0}^{\infty} F_nU\mathfrak{g}/F_{n-1}U\mathfrak{g},\ n \geq 0$$

(where $F_{-1}\mathfrak{g} := 0$) is commutative, whence the morphism of alegebras

$$S^{\cdot}\mathfrak{g} \longrightarrow \mathrm{gr}\, U\mathfrak{g} \tag{1.7.1}$$

extending the identity on $\mathfrak{g}$.

The Poincaré - Birkhoff - Witt theorem. *The map* (1.7.1) *is an isomorphism.*

1.7.1. Exercise. The Poisson structure on $S^{\cdot}\mathfrak{g}$.

1.7.1. Exercise. The Casimir element. Let $\mathfrak{g}$ be a finite dimensonal Lie algebra, and $(,) : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ a nondegenerate symmetric bilinear form. Let $\{x_i\}, \{y_i\}$ be the dual bases of $\mathfrak{g}$, $(x_i, y_j) = \delta_{ij}$. Define

$$c = \sum x_i y_i \in U\mathfrak{g}.$$

Show that c does not depend on the choice of a base. Show that $c \in Z(U\mathfrak{g})$.

Idea. Show that a natural map

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow U\mathfrak{g}$$

is $\mathfrak{g}$-equivariant and remark that c is an image of some element $C \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$.

C. Representations of Lie groups and Lie algebras

1.8. Group representations. Let G be a topological group. A *(complex) representaion* of G is a pair (π, V) where V is a complex Banach vector space and $\pi : G \longrightarrow GL(V)$ a continuous homomorphism.

Terminology: we say also that V is a representation of G, and that V is a G-module. Instead of $\pi(g)x$ we shall write sometimes simply gx.

Notation. $V^G = \{x \in V | \ \forall g \in G \ gx = x\} \subset V$.

We will be mainly concerned with finite dimensional V.

Subreps, irreducible reps.

Standard operations.

Morphisms (intertwining operators). Notation: $Hom_G(V, V')$.

The direct sum and the tensor product of reps. Trivial representation: $\mathbf{1}$.

Dual, or contragradient rep. If (π, V) is a finite dimensional rep, we define its dual $(\pi^\vee, V^\vee)$ as follows: $V^\vee = Hom_{\mathbb{C}}(V, \mathbb{C})$,

$$\langle v, \pi^\vee(g)w\rangle = \langle \pi(g^{-1})v, w\rangle, \ v \in V, w \in V^\vee.$$

More generally, given two finite dimensional reps $(\pi_i, V_i), i = 1, 2$, there is a natural structure of a G-module on the space $Hom(V_1, V_2)$ given by

$$(gf)(x) = g(f(g^{-1}x)).$$

It follows at once that

$$Hom(V_1, V_2)^G = Hom_G(V_1, V_2).$$

Exercise. 1. Find a natural isomorphism of G modules

$$Hom(V_1, V_2) \stackrel{\sim}{=} V_1^\vee \otimes V_2.$$

The finite dimensional reps form a *abelian* $\mathbb{C}$*-linear monoidal category* to be denoted $\mathcal{R}ep(G)$. The unit: the trivial rep $\mathbf{1}$.

Exercise. 2. $G = GL_n(\mathbb{C})$, $V = \mathbb{C}^n$, with the natural action It is called the *fundamental representaton.*

(a) Show that V is irreducible.

(b) Show that

$$V \otimes V = S^2V \oplus \Lambda^2 V$$

is a decompostion of $V \otimes V$ into irreducibles.

(c) For $n = 2$ show that S^kV are irreducible for all $k \geq 0$.

1.9. Representations of a Lie algebra $\mathfrak{g}$ over a field F: a homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ where V is a F-vector space.

Basic operations: $\otimes, M^*, Hom$.

1.10. Example. $\mathfrak{g} = \mathfrak{sl}(2)$, $\lambda \in \mathbb{C}$. The generators e, h, f act on $V(\lambda) = F[x]$ by the differential operators:

$$f = \partial_x,\ h = 2x\partial_x - \lambda,\ e = -x^2\partial_x + \lambda x$$

1.11. From a representation of a Lie group G to the representation of $\mathfrak{g} = Lie(G)$.

A morphism of Lie groups $f : G \longrightarrow G'$ induces the morphism of their Lie algebras $Lie(f) : Lie(G) \longrightarrow Lie(G')$.

In particular, a representation

$$\pi : G \longrightarrow GL(V)$$

gives rise to the representation

$$Lie(\pi) :\ \mathfrak{g} := Lie(G) \longrightarrow \mathfrak{gl}(V).$$

In practical terms: for $X \in \mathfrak{g}$

$$Lie(\pi)(X) = \lim_{t=0} \frac{e^{tX} - \mathrm{Id}_V}{t}.$$

1.12. Example. In the example 1.10, if $\lambda \in \mathbb{N}$, the action of $\mathfrak{g}$ on $V(\Lambda)$ may be integrated to an action of $G = SL(2)$.

