Lecture notes :

Hamilton-Jacobi equations from biology

Sepideh Mirrahimi *

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*Institut Montpelliérain Alexander Grothendieck, UMR5149, CNRS, Université de Montpellier, Place Eugène Bataillon 34090 Montpellier, France

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1 Introduction

The objective of these notes is to present the main elements of an approach based on Hamilton-Jacobi equations with constraint that has been developed since 2005 [DJMP05] to study the evolutionary dynamics of populations structured by quantitative traits.

The ability of an individual to survive and to reproduce depends indeed on genetic or phenotypic parameters called traits. Several mechanisms contribute to the evolution of the living organisms: heredity, i.e. vertical transmission of the ancestral trait to the offspring, mutation which generates variability in the trait values, selection which results from the interaction of individuals with their environment, and horizontal gene transfer, i.e. horizontal exchange of genetic information between individuals during their life time. Is it possible to predict the survival or extinction of a population which is subject to such mechanisms? Can we characterize the phenotypic distribution of such population? Such types of questions emerge for instance in the study of the impact of an environmental change, e. g. climate change, on a population or in the investigation of an efficient therapy avoiding resistance of bacteria or cancer cells to medications [LMR10, CLM10, KM14, BPnMGI17].

When studying large populations, the evolutionary dynamics of phenotypically structured populations, subject to asexual reproduction, selection and mutation, may be described by parabolic Lotka-Volterra type integro-differential models. In Section 1.1 we introduce some pioneer models in this field and we provide explicit solutions in some particular cases. In Section 1.2 we introduce a framework where the Hamilton-Jacobi approach can be applied. This approach allows to study more general situations.

1.1 Models and explicit solutions

Our first model, which is also the most natural one, is the following

$$\begin{cases} \partial_t n = n\, R(z,I) + \int K(y-z)\, b(y,I)\, n(t,y)\, dy, \qquad (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ I(t) = \int_{\mathbb{R}^d} n(t,y) dy, \\ n(0,z) = n_0(z). \end{cases} \tag{1}$$

Here, z corresponds to a phenotypic trait and $n(t,z)$ denotes the density of trait z at time t. The integral term $I(t)$ corresponds to the total population size and can also be interpreted as an indicator of the total consumption of a nutrient. We represent the birth (without mutation) and death rate of the individuals with phenotype z by a growth rate $R(z,I)$ (called also the fitness function) which depends on the phenotype and the environmental feedback $I(t)$. The last integral term models the mutations; the term $b(y, I)$ corresponds to the birth rate of individuals with trait y such that the offspring has a mutated trait and K corresponds to the distribution of mutational effects.

A variant of this model, known as the continuum of alleles model, was first suggested by Kimura [Kim65]. Later such equation was derived rigorously from stochastic individual based models in the limit of large populations by Champagant, Férrière and Méléard [CFM06, CFM08].

Another option is to model the mutations with a diffusion term instead of an integral kernel:

$$\begin{cases} \partial_t n - \sigma^2 \Delta n = n\, R(z, I), & (t, z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ I(t) = \int_{\mathbb{R}^d} n(t, y)\psi(y)dy, \\ n(0, z) = n_0(z). \end{cases} \tag{2}$$

Here, the parameter σ is a positive constant that measures the effect of the mutations. This model was also suggested by Kimura [Kim65] as an approximation of (1) considering small mutational effects (see also [BÖ0, p. 239-241] for a discussion on the range of the validity of the above equation as an approximation of (1)). Later this model was directly derived from a stochastic individual based model considering small effects of mutations but with important mutation rate (or with large birth and death rates) [CFM06, CFM08].

1.1.1 Explicit Gaussian solution for a model with quadratic stabilizing selection

A typical example of the growth rate R is given by

$$R(z, I) = r(z) - \kappa I, \quad \psi \equiv 1, \quad z \in \mathbb{R},$$

$$r(z) = r_{\max} - s(z - \theta)^2,$$

with $r(z)$ known as a quadratic stabilizing selection function. Here, $r_{\max}$ corresponds to the maximal growth rate of individuals. The quadratic term in the expression of $r(z)$ means that the optimal trait is give by $z_0 = \theta$ and having a non-optimal trait has a cost given by $(z - \theta)^2$ times a coefficient s which is called the pressure of the selection. In this example, we consider a uniform competition between the traits, with intensity κ.

For this particular growth rate R, and considering a well-prepared initial condition, (2) has explicit Gaussian solutions:

$$n(t,z) = \frac{\rho(t)}{\sqrt{2\pi\sigma}} \exp\big(- \frac{\sqrt{s}(z-\mu(t))^2}{2\sigma}\big),$$

with

$$\mu'(t) = 2\sqrt{s}\sigma(\theta - \mu),$$

$$\rho'(t) = \rho(t)\big(r_{\max} - \sqrt{s}\sigma - s(\theta-\mu)^2 - \kappa\rho\big).$$

Note that $\rho(t)$ and $\mu(t)$ correspond respectively to the total size of the population, and the mean phenotypic trait at time t. In this example, the variance of the phenotypic density is constant, given by

$$v \equiv \frac{\sigma}{\sqrt{s}}.$$

It is often more practical for biologists to deal with the moments of the phenotypic distribution rather than the phenotypic distribution itself. These quantities are indeed more easily measurable in the biological experiments. Let's give some interpretation of the above expressions :

$$\rho' = \kappa\rho\,\Big(\underbrace{\frac{r_{\max}}{\kappa}}_{\text{maximal total size}} - \underbrace{\frac{\sqrt{s}\sigma}{\kappa}}_{\text{demographic load 1}} - \underbrace{\frac{s}{\kappa}(\theta-\mu)^2}_{\text{demographic load 2}} - \rho\Big).$$

Here, the first term at the r.h.s. $(\frac{r_{\max}}{\kappa})$ corresponds to the size of a population where all the individuals have the optimal trait θ. The second term $(\frac{\sqrt{s}\sigma}{\kappa})$ corresponds to a demographic load due to the mutations. The third term corresponds to a demographic load due to the distance of the phenotypic mean to the optimum. In long time, the population distribution will be centered around the optimal trait (μ converges to θ) and this second demographic load will disappear, while the first load due to the mutations will persist.

The dynamics of the phenotypic mean can be rewritten as below

$$\mu'(t) = \underbrace{\frac{\sigma}{\sqrt{s}}}_{\text{phenotypic variance}} \cdot \underbrace{2s(\theta-\mu)}_{\text{gradient of the fitness } r(z)}.$$

The equation above is known indeed as the Lande's equation in *Quantitative Genetics* [Lus37, Lan79, LA83], obtained under Gaussian assumption on the phenotypic distribution. *Quantitative Genetics* is a theory in evolutionary biology that studies the evolution of continuous traits. It is also related to the so-called canonical equation in *Adaptive Dynamics*, another theory in evolutionary biology, that considers very rare mutations leading to discrete distributions.

1.1.2 Beyond the Gaussian solutions

In the previous subsection, we saw a particular situation where we can find Gaussian explicit solutions for our model. Gaussian distributions may also provide good approximations of the solutions in other situations. Many of the theoretical results in *Quantitative Genetics* are indeed based on such approximations. However, when considering non-Gaussian initial conditions, other fitness functions or in presence of heterogeneity, these approximations are not always satisfying. In particular, when considering more complex models the phenotypic distribution may not be anymore unimodal and several dominant traits may appear in the population. How can one describe such complex distributions?

In what follows we introduce a method based on Hamilton-Jacobi equations that allows to study general models from *Quantitative Genetics*, in a regime where the mutations have small effect (σ small in (2)).

1.2 The regime of small mutations and concentration

The integro-differential equations presented above have the property that in the limit of small diffusion, representing the mutations, and in long time the solution concentrates on one or several evolving points corresponding to dominant traits. In this section, we show how an approach based on Hamilton-Jacobi equations allows to study such phenomena. This approach was first suggested in [DJMP05] and the first results were provided in [BP07, PB08]. Note that related tools were already used to study the propagation phenomena for local reaction-diffusion equations (see for instance [Fre85, ES89]).

To consider small mutation effects, we assume that the variance of the mutational effects is of order ε^2, with ε a small parameter. To take into account this assumption in our models, we replace $K(\cdot)$ by $\frac{1}{\varepsilon^d}K(\frac{\cdot}{\varepsilon})$, in the case of (1) and we take $\sigma = \varepsilon^2$ in the case of (2). Our objective, being to perform an asymptotic analysis, we also make a change of variable in time:

$$t \mapsto \frac{t}{\varepsilon},$$

which accelerates the dynamics such that we can observe the effect of small mutations. We thus define the rescaled functions

$$n_\varepsilon(t,z) = n(\frac{t}{\varepsilon}, z), \quad I_\varepsilon(t) = I(\frac{t}{\varepsilon}).$$

In the first model, the rescaled equation becomes

$$\begin{cases} \varepsilon\partial_t n_\varepsilon = n_\varepsilon\, R(z, I_\varepsilon) + \int \frac{1}{\varepsilon^d} K(\frac{y-z}{\varepsilon})\, b(y, I_\varepsilon)\, n_\varepsilon(t,y)\, dy, \qquad (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ I_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,y)\psi(y)dy, \\ n_\varepsilon(0,z) = n_{\varepsilon,0}(z). \end{cases} \tag{3}$$

In the second model, the rescaled equation is written

$$\begin{cases} \varepsilon\partial_t n_\varepsilon - \varepsilon^2 \Delta n_\varepsilon = n_\varepsilon\, R(z, I_\varepsilon), \qquad (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ I_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,y)\psi(y)dy, \\ n_\varepsilon(0,z) = n_{\varepsilon,0}(z). \end{cases} \tag{4}$$

For both of these equations, we expect that, as $\varepsilon \to 0$, the solution of (4) concentrates on a single trait, corresponding to a dominant trait which evolves in time, i.e.

$$n_\varepsilon(t,z) \rightharpoonup \rho(t)\delta(\overline{z}(t).$$

See Figure 1, for the numerical resolution of (4), for ε small and a choice of parameters compatible with our assumptions below. We observe that the solution concentrates indeed around a dominant trait which evolves in time. The fact that we expect to have a single Dirac mass

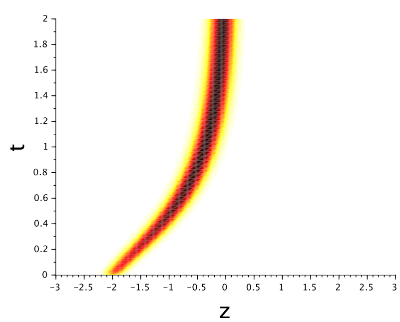

Figure 1 – The dynamics of the phenotypic density. The colors represent different values of the phenotypic density $n_\varepsilon(t,z)$, the solution to (4). The population concentrates on a dominant trait which evolves with time. Here we consider the following parameters $R(z,I) = 7 - z^2 - I$, $\psi(z) \equiv 1$, $n_{\varepsilon,0}(z) = \frac{1}{\sqrt{\varepsilon\pi}} \exp(-\frac{(z+2)^2}{\varepsilon})$ and $\varepsilon = 0.02$.

at the limit is related to the competitive exclusion principle in ecology [Lev70, Sch74]. This principle states indeed that when there are k limiting factors for the population, no more than k distinct traits may coexist. In the present model, we consider only one nutrient that is modeled via the nonlocal term I. Therefore, we expect to have a monomorphic population, except in