1.13. Exercise. Irreducible finite dimensional representations of $\mathfrak{g} = \mathfrak{sl}(2)$. $\mathfrak{g}$ is defined by generators e, f, h subject to relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

The *Verma module* $M(\lambda), \lambda \in \mathbb{C}$: one generator $v = v(\lambda)$, $he = \lambda v, ev = 0$. It admits a base $\{f^i v\}, i \in \mathbb{N}$

(a) Prove that

$$hf^i v = (\lambda - 2i)f^i v, ef^i v = i(\lambda - i + 1)f^i v.$$

(b) Prove that if $\lambda \notin \mathbb{N}$ then $M(\lambda)$ is irreducible.

(c) Suppose that $\lambda = m \in \mathbb{N}$. Then $x = f^{m+1}v$ is a *singular vector*, which means $ex = 0$. Let $M' = \oplus_{i \geq m+1} \mathbb{C} f^i v$. Show that $M' \tilde{=} M(-m-2)$. By definition, $L(m) = M(m)/M'$. It is a $\mathfrak{g}$-module of dimension $m + 1$.

Show that $L(1)$ is the fundamental representation, $L(2)$ is the adjoint representation, $L(m) \tilde{=} S^m L(1)$.

(d) Show that $L(m)$ is irreducible.

Idea. Let $0 \neq L' \subset L(m)$. There exists an eigenvector $x \in L'$ of the operator h.

(e) Let L be a finite dimensional $\mathfrak{g}$-module. Show that $L \cong L(m)$ for some $m \in \mathbb{N}$.

Idea. There exists an eigenvector $y \in L$ of the operator h. By considering the elements $e^i y$ show that there exists $x \in L$, such that $ex = 0$, $hx = \lambda x$, $\lambda \in \mathbb{C}$.

Chapter 2. Representations of compact Lie groups

2.0. Cf. [B].

In this Chapter G will be a compact topological group. All representations will be complex and finite dimensional. We fix a Haar measure dg on G (recall that it is left and right invariant) normalized by

$$\int_G dg = 1.$$

Notation: $\mathcal{R}ep(G)$ the abelian monoidal category of finite dimensional reps.

$C(G)$: the commutative algebra of continuous functions $f : G \longrightarrow \mathbb{C}$.

It is equipped with an Hermitian scalar product

$$(f, g) = \int_G f(x)\overline{g(x)}dx. \tag{2.0.1}$$

2.1. Let $\pi : G \longrightarrow GL(V)$ be a representation. The *character of* π is a map $\chi_\pi : G \longrightarrow \mathbb{C}$ given by $\chi_\pi(g) = \operatorname{tr} \pi(g)$.

We have

$$\chi_\pi(hgh^{-1}) = \chi(g)$$

(one says that χ_π is a *class function*).

2.2. Harmonic (Fourier) analysis on a compact abelian group. Let G be abelian.

Examples. Connected: $T^n = U(1)^n$; disconnected: a finite abelian group.

A character of G is a continuos homomorphism

$$\chi : \ G \longrightarrow U(1)$$

The characters form a discrete abelian group $G^\vee$ (the *Pontryagin dual*).

They form a basis of the Hilbert space $L^2(G)$. Each $f \in L^2(G)$ admits the Fourier expansion

$$f(g) = \sum_{\chi\in G^\vee} a_\chi \chi(g), \ a_\chi = \int_G f(g)\overline{\chi(g)}dg.$$

The *Plancherel formula*

$$\int_G |f(g)|^2 dg = \sum_{\chi\in G^\vee} |a_\chi|^2.$$

All this may be generalized to locally compact abelian groups, cf. [W].

2.3. Let (π, V) be a rep. *An invariant inner product* on V is a positive (hence nondegenerate) Hermitian form (x, y) on V such that for all $g \in G$ $(gx, gy) = (x, y)$.

Starting from an arbitrary Hermitian positive nondegenerate form $(x, y)'$ and setting

$$(x, y) = \int_G (gx, gy)' dg,$$

one gets an invariant inner product.

Theorem. *Let (π, V) be a rep. equipped with an invariant inner product; let $W \subset V$ be a subrep. Then there exists a complement: a subrep $W' \subset V$ such that*

$$W \oplus W' \xrightarrow{\sim} V.$$

One says that $\mathcal{R}ep(G)$ is *semisimple.*

Corollary. *Each representation V is a finite direct sum of irreducibles.*

Exercise. *Show that if G is abelian then every irrep of G is one-dimensional.*

2.4. Schur lemma. Theorem. *Let $f : V \longrightarrow V'$ be an intertwining operator between irreps. Then f is either 0 or an isomorphism.*

2.5. Matrix elements. Definition. *A matrix element of a rep (V, π) is a finite sum of functions $f \in C(G)$ of the form*

$$f(g) = \langle gv, w\rangle$$

where $v \in V, w \in V^\vee$.

Equivalent definition. Fix an invariant hermitian inner product on V. A matrix element of π is a finite sum of functions of the form

$$f(g) = (\pi(g)v, w),\ \ v, w \in V.$$

It is clear that matrix elements of π form a linear subspace $M_\pi \subset C(G)$ of dimension

$$\dim M_\pi \leq (\dim V)^2.$$

If f_i is a matrix element of V_i, $i = 1, 2$, $f_1 + f_2$ (resp. $f_1 f_2$) is a matrix element of $V_1 \oplus V_2$ (resp. of $V_1 \otimes V_2$).

If $f(g)$ is a matrix element of V, $f^\vee(g) := f(g^{-1})$ is a matrix element of $V^\vee$.

It follows that matrix elements of all finite dimensional reps form a commutative subalgebra (with unit)

$$C_{mat}(G) \subset C(G).$$

It is clear that $M_\pi \subset C_{mat}(G)$.

Exercice. *Let* (π, V) *be a rep. Show that its character* $\chi_\pi \in M_\pi$.

Theorem. (i) *If* G *is finite,* $C_{mat}(G) = C(G)$.

(ii) (Peter - Weyl). *For an arbitrary compact* G $C_{mat}(G)$ *is dense in* G.

For a proof of the (ii) see [B], Chapter 3.

2.6. Regular representation. The group G acts on $C(G)$ in two ways: for $f \in C(G)$ we set

$$(\lambda(g)f)(x) = f(g^{-1}x), \ (\rho(g)f)(x) = f(xg)$$

Exercise. *Show that the following conditions on a function* $f \in C(G)$ *are equivalent:*

(i) *The functions* $\lambda(g)f, \ g \in G$, *span a finite dimensional subspace of* $C(G)$. (ii) *The functions* $\rho(g)f, \ g \in G$, *span a finite dimensional subspace of* $C(G)$. (iii) $f \in C_{mat}(G)$.

2.7. Schur orthogonality. Theorem. (i) *Let* $V_i, i = 1, 2$ *be two irreps. If* V_i *are non-isomorphic then every matrix element of* V_1 *is orthogonal to every matrix element of* V_2.

(ii) *Let* V *be an irrep with an invariant inner product* (x, y), $n = \dim V$. *Then*

$$\int_G (gx_1, y_1)\overline{(gx_2, y_2)}dg = \frac{1}{n}(x_1, x_2)(y_1, y_2).$$

for all $x_i, y_i \in V$.

Proof. (i) Fix invariant inner products on V_i. Let $f_i(g) = (\pi_i(g)v_i, w_i), i = 1, 2$. Suppose that $(f_1, f_2) \neq 0$.

Define a linear operator $T : V_1 \longrightarrow V_2$ by

$$T(v) = \int_G (\pi_1(g)v, v_1)\pi_2(g)v_2 dg.$$

It is an intertwining operator (check it!). On the other hand,

$$(w_2, T(w_1)) = (f_1, f_2) \neq 0$$

(check it!). Hence $T \neq 0$ hence T is an isomorphism since V_i are irreducible.

2.8. Characters.

2.8.1. Exercise. Show that

$$\chi_{\pi^\vee} = \bar{\chi}_\pi.$$

2.8.2. Proposition. *If* (π, V) *is an irrep,*

$$\int_G \chi_\pi(g)dg = \begin{cases} 1 & \textit{if } \pi = 1, \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We have

$$\int_G \chi_\pi(g)dg = (\chi_\pi, \chi_1) = 0$$

if $\pi \neq 1$ by Thm. 2.7. □

2.8.3. Corollary. *If* $(\pi, V) \in \mathcal{R}ep(G)$,

$$\int_G \chi_\pi(g) = \dim V^G.$$

Proof. Decompose V into a sum of irreducibles. □

2.9. Schur orthogonality for characters. Theorem. *Let* $(\pi_i, V_i) \in \mathcal{R}ep(G)$, $i = 1, 2$.

(i)

$$(\chi_{\pi_1}, \chi_{\pi_2}) = \dim Hom_G(V_1, V_2).$$

(ii) *If* π_i, $i = 1, 2$ *are irreducible,*

$$(\chi_{\pi_1}, \chi_{\pi_2}) = \begin{cases} 1 & \textit{if } \pi_1 \cong \pi_2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. (i) Apply 2.8.3 to $V = Hom_{\mathbb{C}}(V_1, V_2)$. □

Chapter 3. Complex semisimple Lie algebras:

structure and representations.

Cf. [S], [Ber].