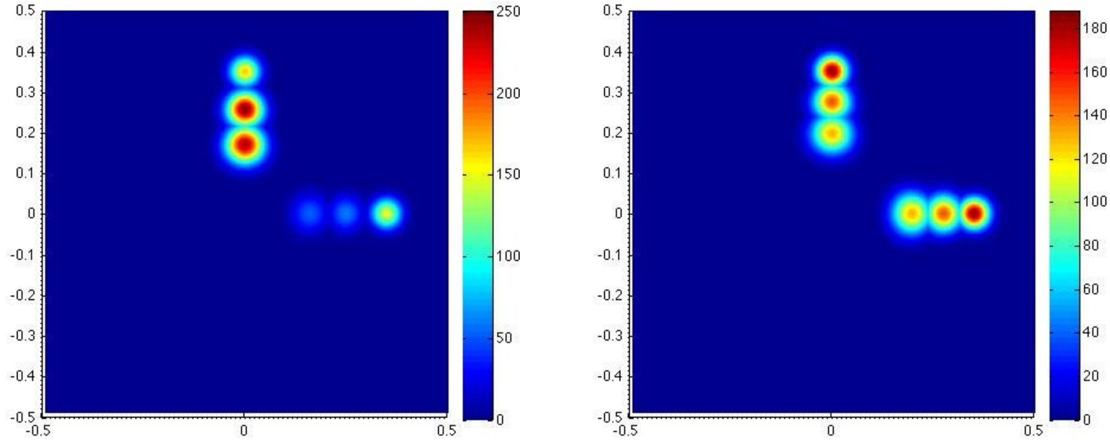

Figure 2 – The numerical resolution of (4) with $z \in \mathbb{R}^2$, $\psi = 1$, $R(z_1, z_2, I) = 3 - 5.6(R_e z_1^2 + z_2^2) - 1.5I$ and $n_{\varepsilon,0} = C_{\text{mass}}\left[\exp\left(-\frac{2.4}{\varepsilon}\left((z_1 - .25\sqrt{2})^2 + z_2^2\right)\right) + \exp\left(-\frac{2.4}{\varepsilon}\left((z_2 - .25\sqrt{2})^2 + z_1^2\right)\right)\right]$ and $\varepsilon = 0.003$. The constant C_{mass} is chosen such that the size of the initial population in the computation domain equal 0.3. At left we consider $R_e = 1.1$ and at right we choose $R_e = 1$. We depict the population density n at three consecutive times. We observe that, only for the completely symmetric case, two Dirac masses can persist and once we perturb a little bit the symmetry, one of the traits disapears.

very particular symmetric cases (see Figure 2).

1.2.1 Regime of interest: evolutionary time scale much larger than ecological time scale

Let's suppose that

$$R(z, I) = r(z) - \kappa I,$$

with $r : \mathbb{R} \to \mathbb{R}$ such that it tales a unique maximum point at $z_0 \in \mathbb{R}$. We rewrite $r(z)$ as below

$$r(z) = r_{\max} - \mu(z),$$

with $\mu(z)$ such that

$$\min_z \mu(z) = \mu(z_0) = 0, \qquad -\frac{1}{2}\mu''(z_0) = s_0.$$

Adimensional parametrization:

$$\widetilde{n}(t, z) = \frac{\kappa}{r_{\max}} n\Big(\frac{t}{r_{\max}}, \frac{z}{\sqrt{s_0/r_{\max}}}\Big), \qquad \widetilde{\mu}(z) = \frac{1}{r_{\max}} \mu\Big(\frac{z}{\sqrt{s_0/r_{\max}}}\Big).$$

Leads to

$$\frac{\partial}{\partial t}\widetilde{n} - \frac{\sigma^2 s_0}{r_{\max}^2}\Delta\widetilde{n} = \widetilde{n}\big(1 - \widetilde{\mu}(z) - \widetilde{\rho}\big),$$

with $\widetilde{\mu}(z)$ such that

$$\min_z \widetilde{\mu}(z) = \widetilde{\mu}(z_0) = 0, \qquad \frac{1}{2}\frac{\partial^2 \widetilde{\mu}}{\partial z^2}(z_0) = 1.$$

The quantity that should be sufficiently small to have a good approximation:

$$\frac{\sigma^2 s_0}{r_{\max}^2} << 1,$$

or equivalently

$$\sigma \sqrt{s_0} << r_{\max}.$$

This means indeed that the evolutionary time scale has to be much larger than ecological time scale.

1.2.2 The Hamilton-Jacobi approach

The main idea in the Hamilton-Jacobi approach, in order to study asymptotically n_ε and in particular to describe such concentration phenomenon, is to perform the Hopf-Cole transformation

$$0 < n_\varepsilon = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} \exp\big(\frac{u_\varepsilon}{\varepsilon}\big). \tag{5}$$

While, as $\varepsilon \to 0$, n_ε converges to a measure, the limit of u_ε is a continuous function which solves a Hamilton-Jacobi equation. The idea is to first obtain the convergence of u_ε to such continuous function u and then use the properties of u to describe the limit n, of n_ε. The identification of u allows us in particular to show that n is a single Dirac mass.

When the growth term R and the initial data $n_{\varepsilon,0}$ are nice, in a sense that we will precise later, one can even go further than just obtaining the convergence of u_ε, but also obtain an asymptotic expansion with respect to ε:

$$u_\varepsilon(t,z) = u(t,z) + \varepsilon v(t,z) + o(\varepsilon).$$

Replacing the above expansion in (5) we obtain an approximation of the population's distribution n_ε for ε small but nonzero:

$$n_\varepsilon(t,z) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} \exp\big(\frac{u(t,z)}{\varepsilon} + v(t,z) + o(\varepsilon)\big).$$

Such approximation is particularly interesting from the biological point of view, since it is more relevant to consider non-vanishing effects of mutations. Moreover, going further in the approximation by using the Lapalce' method for integration, one can estimate the moments of the population's distribution and obtain more quantitative results which could be compared to measurable quantities in biological experiments [MG20, FIM18].

We show below how to derive the convergence of u_ε to u the viscosity solution to a certain Hamilton-Jacobi equation with constraint.

2 Maximum principles for parabolic equations

In this section, we present the maximum principle for linear parabolic equations with Dirichlet boundary conditions.

Consider the following linear elliptic operator, with continuous and bounded coefficients,

$$\mathcal{L} = -\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \Omega.$$

Here, we assume that $(a_{i,j})$ is a symmetric matrix and that the operator is uniformly elliptic, that is there exists $\alpha,\ \beta > 0$, such that

$$\alpha\|\xi\|^2 \le \sum_{i,j} a_{i,j}(t,x)\xi_i\xi_j \le \beta\|\xi\|^2,$$

for all vector $\xi \in \mathbb{R}^d$ and for all $t \ge 0$ and $x \in \Omega$. We introduce the maximum principle for the parabolic equations of the following form:

$$\begin{cases} \partial_t\varphi + \mathcal{L}\varphi = 0, & (t,x) \in (0,\infty) \times \Omega \\ \varphi(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \end{cases} \tag{6}$$

with Ω an open and connected set.

Definition 2.1 *A function $\varphi \in \mathrm{C}^2((0,\infty)\times\Omega) \cap \mathrm{C}([0,\infty)\times\overline{\Omega})$ is called a sub-solution to* (6) *if and only if the following inequalities hold*

$$\begin{cases} \partial_t\varphi + \mathcal{L}\varphi \le 0, & (t,x) \in (0,\infty) \times \Omega \\ \varphi(t,x) \le 0, & (t,x) \in (0,\infty) \times \partial\Omega. \end{cases}$$

A function $\varphi \in \mathrm{C}^2((0,\infty)\times\Omega) \cap \mathrm{C}([0,\infty)\times\overline{\Omega})$ is called a sub-solution to (6) *if and only if the same inequalities hold with opposite signs.*
When Ω is unbounded, we add the condition that, for all $T > 0$,

$$|\varphi(t,x)| \le Ae^{B|x|}, \qquad A, B > 0,$$

for all $(t,x) \in [0,T] \times \Omega$.

Theorem 2.2 (The weak maximum principle) *Let u be a subsolution (resp. supersolution) to* (6)*, such that $u(0,\cdot) \le 0$ (resp. $u(0,\cdot) \ge 0$). Then, for all $t \ge 0$, $u(t,\cdot) \le 0$, (resp. $u(t,\cdot) \ge 0$).*

Theorem 2.3 (The strong maximum principle) *Let u be a subsolution (resp. supersolution) to* (6)*, such that $u(0,\cdot) \le 0$ (resp. $u(0,\cdot) \ge 0$). If there exists $(t_0, x_0) \in (0,\infty) \times \Omega$ such that $u(t_0, x_0) = 0$, then $u \equiv 0$ in $[0, t_0] \times \Omega$.*

Before providing the proofs of the theorems above we first state the following Lemma which will be useful in what follows.

Lemma 2.4 *Let $P = (p_{i,j})_{i,j\in\{1,\cdots,d\}}$ and $Q = (q_{i,j})_{i,j\in\{1,\cdots,d\}}$ be two symmetric definite positive matrices. Then, we have $\mathrm{Tr}\,(PQ) = \sum_{i,j} p_{i,j} q_{i,j} \ge 0$.*

The proof of this lemma is left to the reader as an exercise. We next prove the weak maximum principle in a simple case.

Lemma 2.5 *Let Ω be a bounded set, and u be a strict subsolution of* (6)*, that is*

$$\begin{cases} \partial_t u + \mathcal{L}u < 0, \\ u(t,x) < 0, \quad (t,x) \in \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) < 0, \quad x \in \Omega. \end{cases} \tag{7}$$

Then,

$$u(t,x) < 0, \qquad \text{for all } (t,x) \times \mathbb{R}^+ \times \Omega.$$

Proof of Lemma 2.5. Let $t_0 > 0$ be the first time that u attains the value 0 in Ω, that is there exists $x_0 \in \Omega$ such that $u(t_0, x_0) = 0$. Since this is a maximum point in $[0, t_0] \times \Omega$ we have

$$\partial_t u(t_0, x_0) \ge 0, \qquad \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}(t_0, x_0) = 0, \qquad -D^2 u(t_0, x_0) \ge 0.$$

Moreover, from the latter inequality and Lemma 2.4 we deduce that

$$-\sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(t_0, x_0) \ge 0.$$

We deduce that

$$\partial_t u(t_0, x_0) + \mathcal{L}u(t_0, x_0) \ge 0,$$

which is in contradiction with (7). □

To prove Theorem 2.2 the idea is to bring the problem to the situation above.

Proof the the weak maximum prinicple. (i) Bounded domains. We now prove the result, for bounded domains, in the case where u is not a strict subsolution.

We define $w = ue^{-Kt}$, with K a large constant such that $c + K \geq 1$. We have

$$\partial_t w + \mathcal{L}w + Kw = \partial_t w - \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 w}{\partial x_i \partial x_j}(t_0, x_0) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} w + (c + K)w \leq 0. \tag{8}$$

Note that $w(t, x) \leq 0$ for $(t, x) \in \mathbb{R}^+ \times \partial\Omega \times \{0\} \times \Omega$. Moreover, $w(t, x) \leq 0$ implies that $u(t, x) \leq 0$.

Now let t_0 be the first time that w attains the value $\delta > 0$ in Ω, that is there exists $x_0 \in \Omega$ such that $w(t_0, x_0) = \delta$. Similarly to above, we have

$$\partial_t w(t_0, x_0) \geq 0, \qquad -\sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 w}{\partial x_i \partial x_j}(t_0, x_0) \geq 0, \qquad \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} w(t_0, x_0) = 0.$$

It follows that

$$\partial_t w(t_0, x_0) - \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 w}{\partial x_i \partial x_j}(t_0, x_0) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} w(t_0, x_0) + (c+K)w(t_0, x_0) \geq (c+K)w(t_0, x_0) \geq \delta > 0.$$

This is in contradiction with (8). We deduce that

$$w(t, x) \leq 0, \qquad \text{for all } (t, x) \in \mathbb{R}^+ \times \Omega.$$

(ii) Unbounded domains. Let's suppose now that Ω is unbounded. We define

$$v = u\psi(x),$$

where $\psi \in \mathrm{C}^\infty(\mathbb{R}^d)$ is strictly positive and such that

$$\frac{|\nabla \psi|}{\psi} \in L^\infty(\mathbb{R}^d), \quad \frac{|D^2 \psi|}{\psi} \in L^\infty(\mathbb{R}^d).$$

and

$$\psi(x) = e^{-2B|x|}, \quad \text{for large } |x|.$$

Then v is a subsolution to a linear parabolic equation with bounded coefficients and $v \to 0$ as $|x| \to \infty$. Then, the proof follows similarly to the case (i). $\square$

In order to prove the strong maximum principle, We first state the following lemma.