3.0. In this Chapter, unless specified otherwise, all Lie algebras will be over $\mathbb{C}$ and finite dimensional.

The Killing form:

$$(x, y) = Tr(\operatorname{Ad} x \operatorname{Ad} y).$$

Exercice. Prove that (x, y) is $\mathfrak{g}$-*invariant*, i.e.

$$([x, y], z) + (y, [x, z]) = 0.$$

3.1. Definitions. *A simple Lie algebra* is a Lie algebra $\mathfrak{g}$ which does not contain proper ideals.

Examples: $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$.

A semisimple Lie algebra: a finite direct sum of simple ones.

Other equivalent definitions of semisimple Lie algebras:

(i) $\mathfrak{g}$ does not contain abelian ideals.

(ii) (Killing - Cartan criterion) The Killing form is non-degenerate.

3.2. Cartan subalgebras. Root space decomposition. Let $\mathfrak{g}$ be a Lie algebra.

The *lower central series*: a series of ideals

$$C^1\mathfrak{g} = \mathfrak{g} \supset C^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset \ldots \supset C^i\mathfrak{g} \supset \ldots$$

where

$$C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}].$$

$\mathfrak{g}$ is called **nilpotent** if there exists i such that $C^i\mathfrak{g} = 0$.

If $\mathfrak{g} \subset \mathfrak{g}$ be a subalgebra. Its normalizer $N(\mathfrak{g}') = \{x \in \mathfrak{g} | \ [x, \mathfrak{g}'] \subset \mathfrak{g}'\]$; It is the largest ideal containg $\mathfrak{g}'$.

A **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra such that (i) $\mathfrak{h}$ is nilpotent, and (ii) $\mathfrak{h} = N(\mathfrak{h})$.

Every Lie algebra contains a Cartan subalgebra.

If $\mathfrak{g}$ is semisimple then all Cartan subalgebras are abelian and they are all conjugated.

Example. Let $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{h} \subset \mathfrak{g}$ — the abelian Lie subalgebra of diagonal matrices, it is a Cartan subalgebra; $\mathfrak{n}_\pm \subset \mathfrak{g}$ — the subalgebra of upper (resp. lower) triangular matrices with zeros on the diagonal, they are nilpotent.

Then

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{g} \oplus \mathfrak{n}_+.$$

Moreover,

$$\mathfrak{n}_+ = \oplus_{i<j}\mathbb{C} \cdot E_{ij},\ \ \mathfrak{n}_- = \oplus_{i>j}\mathbb{C} \cdot E_{ij}.$$

From now on $\mathfrak{g}$ will be semisimple. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For a character

$$\lambda \in \mathfrak{h}^* = Hom_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$$

we denote

$$\mathfrak{g}_\chi = \{x \in \mathfrak{g} |\ \forall h \in \mathfrak{h}\ [h, x] = \chi(h)x\}$$

Obviously $\mathfrak{h} \subset \mathfrak{g}_0$.

An element $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$ is called **a root**, and $\mathfrak{g}_\alpha$ is called the root subspace.

All roots form a finite subset $R \subset \mathfrak{h}^*$.

Theorem. $\mathfrak{g}_0 = \mathfrak{h}$ *and*

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in R}\ \mathfrak{g}_\alpha.$$

All root subspaces $\mathfrak{g}_\alpha$ *are one-dimensional.*

The finite subset $R \subset \mathfrak{h}^*$ is remarkable.

3.3. Root systems; Weyl group. Cf. [S], [Bour], [H].

Let V be a real or complex vector space equipped with a symmetric nondegenerate bilinear form $(.,.)$.

For $\alpha \in V$ define $s_\alpha : V \stackrel{\sim}{=} V$ by

$$s_\alpha(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$$

It is the orthogonal reflection wrt hyperplane $\alpha^\perp = \{x \in V |\ (x, \alpha) = 0\}$.

A **root system** in V is a finite subset $R \subset V \setminus \{0\}$ which spans V as a vector space and such that

(a) for each $\alpha \in R$, $s_\alpha(R) \subset R$.

(b) for each $\alpha, \beta \in R$,

$$s_\alpha(\beta) - \beta = n\alpha$$

with $n \in \mathbb{Z}$.

Irreducible and reduced root systems.

Remark. Given a real root system $R \subset V$, $R \subset V \subset V_{\mathbb{C}}$ is a complex root system, and vice versa, any complex root system is a complexification of a unique real root system.

The **Weyl group** W of R is the subgroup of $O(W)$ generated by all $s_\alpha,\ \alpha \in R$.

Since R spans V, W is a subgroup of $Aut(R)$, whence finite.

The bilinear form is W-invariant, i.e.

$$(wx, wy) = (x, y),\ w \in W.$$

Example. The root system of type A_n, $n \geq 1$. $W = S_n$.

Positive and negative roots. Bases.

Now suppose V to be real, $\dim V = r$.

Let $t \in V^*$ be such that for all $\alpha \in R$ $t(\alpha) \neq 0$.

Set

$$R_+ = \{\alpha \in R |\ t(\alpha) > 0\},\ R_- = \{\alpha \in R |\ t(\alpha) > 0\}.$$

Then $R = R_+ \coprod R_-$. Since for all $\alpha \in R$ $-\alpha = s_\alpha(\alpha) \in R$, $R_- = -R_+$.

There exists a unique subset $\{\alpha_1, \ldots, \alpha_r\} \subset R_+$, **a base of** R such that every $\alpha \in R$ is equal to a linear combination

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i,\ n_i \in \mathbb{N}.$$

Dual roots. For $\alpha \in R$ set

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

The **Cartan matrix** $A = (a_{ij})$,

$$a_{ij} = (\alpha_i, \alpha_j^\vee).$$

We have

$$a_{ij} a_{ji} = 4 \cos^2 \phi_{ij} := b_{ij},$$

where ϕ_{ij} is the angle between α_i and α_j.

Since $a_{ij} \in \mathbb{Z}$, $b_{ij} \in \{0, 1, 2, 3, 4\}$.

We have $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$.

Possible cases for $i \neq j$:

a_{ij}	a_{ji}	b_{ij}	m_{ij}
0	0	0	2
-1	-1	1	3
-2	-1	2	4
-3	-1	3	6

Here m_{ij} is the order of $s_i s_j$ where $s_i = s_{\alpha_i}$.

Theorem. *The Weyl group is defined by generators s_i, $1 \leq i \leq r$, and relations $(s_i s_j)^{m_{ij}} = 1$.*

Dynkin diagram. Vertices: in bijection with the sipmle roots.

Two vertices are joined by b_{ij} intervals.

If the lengths of α_i and α_j are different, (which is equivalent to $b_{ij} > 1$), one draws the direction of the arrows from α_i to α_j if $|\alpha_i| < |\alpha_j|$.

List of irreducible reduced root systems:

$$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$$

Example. A_n

3.4. The structure of a simple Lie algebra. Let $\mathfrak{g}$ be a simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, whence the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in R} \ \mathfrak{g}_\alpha.$$

We have the Killing form on $\mathfrak{g}$ which gives by restriction a symmetric nondegenerate bilinear form on $\mathfrak{h}$, and as a consequence, on $\mathfrak{h}^*$.

Theorem. *R is an irreducible reduced (complex) root system in $\mathfrak{h}^*$.*

Choose a base $\{\alpha_1, \dots, \alpha_r\} \subset R$, whence the set of positive roots $R_+ \subset R$.

Set

$$\mathfrak{n}_+ = \mathfrak{n} = \oplus_{\alpha > 0} \mathfrak{g}_\alpha, \ \mathfrak{n}_- = \oplus_{\alpha < 0} \mathfrak{g}_\alpha;$$

these are nilpotent subalgebras of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Denote

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$$

It is a **Borel subalgebra**.

Example. $\mathfrak{g} = \mathfrak{sl}_0 = Lie(SL_n)$, $\mathfrak{b} = Lie(B)$ where $B \subset SL_n$ is the subgroup of upper triangular matrices, the same with $\mathfrak{b}$.

Set

$$\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \alpha > 0.$$

The subspace

$$\mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha \subset \mathfrak{g}$$

is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$.

Let $H_\alpha \in \mathfrak{h}_\alpha$ denote the unique element such that $\alpha(H_\alpha) = 2$.

Set $H_i = H_{\alpha_i},\ i = 1, \ldots r$. Choose any nonzero $E_i \in \mathfrak{g}_{\alpha_i}$ and then define $F_i \in \mathfrak{g}_{-\alpha_i}$ by the condition $[E_i, F_i] = H_i$.

Theorem (Serre). *The Lie algebra $\mathfrak{g}$ may be defined by generators $E_i, H_i, F_i,\ 1 \leq i \leq r$ and relations*

$$[H_i, H_j] = 0,$$
$$[E_i, F_j] = H_i \delta_{ij},$$
$$\mathrm{ad}(E_i)^{-a_{ij}+1}(E_j) = 0,\ i \neq j,$$
$$\mathrm{ad}(F_i)^{-a_{ij}+1}(F_j) = 0,\ i \neq j.$$

The last relations are called the *Serre relations.*

3.4. Center of the enveloping algebra. Harish-Chandra theorem. Cf. [D], Chapitre 7, §7.4.

We fix a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; let $Z(U\mathfrak{g})$ be the center of $U\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ acts by the adjoint representation on $U\mathfrak{g}$. Let

$$U\mathfrak{g}_0 = \{x \in U\mathfrak{g} |\ \forall h \in \mathfrak{h}\ \ \mathrm{ad}(h)(x) = 0\};$$

it is a subalgebra of $U\mathfrak{g}$ containing $Z(U\mathfrak{g})$.