Lemma 2.6 *Let u be a subsolution of* (6) *such that $u(0, x) < 0$ for all $x \in \Omega$. Then, $u(t, x) < 0$ for all $(t, x) \in \mathbb{R}^+ \times \Omega$.*

Proof of Lemma 2.6. Thanks to the weak maximum principle, we can assume without loss of generality that $\Omega = B_\delta(0)$. We then define

$$w = u + \mu(\delta^2 - |x|^2)^2 e^{-\alpha t}.$$

We can choose $\mu > 0$ such taht $w(0,x) < 0$ for all $x \in B_\delta(0)$ and $w(t,x) \leq 0$ for all $(t,x) \in \mathbb{R}^+ \times \partial B_\delta(0)$. We show that we can also choose $\alpha > 0$ such that w is a subsolution. We compute

$$\begin{aligned}
&\partial_t w + \mathcal{L}w = \partial_t u + \mathcal{L}u \\
&+\mu e^{-\alpha t}\big(-\alpha(\delta^2 - |x|^2)^2 - \textstyle\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(\delta^2 - |x|^2)^2 + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}(\delta^2 - |x|^2)^2 + c(\delta^2 - |x|^2)^2\big).
\end{aligned}$$

Note that

$$c(\delta^2 - |x|^2)^2 \leq C_1(\delta^2 - |x|^2)^2.$$

$$\sum_{i=1}^d b_i \frac{\partial}{\partial x_i}(\delta^2 - |x|^2)^2 = -2\sum_{i=1}^d b_i x_i(\delta^2 - |x|^2) \leq C_1\delta(\delta^2 - |x|^2).$$

$$\begin{aligned}
-\textstyle\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(\delta^2 - |x|^2)^2 &= -4\textstyle\sum_{i,j=1}^d a_{i,j}x_i x_j + 2\mathrm{Tr}(a_{i,j})(\delta^2 - |x|^2) \\
&\leq -C_2|x|^2 + 2\mathrm{Tr}(a_{i,j})(\delta^2 - |x|^2).
\end{aligned}$$

One can consequently verify that it is possible to choose $\delta' < \delta$, up to reducing δ if necessary, such that for all $x \in B_\delta(0) \setminus B_{\delta'}(0)$, we have

$$-\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(\delta^2 - |x|^2)^2 + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}(\delta^2 - |x|^2)^2 + c(\delta^2 - |x|^2)^2 \leq 0.$$

We then deduce that w is a subsolution of (6) in $\mathbb{R}^+ \times B_\delta(0) \setminus B_{\delta'}(0)$. Next, we can choose α large enough such that w is a subsolution of (6) in $\mathbb{R}^+ \times B_{\delta'}(0)$. Hence w is a subsolution of (6) in $\mathbb{R}^+ \times B_\delta(0)$. Using the weak maximum principle we obtain that

$$u < w \leq 0.$$

□

Proof of the strong maximum principle. Let $u \not\equiv 0$ in $[0,t_0] \times \Omega$. Then, there exists a point, that we can suppose to be $(0,0)$ without loss of generality, such that $u(0,0) < 0$. By continuity, $u(0,x) < 0$ for all $x \in B_r(0)$, for r small enough. We then suppose that the segment that links 0 to x_0 is included in Ω. Since Ω is an open set, reducing if necessary r, we can also suppose that for all $s \in [0,1]$ any ball of radius r and centered around sx_0 is also included in Ω. We now define

$$w(t,x) = u(t, x + \frac{t}{t_0}x_0).$$

One can verify that w is also a subsolution of an equation of similar form to (6) in $\mathbb{R}^+ \times B_r$. Applying now Lemma 2.6 to w, we deduce that

$$w(t_0, 0) = u(t_0, x_0) < 0,$$

which is a contradiction. By connectedness of the set Ω there always exists an arc which links 0 to x_0. This arc can be chosen as a union of segments. The argument can thus be adapted to the general case. $\square$

Exercise 2.7 *Assume that u is a smooth solution of*

$$\begin{cases} \partial_t u - \Delta u + cu = 0, & \textit{in } (0, +\infty) \times \Omega, \\ u = 0, & \textit{in } (0, +\infty) \times \partial\Omega, \\ u = g & \textit{in } \{0\} \times \Omega, \end{cases}$$

with Ω a bounded domain, and the function c such that $c \geq \gamma > 0$. Prove the following exponential decay estimate

$$|u(t, x)| \leq Ce^{-\gamma t}, \qquad (t, x) \in (0, +\infty) \times \Omega.$$

3 Hamilton-Jacobi equations

The notes in this section are inspired from the books [Bar94] and [Eva98].

3.1 Characteristics

The method of characteristics allows to find local smooth solutions for non-linear first order PDE, when considering smooth solutions on the boundary. The idea of this method is to convert the PDE problem into an appropriate system of ODE. Within this method, we try to compute the value of the solution u at a certain point Z at the interior of the domain, by finding some curve lying within the domain connecting Z with a point Z_0 at the boundary of the domain, and along which we compute u.

Let's consider the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t u + H(z, Du) = 0, & (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,z) = u_0(z), & z \in \mathbb{R}^d. \end{cases} \tag{9}$$

Suppose that the curve along which solve the equation is given by $\mathbf{z}(s)$. Assuming that u is a C^2 solution, we also define

$$\mathbf{u}(s) := u(s, \mathbf{z}(s)), \qquad \mathbf{p}(s) := Du(s, \mathbf{z}(s)).$$

Then the corresponding characteristics equations are given by

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_z H(\mathbf{z}(s), \mathbf{p}(s)) \\ \dot{\mathbf{u}}(s) = D_p H(\mathbf{z}(s), \mathbf{p}(s)) \cdot \mathbf{p}(s) - H(\mathbf{z}(s), \mathbf{p}(s)), \\ \dot{\mathbf{z}}(s) = D_p H(\mathbf{z}(s), \mathbf{p}(s)). \end{cases}$$

Note that in the case of the Hamilton-Jacobi equation, it is indeed enough to solve the Hamilton equations which are decoupled from the dynamics of $\mathbf{u}$:

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_z H(\mathbf{z}(s), \mathbf{p}(s)), \\ \dot{\mathbf{z}}(s) = D_p H(\mathbf{z}(s), \mathbf{p}(s)). \end{cases}$$

The dynamics of $\mathbf{u}$ can then be deduce from the dynamics of $(\mathbf{z}, \mathbf{p})$.

If such ODE system can be solved, with all ending points $\mathbf{z}(t) = z$, in a unique way with an initial condition $(\mathbf{z}(0), \mathbf{u}(0), \mathbf{p}(0))$ which is admissible, that is

$$\mathbf{u}(0) = u_0(\mathbf{z}(0)), \qquad \mathbf{p}(0) = Du_0(\mathbf{z}(0)),$$

then the Hamilton-Jacobi equation can be solved along the characteristic curves and the value of the solution at the point (t, z) is given by $\mathbf{u}(t)$.

However, such admissible initial condition may not exist or may not be unique. When the initial condition u_0 is smooth, the characteristics method is indeed efficient for finding smooth local solutions to the Hamilton-Jacobi equation for small times, since the existence and uniqueness of an admissible initial condition holds locally.

3.2 Viscosity solutions

3.2.1 Looking for a good notion of solutions

As we mentioned in the previous section, the characteristic method could provide us with local smooth solutions for Hamilton-Jacobi equations. However, even starting with a smooth initial condition, after some time the characteristics may cross such that the characteristic curves ending at such crossing point are not unique. In this case, the Hamilton-Jacobi equation does not admit a smooth solution and one should look for a notion of weak solutions. Note however that since the Hamilton-Jacobi equations are not of divergence form, one cannot use the notion of weak solutions in the distribution sense. One should look for another notion of weak solution.

We can for instance look for solutions in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, that is the class of Lipschitz-continuous functions. By the Rademacher's theorem any Lipschitz function u is almost everywhere differentiable. We can hence look for a function $u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ which solves (9) almost everywhere. However, this is not a good choice for the definition of a weak solution since Hamilton-Jacobi equations may have several (and even infinitely many) such solutions.

Example. Consider the following problem

$$\begin{cases} \partial_t u + |\partial_z u|^2 = 0 & \text{in } (0, \infty) \times \mathbb{R} \\ u = 0 & \text{on } \{0\} \times \mathbb{R}. \end{cases} \tag{10}$$

One obvious solution is the following

$$u_0(t, x) \equiv 0.$$

However, one can find infinitely many other solutions:

$$u_c(t,z) := \begin{cases} 0 & \text{if } |z| \geq ct \\ cz - c^2 t & \text{if } 0 \leq z \leq ct \\ -cz - c^2 t & \text{if } -ct \leq z \leq 0 \end{cases}$$

Note that the function u_c is Lipschitz continuous and solves (10) everywhere except on the lines $z = 0$, $z = ct$, $z = -ct$.

Another usual option is the notion of weak solutions in the distribution sense. The idea in this case is to transfer the derivatives by integration by parts on a test function to find a formulation which does not rely on the derivatives of a smooth solution. However this method works for the equations in the divergence form which is not the case of Hamilton-Jacobi equations. The notion of viscosity solutions, introduced by Crandall and Lions in [CL83], relies on a same type of idea, that is to put the derivative on a smooth test function. However, one exploits the maximum principle, and not the integration by parts, to carry out the transfer of the derivatives. We introduce in the following sections the viscosity solutions. By showing existence, uniqueness and stability of such solutions, we will confirm that this is an appropriate notion of solutions for Hamilton-Jacobi equations.

3.2.2 Definition of viscosity solutions

Even if here we are interested in Hamilton-Jacobi equations of first order, we will present the notion of viscosity solutions for non-linear parabolic and elliptic equations of second order since the definition can be introduced in a more natural manner in this general framework. However, in the next sections we will again restrict our studies to the Hamilton-Jacobi equations.

Let's consider the equations of the following types

$$\partial_t u + H(z, u, Du, D^2 u) = 0, \qquad \text{in } \Omega, \tag{11}$$

$$H(z, u, Du, D^2 u) = 0, \qquad \text{in } \Omega, \tag{12}$$

where $\Omega \subset \mathbb{R}^d$ is an open set, and $H : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ is a continuous function. Here, $\mathcal{S}^d$ represents the vector space of $d \times d$ symmetric matrices. We assume that H satisfies the following ellipticity condition

$$H(z, u, p, M_1) \leq H(z, u, p, M_2), \qquad \text{if } M_2 \leq M_1,$$

for all $z \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^d$ and $M_1, M_2 \in \mathcal{S}^d$ and the $\leq$ order on $\mathcal{S}$ being defined as below:

$$M_2 \leq M_1 \iff (M_2 p, p) \leq (M_1 p, p), \qquad \forall p \in \mathbb{R}^d,$$

with $(,)$ the usual scalar product in $\mathbb{R}^d$.

The notion of viscosity solutions relies on the maximum principle. We first state the following theorem based on the maximum principle for equations (11)–(12).

Theorem 3.1 *The function $u \in C^2(\mathbb{R}^+ \times \Omega)$ (resp. $u \in C^2(\Omega)$) is a classical solution of* (11) *(resp.* (12)*) if and only if*
(i) For all $\phi \in C^2(\mathbb{R}^+ \times \Omega)$ (resp. $\phi \in C^2(\Omega)$), if (t_0, z_0) is a local maximum point of $u - \phi$, then

$$\partial_t \phi(t_0, z_0) + H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2\phi(t_0, z_0)) \leq 0,$$

(respectively $H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2\phi(t_0, z_0)) \leq 0$.)
(ii) For all $\phi \in C^2(\mathbb{R}^+ \times \Omega)$ (resp. $\phi \in C^2(\Omega)$), if (t_0, z_0) is a local minimum point of $u - \phi$, then

$$\partial_t \phi(t_0, z_0) + H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2\phi(t_0, z_0)) \geq 0,$$

(respectively $H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2\phi(t_0, z_0)) \geq 0$.)

This theorem provides an equivalent definition of the notion of classical solutions. The particular interest of the second formulation is that it does not use the regularity of u. This formulation is what is used to introduce the notion of viscosity solutions.

Before providing the definition of the viscosity solutions we first prove the theorem above.
Proof. Let us suppose that $u \in C^2(\mathbb{R}^+ \times \Omega)$ is a classical solution of (11). Assume also that $\phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^d)$ and that $u - \phi$ takes a local maximum at (t_0, z_0). We then have

$$\partial_t u(t_0, z_0) = \partial_t \phi(t_0, z_0), \qquad Du(t_0, z_0) = D\phi(t_0, z_0), \qquad D^2 u(t_0, z_0) \leq D^2 \phi(t_0, z_0).$$

Using the ellipticity of (11) we then obtain that

$$\begin{aligned} &\partial_t \phi(t_0, z_0) + H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2\phi(t_0, z_0)) \\ &\leq \partial_t u(t_0, z_0) + H(z_0, u(t_0, z_0), Du(t_0, z_0), D^2 u(t_0, z_0)) = 0. \end{aligned}$$

One can prove the condition (ii), and the corresponding statements for equation (12), following similar arguments.

We next prove the converse statement for equation (11) (the case of equation (12) can be treated similarly). Let $u \in C^2(\mathbb{R}^+ \times \Omega)$ satisfy the statements (i) and (ii). For each of the statements we can consider the test function $\phi = u$. Since any point $(t_0, z_0) \in \mathbb{R}^+ \times \Omega$ is both a maximum and minimum point of $u - u$, we find that

$$0 \leq \partial_t u(t_0, z_0) + H(z_0, u(t_0, z_0), Du(t_0, z_0), D^2 u(t_0, z_0)) \leq 0,$$

which implies that

$$\partial_t u(t_0, z_0) + H(z_0, u(t_0, z_0), Du(t_0, z_0), D^2 u(t_0, z_0)) = 0, \qquad \text{for all } (t_0, z_0) \in \mathbb{R}^+ \times \Omega.$$

□

We are now ready to introduce the notion of viscosity solutions.

Definition 3.2 *(i) A function $u \in C(\mathbb{R}^+ \times \Omega)$ is called a viscosity subsolution of* (11) *(resp.* (12)*), if for all test function $\phi \in C^2(\Omega)$ such that $u - \phi$ takes a local maximum at $(t_0, z_0) \in \mathbb{R}^+ \times \Omega$, we have*

$$\partial_t \phi(t_0, z_0) + H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2 \phi(t_0, z_0)) \leq 0,$$

(resp. $H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2 \phi(t_0, z_0)) \leq 0$).

(ii) A function $u \in C(\mathbb{R}^+ \times \Omega)$ is called a viscosity supersolution of (11) *(resp.* (12)*), if for all test function $\phi \in C^2(\Omega)$ such that $u - \phi$ takes a local minimum at $(t_0, z_0) \in \mathbb{R}^+ \times \Omega$, we have*

$$\partial_t \phi(t_0, z_0) + H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2 \phi(t_0, z_0)) \geq 0.$$

(resp. $H(z_0, u(t_0, z_0), D\phi(t_0, z_0), D^2 \phi(t_0, z_0)) \geq 0$).

(iii) A function $u \in C(\mathbb{R}^+ \times \Omega)$ is called a viscosity solution of (11) *(respectively* (12)*) if it is both viscosity sub and supersolution of* (11) *(resp.* (12)*).*

Proposition 3.3 *We obtain equivalent definitions of viscosity sub and supersolutions, if we replace in the definition 3.2:*

(i) "$\phi \in C^2$" by "$\phi \in C^k$", for $2 < k \leq +\infty$ in the case the equation of second order.

(ii) "$\phi \in C^2$" by "$\phi \in C^k$", for $1 \leq k \leq +\infty$ in the case the equation of first order.

(iii) "local maximum' or "local minimum" by "strict local maximum" or strict local minimum" or by "global maximum' or "global minimum" or by "strict global maximum' or "strict global minimum".

(iv) We obtain also an equivalent definition of viscosity sub and supersolutions, if we impose to the test functions that $u(t_0, z_0) = \phi(t_0, z_0)$.

This proposition is very useful since it allows to simplify many proofs. The proof of this proposition relies on classical arguments from the functional analysis and we let it as an exercise.

3.2.3 An equivalent definition using sub- and superdifferentials

In this section, we provide an equivalent definition of viscosity sub- and supersolutions which may be useful to identify the viscosity solutions. We introduce this equivalent definition in the case of time independent first order equations

$$H(z, u, Du) = 0, \qquad z \in \Omega. \tag{13}$$

Similar types of definitions exist also for time dependent Hamilton-Jacobi equations (9) and also for second order elliptic or parabolic equations of type (11) or (12).

Definition 3.4 (Sub- and superdifferentials of a continuous function) *Let $u \in C(\Omega)$, with $\Omega \subset \mathbb{R}^d$ an open set. Fix $x \in \Omega$. We define the superdifferential of u at point x as*

$$D^+u(x) = \{p \in \mathbb{R}^d;\ \limsup_{y \to x} \frac{u(y) - u(x) - (p, y - x)}{|y - x|} \leq 0\}.$$

We define the subdifferential of u at point x as

$$D^-u(x) = \{p \in \mathbb{R}^d;\ \liminf_{y \to x} \frac{u(y) - u(x) - (p, y - x)}{|y - x|} \geq 0\}.$$

Exercise 3.5 *Show that if u is differentiable in z, then $D^+u(z) = D^-u(z) = \{Du(z)\}$.*

Exercise 3.6 *Let $z \in \mathbb{R}$. Compute the sub- and superdifferentials of the mapping $z \to |z|$. Consider next $z \in \mathbb{R}^d$ and compute the sub- and and superdifferentials of the mapping $z \to |z|$ when $|\cdot|$ corresponds to the euclidean norm.*

Theorem 3.7 *(i) $u \in C(\Omega)$ is a viscosity subsolution of (9), if and only if for all $z \in \Omega$:*

$$\forall p \in D^+u(z), \quad H(z, u(z), p) \leq 0.$$

(ii) $u \in C(\Omega)$ is a viscosity supersolution of (9), if and only if for all $z \in \Omega$:

$$\forall p \in D^-u(z), \quad H(z, u(z), p) \geq 0.$$

3.2.4 Examples

Example 1. Let's consider equation (10) introduced in Section 3.2.1. One can easily verify that u_0 is a viscosity solution to (10). We show that the functions u_c, for all $c > 0$, are not viscosity solutions of (10). We show indeed that the viscosity supersolution criterion does not hold at the points $(t,0)$, for all $t \geq 0$. In order to do so, let's consider the test functions

$$\phi(t,x) = -c^2 t.$$

One can verify that

$$u_c(t,x) \geq \phi(t,x), \qquad u_c(t,0) = \phi(t,0).$$

Therefore, $u_c(t,x) - \phi(t,x)$ has minimum points at the points $(t,0)$, for all $t \geq 0$. For u_c to be a viscosity supersolution of (10) one should have

$$\partial_t \phi(t,0) + |\partial_x \phi(t,x)|^2 \geq 0.$$

However, this cannot hold since the left hand side term equals to $-c^2$.

Example 2. We now consider the following Eikonal equation:

$$\begin{cases} |Du|^2(z) = 1, & z \in (-1,1), \\ u(z) = 0, & z = \pm 1. \end{cases} \tag{14}$$

This equation admits infinitely many Lipschitz continuous solution which satisfy the equation almost everywhere. Here are some of these solutions:

$$u_1(z) = 1 - |z|,$$

$$u_2(z) = -1 + |z|,$$

$$u_3(z) = \begin{cases} z+1 & \text{for } z \in [-1, -\frac{1}{2}), \\ -z & \text{for } z \in [-\frac{1}{2}, \frac{1}{2}), \\ z-1 & \text{for } z \in [\frac{1}{2}, 1]. \end{cases}$$

However, only the function u_1 is a viscosity solution of (14). One can also verify that any function that has an angle towards the bottom (as the function u_2 at the point $z = 0$ or the function u_3 at the point $\frac{1}{2}$) is not a viscosity supersolution of the problem. We state these properties in the following exercise.

Exercise 3.8 *(i) Show that $u_1(z)$ is a viscosity solution of* (14).

(ii) Show that $u_2(z) = -u_1(z)$ and u_3 are not viscosity solutions of (14).

Exercise 3.9 *Assume that u is a viscosity solution of*

$$\partial_t u + H(z, Du) = 0, \quad \textit{in } \mathbb{R}^+ \times \mathbb{R}^d.$$

Show that $\widetilde{u} := -u$ *is a viscosity solution of*

$$\partial_t \widetilde{u} + \widetilde{H}(z, D\widetilde{u}) = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d,$$

with $\widetilde{H}(z,p) := -H(z,-p)$.

3.2.5 The vanishing viscosity method

The ellipticity of H was a crucial condition in the definition above. In the case of first order Hamilton-Jacobi equations (9) without the term D^2u it is not clear where the inequalities above come from. A possible way to justify these inequalities is to obtain such solutions by the method of vanishing viscosity, and this is where the terminology viscosity solution comes from. Let us introduce an approximate problem by adding a small viscosity term to the equation (9):

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(z, Du_\varepsilon) = 0, & (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u_\varepsilon(0,z) = u_0(z) & z \in \mathbb{R}^d. \end{cases} \tag{15}$$

Note that while (9) involves a fully nonlinear first order PDE, (15) is a quasilinear parabolic equation which turns out to have a smooth solution. The term $-\varepsilon\Delta$ regularizes indeed the Hamilton-Jacobi equation. The idea is then to let $\varepsilon \to 0$ and to hope that u_ε converges to some function u which would be a weak solution of the Hamilton-Jacobi equation. This is called the vanishing viscosity method.

However, one could expect to loose control over the regularity estimates on u_ε as $\varepsilon \to 0$ since these regularities rely strongly on the regularizing effects of the vanishing viscosity term $\varepsilon\Delta$. Fortunately, it turns out that most of the time one can prove at least that the family of solutions (u_ε) is equicontinuous on compact sets of $\mathbb{R}^+ \times \mathbb{R}^d$. Then, using the Arzela-Scoli Theorem one can deduce that, as $\varepsilon \to 0$ and along subsequences, (u_ε) converges locally uniformly to a continuous function u (see for instance Section 5 where such property is proved in the study of problem (4)). We show below that such a limiting function u is indeed a viscosity solution of (9).

To do so, we will use the following lemma whose proof is left as an exercise to the reader.

Lemma 3.10 *Let* (v_e) *be a family of continuous functions in an open set* Ω *which converges in* $C(\Omega)$ *to* v*. If* $x_0 \in \Omega$ *is a strict local maximum point of* v*, then there exists a sequence of local maximum points of* v_ε*, denoted by* (x_ε)*, which converges to* x_0*.*

We prove that such a limiting function u is a viscosity subsolution of (9). The proof of the supersolution criterion follows similar arguments.