Fix a base $\Delta \subset R$, whence $\mathfrak{n}_\pm \subset \mathfrak{g}$.

Lemma. (i) *The subspace*

$$L = U\mathfrak{g}_0 \cap \mathfrak{n}_+ U\mathfrak{g} = U\mathfrak{g}_0 \cap \mathfrak{n}_- U\mathfrak{g}$$

is a two-sided ideal in $U\mathfrak{g}_0$.

(ii)

$$U\mathfrak{g}_0 = U\mathfrak{h} \oplus L.$$

This lemme is an easy corollary of the PBW theorem, cf. [D], Lemma 7.4.2.

Let

$$j:\ U\mathfrak{g}_0 \longrightarrow U\mathfrak{h}$$

denote the projection.

We can identify

$$U\mathfrak{h} = S\mathfrak{h}$$

with the algebra $\mathbb{C}[\mathfrak{h}^*]$ of polynomial functions on $\mathfrak{h}^*$.

Let

$$d: \ \mathbb{C}[\mathfrak{h}^*] \longrightarrow \mathbb{C}[\mathfrak{h}^*]$$

("décalage", or shift) denote the homomorphism which takes $p(\lambda)$ to $p(\lambda - \rho)$.

Consider the composition

$$HC: \ Z(U\mathfrak{g}) \subset U\mathfrak{g}_0 \longrightarrow U\mathfrak{h} \xrightarrow{d} U\mathfrak{h}$$

Theorem. *The map HC does not depend on a choice of a base Δ and induces an algebra isomorphism*

$$HC: \ Z(U\mathfrak{g}) \stackrel{\sim}{=} U\mathfrak{h}^W.$$

Example. $\mathfrak{sl}_2$.

3.5. Verma modules. Irreducible representations.

Induced representations. Let $\mathfrak{a} \subset \mathfrak{b}$ be a Lie algebra and a Lie subalgebra, and M a representation of $\mathfrak{a}$.

$$\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{b}} M = U\mathfrak{b} \otimes_{U\mathfrak{a}} M$$

Verma modules. We fix $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, with $\mathfrak{g}$ semisimple, $\mathfrak{b}$ a Borel, $\mathfrak{h}$ a Cartan.

Let $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ be a character. It gives rise to a one-dimensional $\mathfrak{h}$-module $\mathbf{1}_\lambda$, with

$$h \cdot 1 = \lambda(h) \cdot 1$$

and hence, by restriction, using the projection $\mathfrak{b} \longrightarrow \mathfrak{h}$, a $\mathfrak{b}$-module $\mathbf{1}_\lambda$. By definition,

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbf{1}_{\lambda - \rho}.$$

We shall denote 1_λ the highest vector of $M(\lambda)$. The map $x \mapsto x \cdot 1_\lambda$ idnuces an ismorphism

$$U\mathfrak{n}_- \xrightarrow{\sim} M(\lambda).$$

3.5.1. Example. $\mathfrak{g} = \mathfrak{sl}_2$, E, F, H the standard basis. $R = \{\alpha, -\alpha\}$, $\alpha(H) = 2$, $\omega = \alpha/2 = \rho$. Let $\lambda(H) = a \in \mathbb{N}$. Thus, $H1_\lambda = a - 1$.

A base of $M(\lambda)$: $\{F^i 1_\lambda, i \geq 0\}$, and $F^i 1_\lambda \in M(\lambda)_{\lambda - (2i+1)\omega}$.

$$EF^i 1_\lambda = i(???) \cdot F^{i-1} 1_\lambda$$

Suppose $a \in \mathbb{N}^*$, thus $\lambda = a\rho$. The only nontrivial element of the Weil group $s(\alpha) = -\alpha$, $s(\lambda) = -\lambda$.

The vector

$$x = F^a 1_\lambda \in M(\lambda)_{s(\lambda)-\omega}$$

is *singular*, which means by definition $Ex = 0$. Thus we can define an embedding

$$f: \ M(s\lambda) \longrightarrow M(\lambda), \ f(1_{s\lambda}) = x.$$

The quotient

$$L(\lambda) := M(\lambda)/f(M(s\lambda))$$

is the irreducible representation of dimension a.

So we have an exact sequence of reps

$$0 \longrightarrow M(s\lambda) \xrightarrow{f} M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

If $a \notin \mathbb{N}$, $M(\lambda)$ does not contain singular vectors; it is irreducible. $\square$

Let us pass to the general case.

Weight lattice:

$$P(R) = \{\lambda \in \mathfrak{h}^* | \forall \alpha \in R \ (\lambda, \alpha^\vee) \in \mathbb{Z}\},$$

it is a free abelian group of rank r.

Fix a base $B = \{\alpha_1, \ldots, \alpha_r\} \subset R\}$, denote $s_i = s_{\alpha_i}$ (the simple reflections).