Let $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be a test function such that $u - \phi$ attains a local maximum at the point (t_0, z_0). Since $u_\varepsilon - \phi$ converges locally uniformly along a subsequence to u in $\mathbb{R}^+ \times \mathbb{R}^d$, we deduce thanks to Lemma 3.10 that, along this subsequence, $u_\varepsilon - \phi$ attains a local maximum at a point $(t_\varepsilon, z_\varepsilon)$, with $(t_\varepsilon, z_\varepsilon) \to (t_0, z_0)$ as $\varepsilon \to 0$. Since $(t_\varepsilon, z_\varepsilon)$ is a local maximum point of $(u_\varepsilon - \phi)$ we obtain that

$$\partial_t u_\varepsilon(t_\varepsilon, z_\varepsilon) = \partial_t \phi(t_\varepsilon, z_\varepsilon), \quad Du_\varepsilon(t_\varepsilon, z_\varepsilon) = D\phi(t_\varepsilon, z_\varepsilon), \quad -\varepsilon \Delta u_\varepsilon(t_\varepsilon, z_\varepsilon) \geq -\varepsilon \Delta \phi(t_\varepsilon, z_\varepsilon).$$

Combining this with equation (15) we then obtain that

$$\partial_t \phi(t_\varepsilon, z_\varepsilon) - \varepsilon \Delta \phi(t_\varepsilon, z_\varepsilon) + H(z, D\phi(t_\varepsilon, z_\varepsilon)) \leq 0.$$

Letting now $\varepsilon \to 0$ we deduce that

$$\partial_t \phi(t_0, z_0) + H(z, D\phi(t_0, z_0)) \leq 0,$$

which is the viscosity subsolution criterion.

Exercise 3.11 *Let $(u_k)_{k=1}^\infty$, be viscosity solutions of the Hamilton-Jacobi equations*

$$\partial_t u_k + H(z, Du_k) = 0, \qquad \text{in } (0, +\infty) \times \mathbb{R}^d,$$

with H a continuous function. Suppose also that $u_k \to u$ locally uniformly. Show that u is a viscosity solution of

$$\partial_t u + H(z, Du) = 0, \qquad \text{in } (0, +\infty) \times \mathbb{R}^d.$$

3.2.6 Consistency

Let us now check whether the notion of viscosity solutions is consistent with that of the classical solutions. We have seen in Section 3.2.2 that any classical solution of (11) or (12) (and consequently any classical solution of (9)) is also a viscosity solution. We now prove the following.

Theorem 3.12 (Consistency) *Let u be a viscosity solution of* (9) *and that u is differentiable at some point $(t_0, z_0) \in \mathbb{R}^+ \times \mathbb{R}^d$. Then, we have*

$$\partial_t u(t_0, z_0) + H(Du(t_0, z_0), z_0) = 0.$$

The proof of the theorem above relies on the next lemma.

Lemma 3.13 (Touching by a C^1 function) *Assume that $v : \mathbb{R}^k \to \mathbb{R}$ is a continuous function which is differentiable at the point x_0. Then, there exists a function $\psi \in C^1(\mathbb{R}^k)$ such that*

$$v(x_0) = \psi(x_0),$$

and

$$v - \psi \text{ has a strict local maximum at } x_0.$$

We postpone the proof of this lemma to the end of this section and proceed with the proof of Theorem 3.12.

Proof. [Proof of Theorem 3.12] Applying Lemma 3.13 to u, with $k = d+1$ and $x_0 = (t_0, z_0)$ we obtain that there exists a function $\psi \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$ such that

$$u - \psi \text{ has a strict maximum at } (t_0, z_0).$$

On the one hand, from the definition of the viscosity solutions we deduce that

$$\partial_t \psi(t_0, z_0) + H(D\psi(t_0, z_0), z_0) \leq 0.$$

On the other hand, since u and ψ are C^1 functions, we find that

$$\partial_t u(t_0, z_0) = \partial_t \psi(t_0, z_0), \qquad Du(t_0, z_0) = D\psi(t_0, z_0).$$

It follows that

$$\partial_t u(t_0, z_0) + H(Du(t_0, z_0), z_0) \leq 0.$$

One can prove following similar arguments that

$$\partial_t u(t_0, z_0) + H(Du(t_0, z_0), z_0) \geq 0.$$

which completes the proof. □

We finally prove Lemma 3.13.

Proof. [Proof of Lemma 3.13] (i) First note that, up to changing the function v to $v(x + x_0) - v(x_0) - Dv(x_0) \cdot x$, we can suppose that

$$x_0 = 0, \quad v(x_0) = 0, \quad Dv(x_0) = 0.$$

(ii) We can then write

$$v(x) = |x| \rho_1(x),$$

wit $\rho_1 : \mathbb{R}^k \to \mathbb{R}$ a continuous function and $\rho_1(0) = 0$.

We next define

$$\rho_2(r) = \max_{x \in B_r(0)} \{|\rho_1(x)|\}, \quad r \geq 0.$$

The function $\rho_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing and $\rho_2(0) = 0$.

(iii) We are now ready to define, for all $x \in \mathbb{R}^k$,

$$\psi(x) = \int_{|x|}^{2|x|} \rho_2(r)dr + |x|^2.$$

Since $0 \leq \psi(x) \leq |x|\rho_2(2|x|) + |x|^2$, we find that

$$\psi(0) = 0, \quad D\psi(0) = 0.$$

Moreover, for all $x \neq 0$, we have

$$D\psi(x) = \frac{2x}{|x|}\rho_2(2|x|) - \frac{x}{|x|}\rho_2(|x|) + 2x,$$

which implies that $\psi \in C^1(\mathbb{R}^k)$.

(iv) Finally note that

$$\begin{aligned} v(x) - \psi(x) &= |x|\rho_1(x) - \textstyle\int_{|x|}^{2|x|} \rho_2(r)dr - |x|^2 \\ &\leq |x|\rho_2(|x|) - \textstyle\int_{|x|}^{2|x|} \rho_2(r)dr - |x|^2 \\ &\leq -|x|^2 < 0 = v(0) - \psi(0). \end{aligned}$$

Therefore $v - \psi$ has a strict local maximum at 0. □

3.2.7 Comparison principle and uniqueness

There are different variants of results providing a comparison principle for Hamilton-Jacobi equations depending on the assumptions that one makes on the Hamiltonian and on the domain. Here, via a particular framework we show the main idea to obtain such results, that is the method of doubling the number of variables.

Let Ω be a bounded and open set of $\mathbb{R}^d$ and H a continuous Hamiltonian which is Lipschitz continuous with respect to z such that

$$|\partial_z H(z, p)| \leq C(1 + |p|). \tag{16}$$

We consider the following Hamilton-Jacobi equations

$$u + H(Du, z) = 0, \quad z \in \Omega, \tag{17}$$

and

$$\partial_t u + H(Du, z) = 0, \quad (t, z) \in \mathbb{R}^+ \times \Omega. \tag{18}$$

Theorem 3.14 (Comparison principle for (17)**)** *Assume* (16)*. Let u and v be respectively a viscosity subsolution and supersolution of* (17)*. Assume also that $u \le v$ on $\partial\Omega$. Then, $u \le v$ in Ω.*

Theorem 3.15 (Comparison principle for (18)**)** *Assume* (16)*. Let u and v be respectively a viscosity subsolution and supersolution of* (18)*. Assume also that $u \le v$ on $\{0\} \times \Omega$ and on $\mathbb{R}^+ \times \partial\Omega$. Then, $u \le v$ in $\mathbb{R}^+ \times \Omega$.*

It is immediate that the comparison principle 3.14 leads to a uniqueness result for the following problem:

$$\begin{cases} u + H(Du, z) = 0, & z \in \Omega, \\ u(z) = g(z), & z \in \partial\Omega, \end{cases}$$

and that the comparison principle 3.15 leads to a uniqueness result for:

$$\begin{cases} \partial_t u + H(Du, z) = 0, & (t, z) \in \mathbb{R}^+ \times \Omega, \\ u(t, z) = g(t, z), & (t, z) \in \mathbb{R}^+ \times \partial\Omega, \\ u(0, z) = u_0(z), & z \in \Omega. \end{cases}$$

Proof. [Proof of Theorem 3.14] Define $M = \sup_{\overline{\Omega}}(u - v)$. Since u and v are continuous functions and since Ω is bounded. This supremum is attained and finite. Our objective is to show $M \le 0$. We hence assume that $M > 0$ and try to find a contradiction. To this end, we use the method of doubling the number of variables, that is we define

$$M_\delta = \sup_{(x,y) \in \bar{\Omega}^2} \psi_\delta(x, y), \qquad \psi_\delta(x, y) = u(x) - v(y) - \frac{1}{\delta^2}|x - y|^2.$$

Still from the continuity of u and v we obtain that this suppremum is attained and finite. We then state the following lemma.

Lemma 3.16 *(i) As $\delta \to 0$, $M_\delta \to M$.*
(ii) If (x_δ, y_δ) is a maximum point of ψ_δ, then we have

$$\frac{|x_\delta - y_\delta|^2}{\delta^2} \to 0, \quad \textit{as } \delta \to 0,$$

$$u(x_\delta) - v(y_\delta) \to M, \quad \textit{as } \delta \to 0,$$

(iii) If δ is small enough, then x_δ, $y_\delta \in \Omega$.

The proof of this lemma is left as an exercise.

We now use the viscosity sub and supersolution criteria. Let's choose δ small enough such that Lemma 3.16–(iii) holds. Since u is a viscosity subsolution of (17) and since $u(x) - v(y_\delta) - \frac{1}{\delta^2}|x - y_\delta|^2$ has a maximum point at $x_\delta \in \Omega$, we obtain that

$$u(x_\delta) + H\big(x_\delta, \frac{2(x_\delta - y_\delta)}{\delta^2}\big) \leq 0.$$

Similarly, since v is a supersolution of (17) and since $v(y) - u(x_\delta) + \frac{1}{\delta^2}|x_\delta - y|^2$ has a maximum point at $y_\delta \in \Omega$, we obtain that

$$v(y_\delta) + H\big(y_\delta, \frac{2(x_\delta - y_\delta)}{\delta^2}\big) \geq 0.$$

Combining the inequalities above we obtain that

$$u(x_\delta) - v(y_\delta) \leq H\big(y_\delta, \frac{2(x_\delta - y_\delta)}{\delta^2}\big) - H\big(x_\delta, \frac{2(x_\delta - y_\delta)}{\delta^2}\big).$$

Using assumption (16) we deduce that

$$u(x_\delta) - v(y_\delta) \leq C(1 + \frac{2|x_\delta - y_\delta|}{\delta^2})|x_\delta - y_\delta|.$$

Thanks to Lemma 3.16 we deduce that the right hand side of the equality above tends to 0, as $\delta \to 0$, while the left hand side converges to $M > 0$, which is a contradiction. We conclude that $M \leq 0$. □

Proof. [Proof of Theorem 3.15] The proof of Theorem 3.15 follows similar types of arguments. However several new difficulties arise:

(i) A first difficulty comes from the fact that there is no dependence on u in (18). This dependence was used in the proof of 3.14 to obtain a contradiction at the very last step from a strict inequality. To overcome this difficulty we will modify the function u as below

$$u_\alpha = u - \alpha t.$$

It is indeed enough to prove that $u_\alpha \leq v$ in $\mathbb{R}^+ \times \overline{\Omega}$ for all $\alpha \geq 0$.

Let us fix $T > 0$ and assume that $\max_{[0,T]\times\overline{\Omega}}\{u - v\} > 0$. Then, for α small enough, we would also have that $\max_{[0,T]\times\overline{\Omega}}\{u - \alpha t - v\} > 0$. Let's first assume that the maximum is attained in

$(0,T)\times\Omega$. We next consider the following function

$$\psi_{\delta,\alpha}(t,s,x,y) = u(t,x) - \alpha t - v(s,y) - \frac{1}{\delta^2}|x-y|^2 - \frac{1}{\delta^2}|s-t|^2.$$

Following similar arguments as in the proof of Theorem 3.14 we would obtain that for δ small enough the maximum of the above function is attained at some point $(t_{\alpha,\delta}, s_{\alpha,\delta}, x_{\alpha,\delta}, y_{\alpha,\delta}) \in (0,T)\times\Omega$ and we have

$$0 < \alpha \leq H\big(y_{\alpha,\delta}, \frac{2(x_{\alpha,\delta}-y_{\alpha,\delta})}{\delta^2}\big) - H\big(x_{\alpha,\delta}, \frac{2(x_{\alpha,\delta}-y_{\alpha,\delta})}{\delta^2}\big) \leq C(1+\frac{2|x_{\alpha,\delta}-y_{\alpha,\delta}|}{\delta^2})|x_{\alpha,\delta}-y_{\alpha,\delta}|.$$

Letting $\delta \to 0$ we then obtain a contradiction since the right hand side of the inequality tends to 0.

(ii) A second difficulty comes from the boundary condition. From the arguments above we deduce that for α small enough, the maximum of $u(t,x) - \alpha t - v(t,x)$ is attained on the boundary of the domain. If the maximum is attained on the parabolic boundary, that is $[0,T]\times\partial\Omega\cup\{0\}\times\bar{\Omega}$, then we can use the fact that $u \leq v$ on this set by assumption and obtain a contradiction by letting α tend to 0.

Now, what can we say on the maximum when the maximum point is attained at a point (T,x_0), which does not belong to the parabolic boundary? To overcome this difficulty we prove that u (resp. v) is indeed a viscosity subsolution (resp. supersolution) up to the boundary $\{T\}\times\Omega$. That is, for instance in the case of viscosity subsolution, we prove that for all $\phi \in C^1((0,T]\times\Omega)$, if (T,x_0) is a local maximum point of $u-\phi$ in $[0,T]\times\Omega$, then we have

$$\frac{\partial\phi}{\partial t}(T,x_0) + H(x_0, D\phi(T,x_0)) \leq 0.$$

Lemma 3.17 *Let $u \in C([0,T]\times\overline{\Omega})$ be a subsolution (resp. supersolution) of* (18) *in* $(0,T)\times\Omega$*, then u is a subsoltion (resp. supersolution) of* (18) *in* $\Omega\times(0,T]$.

This lemma completes the proof of Theorem 3.15. □

Proof. [Proof of Lemma 3.17] We provide the proof in the case where u is a subsolution. The supersolution case can be treated following similar arguments.‘

Let $\phi \in C^1(0,T]\times\Omega)$ and assume that $u-\phi$ has a local maximum point at (T,x_0), with $x_0\in\Omega$. Replacing if necessary ϕ by $\phi+|x-x_0|^2+(t-T)^2$ we can assume that this is indeed a local strict maximum point. Then the strategy is to move this maximum point to the interior of the domain $(0,T)\times\Omega$. Let's consider the function

$$\chi_\eta(t,x) = u(t,x) - \phi(t,x) - \frac{\eta}{T-t}.$$

One can prove that there exists a sequence of points (t_η, x_η) which are local maximum points of χ_η and which tend to (T, x_0) as $\eta \to 0$ (this is let as an exercise). Moreover, since $\lim_{t\to T} \chi_\eta(t,x) = +\infty$, we necessarily have $t_\eta < T$. Therefore, for η small enough, we can write the viscosity subsolution criterion at the point (t_η, x_η) to obtain

$$\frac{\eta}{(T-t_\eta)^2} + \partial_t \phi(t_\eta, x_\eta) + H(x_\eta, D\phi(t_\eta, x_\eta)) \le 0,$$

and consequently

$$\partial_t \phi(t_\eta, x_\eta) + H(x_\eta, D\phi(t_\eta, x_\eta)) \le 0.$$

We then conclude by letting $\eta \to 0$. $\square$

3.3 Variational solutions

Let's consider the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t u + H(Du, z) = 0, & (t,z) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,z) = u_0(z). \end{cases} \tag{19}$$

Assume that H is smooth and that for all $z \in \mathbb{R}^d$,

$$\text{the mapping } p \to H(p,z) \text{ is convex} \tag{20}$$

and

$$\lim_{|p|\to\infty} \frac{H(p,z)}{|p|} = +\infty. \tag{21}$$

In this framework we can also introduce another type of weak solution to (19) based on a variational formulation.

3.3.1 Lagrangian formulation

We define the Lagrangian L associated with the Hamiltonian H as

$$L(v,z) = \sup_{p\in\mathbb{R}^d} \{p \cdot v - H(p,z)\}.$$

Note that this corresponds indeed to the Legendre transform of H. Note also that in view of (21) the sup above is indeed a max.

One can verify that L is itself a convex and super-linear function of v. Moreover, H is also equal to the Legendre transform of L.

The variational formulation of the problem is then given below

$$u(t,z) = \inf_{\gamma \in \mathcal{A}(t,z)} \int_0^t L(\dot{\gamma}(s), \gamma(s))ds + u_0(\gamma(0)), \tag{22}$$

with $\mathcal{A}$ the set of all admissible trajectories

$$\mathcal{A}(t,z) = \{\gamma(\cdot) \in C^2([0,t];\mathbb{R}^d) \mid \gamma(t) = z\}.$$

Under suitable assumptions such variational solution is indeed the unique viscosity solution of the problem and hence provides a representation formula for the unique viscosity solution of the problem. We will not show such property here (see for instance [Lio82, FS06, BCD97]). We show however, the relation between such variational problem with the characteristic ODEs.

Let's assume that the infimum value in the variational problem is indeed attained for a certain trajectory $\mathbf{z} \in \mathcal{A}$. We next deduce some of its properties.

Theorem 3.18 (Euler-Lagrange equation) *The function $\mathbf{z}(\cdot)$ solves the system of Euler-Lagrange equations*

$$-\frac{d}{ds}(D_v L(\dot{\mathbf{z}}(s), \mathbf{z}(s)) + D_z L(\dot{\mathbf{z}}(s), \mathbf{z}(s)) = 0, \qquad 0 \le s \le t. \tag{23}$$

Proof. Choose a smooth function $y : [0,t] \to \mathbb{R}^d$ such that

$$y(0) = y(t) = 0. \tag{24}$$

We define for all $h \in \mathbb{R}$, $x^h(\cdot) = \mathbf{z}(\cdot) + hy(\cdot)$. Then $x^h \in \mathcal{A}(t,z)$. Therefore, the function $J(h) = \int_0^t L(\dot{x}^h(s), x^h(s))ds + u_0(\mathbf{z}(0))$ has a minimum at $h = 0$, and consequently $J'(0) = 0$. We then compute this derivative:

$$J'(h) = \int_0^t \sum_{i=1}^d \big(\frac{\partial}{\partial v_i} L(\dot{x}_i^h(s), x_i^h(s))\dot{y}_i(s) + \frac{\partial}{\partial z_i} L(\dot{x}_i^h(s), x_i^h(s))y_i(s)\big)ds.$$

We then integrate by parts and use (24) to obtain

$$J'(h) = \int_0^t \sum_{i=1}^d \Big(-\frac{d}{ds}\big(\frac{\partial}{\partial v_i} L(\dot{x}_i^h(s), x_i^h(s))\big) + \frac{\partial}{\partial z_i} L(\dot{x}_i^h(s), x_i^h(s))\Big)y_i(s)ds.$$

We evaluate the above equality at $h = 0$ and find

$$J'(0) = \int_0^t \sum_{i=1}^d \Big(- \frac{d}{ds}\big(\frac{\partial}{\partial v_i} L(\dot{\mathbf{z}}(s), \mathbf{z}(s))\big) + \frac{\partial}{\partial z_i} L(\dot{\mathbf{z}}(s), \mathbf{z}(s))\Big) y_i(s) ds.$$

This identity holds for all smooth function $y(\cdot)$ satisfying the boundary condition (24). We conclude that, for all $1 \le i \le d$,

$$-\frac{d}{ds}\big(\frac{\partial}{\partial v_i} L(\dot{\mathbf{z}}(s), \mathbf{z}(s))\big) + \frac{\partial}{\partial z_i} L(\dot{\mathbf{z}}(s), \mathbf{z}(s)) = 0, \qquad \text{for all } 0 \le s \le t.$$

3.3.2 Hamilton's equations

We now transform the Euler-Lagrange equation into Hamilton's equations. Let $\mathbf{z}$ be a C^2 function that is a critical point of the action functional so that it solves the Euler-Lagrange equation (23).

We define

$$\mathbf{p}(s) := D_v L(\dot{\mathbf{z}}(s), \mathbf{z}(s)), \qquad 0 \le s \le t. \tag{25}$$

Here, $\mathbf{p}(\cdot)$ is known as the generalized momentum corresponding to the position $\mathbf{z}(\cdot)$ and velocity $\dot{\mathbf{z}}(\cdot)$.

Then, we have the following result.

Theorem 3.19 *The functions $\mathbf{z}(\cdot)$ and $\mathbf{p}(\cdot)$ satisfy Hamilton's equations:*

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_z H(\mathbf{p}(s), \mathbf{z}(s)), \\ \dot{\mathbf{z}}(s) = D_p H(\mathbf{p}(s), \mathbf{z}(s)). \end{cases}$$

Moreover, the value function $\mathbf{u}(s) = u(s, \mathbf{z}(s))$ satisfies

$$\dot{\mathbf{u}}(s) = D_p H(\mathbf{p}(s), \mathbf{z}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{z}(s)).$$

Proof. To simplify the notations we prove the result in one dimension $d = 1$. The multi-dimensional case can be treated following similar arguments.

Note first that since H is the Legendre transform of L, we have

$$H(p, z) = \sup_{v \in \mathbb{R}^d} \{p \cdot v - L(v, z)\}.$$

Note that such sup is also attained for some point $\mathbf{v}(p, z)$, such that

$$p = \frac{\partial}{\partial v} L(\mathbf{v}(p, z), z). \tag{26}$$

Since L is convex and super-linear, the function $\mathbf{v}(p)$ is uniquely determined. Now from the definition of $\mathbf{p}(s)$ in (25) we deduce that

$$\dot{\mathbf{z}}(s) = \mathbf{v}(\mathbf{p}(s), \mathbf{z}(s)). \tag{27}$$

Using

$$H(p, z) = \mathbf{v}(p, z) \cdot p - L(\mathbf{v}(p, z), z)\}.$$

we obtain

$$\begin{aligned} \tfrac{\partial}{\partial z} H(p, z) &= p \tfrac{\partial}{\partial z} \mathbf{v}(p, z) - \tfrac{\partial}{\partial v} L(\mathbf{v}(p, z), z) \tfrac{\partial}{\partial z} \mathbf{v}(p, z) - \tfrac{\partial}{\partial z} L(\mathbf{v}(p, z), z) \\ &= -\tfrac{\partial}{\partial z} L(\mathbf{v}(p, z), z). \end{aligned} \tag{28}$$

We also compute

$$\begin{aligned} \tfrac{\partial}{\partial p} H(p, z) &= p \tfrac{\partial}{\partial p} \mathbf{v}(p, z) + \mathbf{v}(p, z) - \tfrac{\partial}{\partial v} L(\mathbf{v}(p, z), z) \tfrac{\partial}{\partial p} \mathbf{v}(p, z) \\ &= \mathbf{v}(p, z). \end{aligned} \tag{29}$$

We obtain Hamilton's equations by combining (23), (26), (27), (28) and (29).

Note also that the characteristic equation for the value function $\mathbf{u}$ is given by

$$\begin{aligned} \dot{\mathbf{u}}(s) &= L(\dot{\mathbf{z}}(s), \mathbf{z}(s)) \\ &= \dot{\mathbf{z}}(s) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{z}(s)), \\ &= D_p H(\mathbf{p}(s), \mathbf{z}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{z}(s)). \end{aligned}$$

□

4 Well-posedness of the problem (4)

4.1 Choice of the growth rate and the initial population

In this section, we provide the main assumptions on the different parameters of the model.