The base of P: $\{\omega_1, \ldots, \omega_r\}$,

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}.$$

ω_i are called **the fundamental weights**.

Remark. Let $Q(R) \subset \mathfrak{h}^*$ (*the root lattice*) be the abelian generated by R. It is also a free abelian group of rank r, and $Q(R) \subset P(R)$.

The cone of **dominant weights**:

$$P_{++} = \{\lambda \in P | \ \forall \ i \ (\alpha_i^\vee, \lambda) \in \mathbb{N}\} = \oplus_{i=1}^r \mathbb{N}\omega_i.$$

$$\rho = \frac{1}{2} \sum_{\alpha \in \ R_+} \alpha = \sum_{i=1}^r \omega_i.$$

Let $\lambda \in P_{++} + \rho$. For each $1 \leq i \leq r$ there is an inclusion

$$f_i : M(s_i\lambda) \hookrightarrow M(\lambda),$$

where

$$f_i(1_{s_i\lambda}) = F_i^{(\lambda,\alpha_i) CHECK!} 1_\lambda,$$

We define

$$L(\lambda) = M(\lambda)/\sum_i f(M(s_i\lambda)),$$

so that we have an exact sequence

$$\oplus_{i=1}^{r} M(s_i\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0 \qquad (3.5.1)$$

Theorem. *$L(\lambda)$ is a finite dimensional irreducible representation (of highest weight $\lambda - \rho$).*

All irreducible finite dimensional representations are of the form $L(\lambda)$ for some $\lambda \in P_{++} + \rho$; they are pairwise nonisomorphic.

Any finite dimensional representation of $\mathfrak{g}$ is a direct sum of irreducibles.

3.6. BGG resolution.

Length on the Weyl group. For $w \in W$ we write $\ell(w)$ for the minimal number of simple reflections in a decomposition

$$w = s_{i_1} \dots s_{i_n}.$$

The sign:

$$\epsilon(w) = (-1)^{\ell(w)} = \det w.$$

The maximal length is

$$N := \max_{w\in W}\{\ell(w)\} = \dim \mathfrak{n} = |R|/2.$$

Denote

$$W_i = \{w \in W |\ \ell(w) = i\},\ n_i = |W_i|,$$

so that $n_0 = 1$, $n_1 = r$. We have the *symmetry property*

$$n_i = n_{N-i}.$$

Theorem (Chevalley)

$$\sum_{i=0}^{N} n_i t^i = \prod_{j=1}^{r} \frac{t^{d_j} - 1}{t - 1}.$$

The numbers $d_1, \dots, d_r$ are called *exposants*.

Example. For $W = S_{r+1}$, $d_j = j + 1$.

The exact sequence (3.5.1) may be prolonged.

Theorem, [BGG]. *Let $\lambda \in P_{++} + \rho$. There is an exact sequence*

$$0 \longrightarrow C_N(\lambda) \longrightarrow \dots \longrightarrow C_1(\lambda) \longrightarrow C_0(\lambda) \longrightarrow L(\lambda) \longrightarrow 0 \qquad (3.6.1)$$

with

$$C_i(\lambda) = \oplus_{w\in W_i} M(w\lambda).$$

Let us explain the differentials in this sequence. Let us write $w \longrightarrow w'$ if $\ell(w') = \ell(w) + 1$ and $w' = sw$ for some simple reflection $s = s_i$. In this case we have a natural inclusion

$$f_{w,w'} = f_s : \ M(sw\lambda) \hookrightarrow M(w\lambda)$$

as in 3.5. If $w \longrightarrow w' \longrightarrow w''$ then

$$f_{w,w'} f_{w',w''} = f_{w,w''}.$$

Lemma 1. *If* $\ell(w'') = \ell(w) + 2$ *then the number of* w' *such* $w \longrightarrow w' \longrightarrow w''$ *is either* 0 *or* 2.

In the last case let $w \longrightarrow w'_i \longrightarrow w''$, $i = 1, 2$ be the two chains, let us call such a situation *a square*. Then

$$f_{w,w'_1} f_{w'_1,w''} = f_{w,w'_2} f_{w'_2,w''}.$$

Lemma 2. *We can find signs*

$$b(w, w') = \pm 1, \ w \longrightarrow w'$$

in such a way that for each square as above

$$b(w, w'_1) b(w'_1, w'') = -b(w, w'_2) b(w'_2, w'').$$

We define the differentials

$$d_i : \ C_i(\lambda) \longrightarrow C_{i-1}(\lambda)$$

to have nonzero components

$$d_{i;w,w'} = b(w, w') f_{w,w'} : \ M(w'\lambda) \longrightarrow M(w\lambda)$$

for each $w \longrightarrow w'$. Lemma 2 implies that $d^2 = 0$.