We assume that R is smooth with respect to z and differentiable with respect to I and that there exist positive constants K_i such that, for any $z \in \mathbb{R}^d$ and $I \in \mathbb{R}^+$,

$$- K_1 \leq \frac{\partial}{\partial I} R(z, I) \leq -K_2. \tag{30}$$

Note that the above assumption means that the growth rate of any individual is a decreasing function of the total consumption of the resources, which is natural from the biological point of view. We also assume that the resources are limited such that if the the total consumption $I(t)$ reaches a certain threshold I_M, with I_M a positive constant, then the growth rate becomes negative everywhere, in other terms,

$$\sup_{z \in \mathbb{R}^d} R(z, I_M) = 0. \tag{31}$$

We furthermore, make the following technical assumptions on R:

$$- K_3 \leq \Delta(\psi R), \quad -K_4(1 + |z|^2) \leq R(z, I), \qquad \text{for all } 0 \leq I \leq 2I_M. \tag{32}$$

Note that assumptions (30)–(31) imply also that

$$\sup_{\substack{z \in \mathbb{R}^d \\ 0 \leq I \leq 2I_M}} R(z, I) \leq K_5.$$

We assume that the consumption rate ψ is a smooth function such that, for ψ_m and ψ_M some positive constants, we have

$$\psi_m \leq \psi(z) \leq \psi_M, \qquad \|\psi\|_{W^{2,\infty}(\mathbb{R}^d)} \leq K_6. \tag{33}$$

To provide the assumptions on the initial condition, we first define

$$u_{\varepsilon,0} = \varepsilon \log \left((2\pi\varepsilon)^{\frac{d}{2}} n_{\varepsilon,0} \right).$$

We then assume that there exist positive constants A_i and I_0 such that

$$-A_1 - A_2|z|^2 \leq u_{\varepsilon,0}(z) \leq A_3 - A_4|z| \quad \text{for all } z \in \mathbb{R}^d, \tag{34}$$

$$I_\varepsilon(0) \to I_0 > 0, \qquad \text{as } \varepsilon \to 0, \tag{35}$$

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^d} \psi(z) R(z, I_\varepsilon(0)) n_{\varepsilon,0}(z) dz \geq o(1), \qquad \text{as } \varepsilon \to 0. \tag{36}$$

Assumption (34) is to consider a well-prepared initial condition. The left inequality in (34) can indeed be relaxed, however, we keep it to make the computations more straight forward. The right inequality is to consider small population at infinity. Assumption (35) means that the initial population has a non-negligeable size. Assumption (36) indicates that initially the population is not very maladapted to the environment and it guarantees the non-extinction of the population. However, condition (36) is not necessary for the population to persist. A precise criterion for extinction versus survival of the population is given in [CEM21].

Example of admissible growth rate and initial condition: a typical example for the growth rate R, for $z \in \mathbb{R}$, is the following

$$R(z, I) = r - g(z-\theta)^2 - \kappa I.$$

Here, r is the maximal growth rate and θ corresponds to the optimal trait. Individuals with non-optimal traits suffer from a decreased growth rate depending on their distance with the optimal trait and proportionally to g, a positive constant that denotes the selection pressure. Finally, κ denotes the intensity of the competition.

For the consumption rate, we consider $\psi \equiv 1$.

A typical example for compatible initial condition is the following:

$$n_{\varepsilon,0}(z) = \frac{I_0}{\sqrt{2\pi\varepsilon}} e^{-\frac{(z-z_0)^2}{\varepsilon}},$$

with (z_0, I_0) such that

$$r - g(z_0 - \theta)^2 - I_0 > 0.$$

The above condition guarantees that Assumption (36) is satisfied, i. e. the initial population is not maladapted.

However, there is no reason to consider an initial condition which concentrates as a Dirac mass as $\varepsilon \to 0$, even if this will be the case for all positive times. One can for instance consider also the following initial condition

$$n_{\varepsilon,0}(z) = \begin{cases} C & \text{for } |z-z_0| < \delta, \\ \frac{1}{\sqrt{2\pi\varepsilon}} e^{\frac{\varphi_\varepsilon(z)}{\varepsilon}} & \text{for } 1 \le |z-z_0| < 2\delta, \\ \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{(z-z_0)^2}{\varepsilon}} & \text{for } 2\delta < |z-z_0|. \end{cases}$$

To satisfy the assumptions, we choose φ a positive and smooth function such that

$$\varphi_\varepsilon(z) = \varepsilon \log(C\sqrt{2\pi\varepsilon}), \quad \text{for } |z-z_0| = \delta, \quad \text{and} \quad \varphi_\varepsilon(z) = -4\delta^2, \quad \text{for } |z-z_0| = 2\delta.$$

We also choose C and φ_ε such that, as $\varepsilon \to 0$, $\int_{\mathbb{R}} n_{\varepsilon,0}(z)dz \to I_0$, with (z_0, I_0) such that

$$r - g(y-\theta)^2 - I_0 \ge 0, \quad \text{for all } |y - z_0| \le 2\delta.$$

The results presented below hold under the above assumptions. However, in order to simplify the computations during the lecture we may replace assumption (32) by the following more restrictive assumptions

$$\sup_{0 \le I \le 2I_M} \|R(\cdot, I)\|_{W^{2,\infty}(\mathbb{R}^d)} < K_5, \tag{37}$$

and

$$\exists I_m > 0, \quad \text{s.t.} \quad \inf_{z\in\mathbb{R}^d} R(z, I_m) = 0. \tag{38}$$

4.2 Well-posedness

Theorem 4.1 *Under assumptions* (30)–(33) *and* (35)–(36)*, there is a unique solution* $n_\varepsilon \in \mathrm{C}(\mathbb{R}^+; L^1(\mathbb{R}^d))$ *to* (4)*. This solution is nonnegative for all* $t \ge 0$*. Moreover, assuming additionally* (38)*, there exists* $\varepsilon_0 > 0$ *such that we have*

$$I'_m = \min(\frac{I_m}{2}, \frac{I_0}{2}) \le I_\varepsilon(t) \le I'_M = 2I_M, \quad \textit{for all } \varepsilon \le \varepsilon_0. \tag{39}$$

Remark 4.2 *Assumption* (38) *has been made in the theorem above to prove the uniform lower bound on* I_ε*. However, this assumption may be relaxed. In [CEM21] rather general conditions are provided for the population size to remain uniformly bounded away from* 0*.*

Proof. We prove the result under additional assumptions (37)–(38) using the Banach-Picard fixed point theorem. We do this in several steps.

Step 1. We first prove that there exists a unique solution to an auxiliary problem, where R

is replaced by $\widetilde{R}$, defined as below

$$\widetilde{R}(z,I) = \begin{cases} R(z,I) & \text{if} \quad I'_m < I < I'_M, \\ R(z,I'_M) & \text{if} \quad I'_M \le I, \\ R(z,I'_m) & \text{if} \quad I \le I'_m, \end{cases}$$

and in a short interval of time.

Let $T>0$ be given and A be the following closed subset:

$$\mathrm{A} = \{u \in \mathrm{C}\big([0,T], L^1(\mathbb{R}^d)\big),\ u \ge 0,\ \|u\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} \le a\},$$

where $a = \left(\int_{\mathbb{R}^d} n_{\varepsilon,0} dz\right) e^{\frac{K_5 T}{\varepsilon}}$. Let Φ be the following application:

$$\Phi : \mathrm{A} \to \mathrm{A}$$

$$u \mapsto v,$$

where v is the solution to the following equation

$$\begin{cases} \partial_t v - \varepsilon \Delta v = \frac{v}{\varepsilon} \widetilde{R}(z, I_u(t)), \quad z \in \mathbb{R}^d,\ t \ge 0, \\ v(t=0) = n_{\varepsilon,0}. \end{cases} \tag{40}$$

$$I_u(t) = \int_{\mathbb{R}^d} \psi(z) u(t,z) dz. \tag{41}$$

Note that from the assumptions that $n_{\varepsilon,0} \in L^2(\mathbb{R}^d)$. Therefore, the existence of a weak solution v to the equation above follows for instance from the Galerkin method. Moreover, from the regularizing effects of the heat operator, we obtain that such solution is indeed smooth and classical (see for instance [Eva98] for more details).

We prove that

(a) Φ defines a mapping of A into itself,

(b) Φ is a contraction for T small.

With these properties, we can apply the Banach-Picard fixed point theorem to prove that there exists a unique solution to (40) for $t \in [0,T]$.

Assume that $u \in \mathrm{A}$. In order to prove (a) we show that v, the solution to (40), belongs to A.

By the maximum principle we know that $v \geq 0$. To prove the L^1 bound we integrate (40)

$$\frac{d}{dt}\int_{\mathbb{R}^d} v dz = \int_{\mathbb{R}^d} \frac{v}{\varepsilon}\widetilde{R}(z, I_u(t))dz \leq \frac{1}{\varepsilon}\sup_{z\in\mathbb{R}^d} \widetilde{R}(z, I_u(t))\int_{\mathbb{R}^d} v dz \leq \frac{K_5}{\varepsilon}\int_{\mathbb{R}^d} v dz.$$

Note that to do the integration rigorously one could multiply first the equation by χ_L, a smooth function with a compact support such that $\chi_L|_{\mathrm{B}(0,L)} \equiv 1$, $\chi_L|_{\mathbb{R}\setminus\mathrm{B}(0,2L)} \equiv 0$. Then integrate to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \psi_L v dz = \int_{\mathbb{R}^d} \psi_L \frac{\partial}{\partial t} v dz = \int \Delta\psi_L v dz + \int_{\mathbb{R}^d} \frac{v}{\varepsilon}\widetilde{R}(z, I_u(t))dz.$$

The integration above then follows by letting L tend to $+\infty$.

We conclude from the Gronwall Lemma that

$$\|v\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} \leq \left(\int_{\mathbb{R}^d} n_{\varepsilon,0} dz\right) e^{\frac{K_5 T}{\varepsilon}} = a.$$

Thus (a) is proved.

It remains to prove (b). Let $u_1,\ u_2 \in \mathrm{A}$, $v_1 = \Phi(u_1)$ and $v_2 = \Phi(u_2)$. We have

$$\partial_t(v_1 - v_2) - \varepsilon\Delta(v_1 - v_2) = \frac{1}{\varepsilon}\left[(v_1 - v_2)\widetilde{R}(z, I_{u_1}) + v_2\left(\widetilde{R}(z, I_{u_1}) - \widetilde{R}(z, I_{u_2})\right)\right]. \tag{42}$$

It is then tempting to multiply the equation above by $\mathrm{sgn}(v_1 - v_2)$ and integrate with respect to z to obtain an inequality on $\|(v_1 - v_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)}$ which is the quantity that we would like to control. However, to do this we encounter the technical difficulty that the function $\mathrm{sgn}(v_1 - v_2)$ is not smooth. We hence use a sequence of smooth functions (S_δ) which approach the sign function in the following way

$$\begin{cases} S_\delta \text{ is smooth, even and convex}, \qquad \max(0, |x| - \delta) \leq S_\delta(x) \leq |x|, \\ 0 \leq \mathrm{sgn}(x) S'_\delta(x) \leq 1, \qquad 0 \leq \frac{x}{2} S'_\delta(x) \leq S_\delta(x). \end{cases}$$

We then multiply (42) by $S'_\delta(v_1 - v_2)$ to obtain

$$\begin{aligned}\partial_t S_\delta(v_1 - v_2) - \varepsilon\Delta S_\delta(v_1 - v_2) &= -\varepsilon S''_\delta(v_1 - v_2)|\nabla v_1 - \nabla v_2|^2 \\ &\quad + \tfrac{1}{\varepsilon} S'_\delta(v_1 - v_2)\left[(v_1 - v_2)\widetilde{R}(z, I_{u_1}) + v_2\left(\widetilde{R}(z, I_{u_1}) - \widetilde{R}(z, I_{u_2})\right)\right].\end{aligned}$$

Next, thanks to the assumptions on S_δ, we obtain that

$$\partial_t S_\delta(v_1 - v_2) - \varepsilon\Delta S_\delta(v_1 - v_2) \leq \frac{1}{\varepsilon}\left[2S_\delta(v_1 - v_2)|\widetilde{R}(z, I_{u_1})| + v_2\left|\widetilde{R}(z, I_{u_1}) - \widetilde{R}(z, I_{u_2})\right|\right].$$