3.7. Hermann Weyl character formula. Cf. [Bour], Ch. VIII, §9.

Laurent series wrt positive weights. We consider the group ring $\mathbb{Z}[P]$ with the base $e^\lambda, \ \lambda \in P$,

$$e^\lambda \cdot e^{\lambda'} = e^{\lambda+\lambda'}$$

The elements of $\mathbb{Z}[P]$ are finite sums $\sum_{\mu \in P} a(\mu) e^\mu$ functions $a : P \longrightarrow \mathbb{Z}$ with finite support, so

$$\mathbb{Z}[P] \subset \mathbb{Z}^P \subset \mathbb{Z}^{\mathfrak{h}^*}$$

where $X^Y = \{f : \ X \longrightarrow Y\}$; if X is an abelian group then X^Y is an abelian.

Let $\mathcal{P}_-(\mathfrak{h}^*)$ denote the set of subsets $S \subset \mathfrak{h}^*$ contained in a finite union of subsets of the form $\mu - P_{++}$, $\mu \in \mathfrak{h}^*$.

Let

$$\mathbb{Z}[[P]] \subset \mathbb{Z}^{\mathfrak{h}^*}$$

denote the subgroup of functions a whose support $\mathrm{Supp}(a) \in \mathcal{P}_-(\mathfrak{h}^*)$.

Define the multiplication in $\mathbb{Z}[[P]]$ as the convolution

$$(ab)(\mu) = \sum_\nu a(\nu)b(\mu - \nu),$$

the sum being finite due to the condition on the support; this multiplication extends the multiplication in $\mathbb{Z}[P]$.

We can regard elements $a \in \mathbb{Z}[[P]]$ as formal series

$$a = \sum_{\lambda \in \mathfrak{h}^*} a(\lambda)e^\lambda;$$

they are "Laurent series" with finitely many positive terms but possibly infinitely many egative ones.

Discrminant, or Weyl denominator.

$$D = \prod_{\alpha > 0}(e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha > 0}(1 - e^{-\alpha}).$$

Theorem.

$$D = \sum_{w \in W} \epsilon(w)e^{w\rho}.$$

For $R = A_n$ this is the Vandermonde determinant.

Kostant partition function.

Let

$$Q_+ = P_{++} \cap Q = \oplus_{i=1}^r \mathbb{N}\alpha_i.$$

For $\lambda \in Q_+$ define

$$K(\lambda) = \mathrm{Card}\{(n_\alpha)_{\alpha>0} |\ \lambda = \sum n_\alpha \alpha\}$$

and set

$$K = \sum_{\lambda \in Q_+} K(\lambda)e^{-\lambda} \in \mathbb{Z}[[P]]$$

Category $\mathcal{O}$. Characters.

We define the category $\mathcal{O}$ as the category of $\mathfrak{g}$-modules M which are

(a) $\mathfrak{h}$-diagonalizable, so

$$M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$

where

$$M_\lambda = \{x \in M |\ \forall h \in \mathfrak{h}\ hx = \lambda(h)x\}.$$

(b)

$$\forall\lambda \in \mathfrak{h}^* \dim M_\lambda < \infty$$

Thus we get a function

$$ch(M):\ \mathfrak{h}^* \longrightarrow \mathbb{N},\ ch(M)(\lambda) = \dim M_\lambda.$$

The characters λ such that $\dim M_\lambda \neq 0$ are called the *weights* of M, and $\dim M_\lambda$ - the multiplicity of λ.

Example. The adjoint representation $\mathfrak{g}$, its weights = the roots, all multiplicities = 1.

(c)

$$\mathrm{Supp}\, ch(M) \in \mathcal{P}_-(\mathfrak{h}_*).$$

Thus we can consider ch_M as an element of $\mathbb{Z}[[P]]$,

$$ch(M) = \sum_{\lambda\in\mathfrak{h}^*} \dim M_\lambda e^\lambda.$$

Examples: Verma modules, irreducible modules.

Examples. Let $\mathfrak{g} = \mathfrak{sl}_2$, $L(m)$ - the irreducible $\mathfrak{g}$-module with highest weight $m\omega$, $m \in \mathbb{N}$. We have $\dim L(m) = m+1$ its nonzero weights are

$$m\omega, (m-2)\omega, \ldots, -m\omega,$$

each weight has multiplicity 1. Thus

$$ch(L(m)) = \sum_{i=0}^{m} e^{(-m+2i)\alpha/2} = \frac{e^{(m+1)\alpha/2} - e^{-(m+1)\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}}$$

Theorem (Hermann Weyl character formula).

$$ch(L(\lambda)) = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda)}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)}}.$$

Attention: the highest weight of $L(\lambda)$ is $\lambda - \rho$.

Theorem

$$\dim(L(\lambda)) = \prod_{\alpha>0} \frac{(\lambda,\alpha)}{(\rho,\alpha)}.$$

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