Noting that $|\widetilde{R}(z, I_{u_1}) - \widetilde{R}(z, I_{u_2})| \leq K_1 |I_{u_1} - I_{u_2}| \leq K_1 \psi_M \|u_1 - u_2\|_{L^\infty([0,T];L^1(\mathbb{R}^d))}$ we find that

$$\partial_t S_\delta(v_1 - v_2) - \varepsilon \Delta S_\delta(v_1 - v_2) \leq \frac{2K_5}{\varepsilon} S_\delta(v_1 - v_2) + \frac{v_2(z) K_1 \psi_M}{\varepsilon} \|(u_1 - u_2)(t, \cdot)\|_{L^1(\mathbb{R}^d)}.$$

We then integrate the equation above following similar arguments as above, using truncation functions χ_L, and recalling that $\|v_2\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} \leq a$, to obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} S_\delta(v_1 - v_2)(t, z) dz \leq \frac{2K_5}{\varepsilon} \int_{\mathbb{R}^d} S_\delta(v_1 - v_2)(t, z) dz + \frac{a K_1 \psi_M}{\varepsilon} \|(u_1 - u_2)(t, \cdot)\|_{L^\infty([0,t];L^1(\mathbb{R}^d))}.$$

Applying the Grönwall lemma and using the fact that $v_1(0, \cdot) = v_2(0, \cdot)$ we deduce that

$$\int_{\mathbb{R}^d} S_\delta(v_1 - v_2)(t, z) dz \leq \frac{a K_1 \psi_M}{2K_5} (e^{\frac{2K_5 t}{\varepsilon}} - 1) \|u_1 - u_2\|_{L^\infty([0,t];L^1(\mathbb{R}^d))}.$$

We then let $\delta \to 0$ to find

$$\|v_1 - v_2\|_{L^\infty([0,T];L^1(\mathbb{R}^d))} \leq \frac{a K_1 \psi_M}{2K_5} (e^{\frac{2K_5 T}{\varepsilon}} - 1) \|u_1 - u_2\|_{L^\infty([0,T];L^1(\mathbb{R}^d))}.$$

Thus, for T small enough such that $e^{\frac{K_5 T}{\varepsilon}} (e^{\frac{2K_5 T}{\varepsilon}} - 1) < \frac{K_5}{K_1 \psi_M \int \widetilde{n}_{\varepsilon,0} dz}$, Φ is a contraction. Therefore Φ has a fixed point and there exists $\widetilde{n}_\varepsilon \in \mathrm{A}$ a solution to the following equation, in $[0, T] \times \mathbb{R}^d$,

$$\begin{cases} \partial_t \widetilde{n}_\varepsilon - \varepsilon \Delta \widetilde{n}_\varepsilon = \frac{\widetilde{n}_\varepsilon}{\varepsilon} \widetilde{R}(z, \widetilde{I}_\varepsilon(t)), \\ \widetilde{I}_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(z) \widetilde{n}_\varepsilon(t, z) dz, \\ \widetilde{n}_\varepsilon(t = 0) = n_{\varepsilon,0}. \end{cases} \tag{43}$$

Exercise 4.3 *Let m solve the following equation*

$$\begin{cases} \partial_t m(t, x) - \sigma \Delta m(, x) = f(t, x), \qquad x \in \mathbb{R}^d \\ m(0, x) = m_0(x), \end{cases}$$

with $m_0 \in L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$. Prove that

$$\|m(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d)} ds + \|m_0\|_{L^2(\mathbb{R}^d)}.$$

Step 2. We next prove that for any solution $(n_\varepsilon, I_\varepsilon)$ to (4) and for any solution $(\widetilde{n}_\varepsilon, \widetilde{I}_\varepsilon)$ to (43) we have, for $\varepsilon \leq \varepsilon_0$ with ε_0 a small positive constant,

$$I'_m \leq \widetilde{I}_\varepsilon(t),\ I_\varepsilon(t) \leq I'_M, \qquad t \in [0, \infty).$$

This property implies that $\widetilde{n}_\varepsilon$ is indeed the unique solution to (4) and that (39) holds.

This property can be proved following similar arguments for I_ε and $\widetilde{I}_\varepsilon$. We prove the property for I_ε.

To this end, we define $\psi_L = \chi_L \cdot \psi \in \mathbf{W}^\infty_{2,c}(\mathbb{R}^d)$, where χ_L is a smooth function with a compact support such that $\chi_L|_{\mathrm{B}(0,L)} \equiv 1$, $\chi_L|_{\mathbb{R}\setminus\mathrm{B}(0,2L)} \equiv 0$, $\|\chi_L\|_{W^{2,\infty}} \leq C$. We also define

$$I_{\varepsilon,L} = \int_{\mathbb{R}^d} \psi_L(z) n_\varepsilon(t,z)dz.$$

We next multiply the equation in (4) by $\psi_L(z)$ and then integrate with respect to z, performing an integration by parts, to obtain

$$\frac{d}{dt} I_{\varepsilon,L} = \int_{\mathbb{R}^d} \psi_L(z) \frac{\partial}{\partial t} n_\varepsilon(t,z)dz = \varepsilon \int_{\mathbb{R}^d} n_\varepsilon(t,z)\Delta\psi_L(z)dz + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \psi_L(z) n_\varepsilon(t,z) R(z, I_\varepsilon(t))dz.$$

We then let $L \to \infty$ to obtain

$$\frac{dI_\varepsilon}{dt} = \varepsilon \int_{\mathbb{R}^d} n_\varepsilon(t,z)\Delta\psi(z)dz + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \psi(z) n_\varepsilon(t,z) R(z, I_\varepsilon(t))dz.$$

It follows that

$$-\varepsilon \frac{K_6}{\psi_m} I_\varepsilon + \frac{1}{\varepsilon} I_\varepsilon \inf_{z\in\mathbb{R}^d} R(z, I_\varepsilon) \leq \frac{dI_\varepsilon}{dt} \leq \varepsilon \frac{K_6}{\psi_m} I_\varepsilon + \frac{1}{\varepsilon} I_\varepsilon \sup_{z\in\mathbb{R}^d} R(z, I_\varepsilon).$$

It follows that, for C chosen large enough and using assumptions (30), (31) and (38), if $I_\varepsilon \geq I_M + C\varepsilon^2$, we then have $\frac{dI_\varepsilon}{dt} < 0$. Similarly, if $I_\varepsilon \leq I_m - C\varepsilon^2$, we then have $\frac{dI_\varepsilon}{dt} > 0$. Furthermore, thanks to assumptions (30), (31) and (36), we obtain that $I_\varepsilon(0) \leq I'_M$ for ε small enough. Hence (39).

Step 3. We now fix T small enough such that $e^{\frac{K_5 T}{\varepsilon}}(e^{\frac{2K_5 T}{\varepsilon}} - 1) < \frac{K_5}{K_1 \psi_M I'_m}$. The previous steps imply that there exist a unique solution to (4) in $[0,T] \times \mathbb{R}^d$. Moreover, since $I'_m \leq \int n_\varepsilon(t,z)dz$ for all t thanks to Step 2, one can iterate the procedure above to find a unique solution n_ε in $[0,\infty) \times \mathbb{R}^d$.

5 Asymptotic analysis of the problem (4)

In this section, we provide the main results on the asymptotic analysis of equation (4), that we recall below

$$\begin{cases} \varepsilon\partial_t n_\varepsilon - \varepsilon^2\Delta n_\varepsilon = n_\varepsilon\, R(z, I_\varepsilon), & (t,z)\in\mathbb{R}^+\times\mathbb{R}^d,\\ I_\varepsilon(t) = \int_{\mathbb{R}^d} n_\varepsilon(t,y)\psi(y)dy, \\ n_\varepsilon(0,z) = n_{\varepsilon,0}(z). \end{cases}$$

Our objective is to prove that the solution n_ε converges, as $\varepsilon\to 0$, to a Dirac mass. To this end, we use the Hopf-Cole transformation (5) which, when combined by the above equation, leads to the following equation on u_ε :

$$\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon = |\nabla u_\varepsilon|^2 + R(z, I_\varepsilon). \tag{44}$$

To study the limit of n_ε as $\varepsilon\to 0$, we first provide an asymptotic analysis of u_ε. We prove that

Theorem 5.1 (Convergence to a Hamilton-Jacobi equation with constraint) *Assume* (30)–(36). *Let n_ε be the solution of* (4)*, and u_ε be given by* (5)*. Assume additionally that $(u_{\varepsilon,0})_\varepsilon := \log(n_{\varepsilon,0})$ is a sequence of locally uniformly Lipschitz continuous functions which converges locally uniformly to u_0. Then, after extraction of a subsequence, $(I_\varepsilon)_\varepsilon$ converges a.e. to $I\in \mathrm{BV}(\mathbb{R}^+)$ and $(u_\varepsilon)_\varepsilon$ converges locally uniformly to $u\in \mathrm{C}([0,\infty)\times\mathbb{R}^d)$, a viscosity solution to the following equation:*

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(z, I(t)), \\ \max\limits_{z\in\mathbb{R}^d} u(t,z) = 0, \quad \forall t>0, \\ u(0,z) = u_0(z). \end{cases} \tag{45}$$

In the above equation, I can be seen as a Lagrange multiplier associated with the constraint $\max_{z\in\mathbb{R}^d} u(t,z) = 0$. The above theorem was proved in [PB08] with the uniform continuity assumption of the initial data $(u_{\varepsilon,0})_\varepsilon$ and without such assumption in [BMP09]. The above theorem provides the convergence along subsequences and does not guarantee the uniqueness of the limit. The uniqueness property was proved later in [CL]:

Theorem 5.2 (Uniqueness property for the Hamilton-Jacobi equation with constraint) *Let $(u_i, I_i)\in \mathrm{C}(\mathbb{R}^+\times\mathbb{R}^d)\times \mathrm{BV}(\mathbb{R}^+)$, for $i=1,2$ such that (u_i, I_i) solves*

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(z, I(t)), \\ \max\limits_{z\in\mathbb{R}^d} u(t,z) = 0, \quad \forall t>0, \\ u(0,z) = u_0(z). \end{cases} \tag{46}$$

in the viscosity sense. Then, $I_1 = I_2$ *a.e. and* $u_1 = u_2$.

A consequence of the above result is that, in the case of locally uniformly Lipschitz initial data, the whole sequence $(u_\varepsilon)_e$ converges to the unique limit u. Such uniqueness result, before being proved in the above general form, was first proved in [PB08] for a particular form of growth rate, and next in [MR16] in a concave framework, where R and u_0 are strictly concave with respect to z. In this latter case, one can go further than the uniqueness result and obtain the regularity of the solution (u, I) and also an asymptotic expansion for the solution u_ε in terms of ε.

Once the limit of $(u_\varepsilon)_\varepsilon$ identified, we can obtain some information on n, the limit of n_ε, which allows us to completely characterize n in some cases.

Theorem 5.3 (The inclusion properties) *Assume* (30)–(36)*. Then, as* $\varepsilon \to 0$ *and along subsequences,* n_ε *converges weakly in the sense of measures to* $n \in L^\infty(\mathbb{R}^+; M^1(\mathbb{R}^d))$ *such that, for a.e.* t,

$$\int \psi(z) n(t,z) dz = I(t) \quad a.e., \tag{47}$$

and

$$\text{supp } n(t,\cdot) \subset \{z \,|\, u(t,z) = 0\} \subset \{z \,|\, R(z, I(t)) = 0\}, \qquad \textit{for a.e. } t. \tag{48}$$

The inclusion property (48) is a key point in the description of the limit n. In particular, if one of the sets $\{z \,|\, u(t,z) = 0\}$ or $\{z \,|\, R(z, I(t)) = 0\}$ has a single point, then (48) implies that the measure n has to be a Dirac mass at that point. Such property holds for instance in the case where $z \in \mathbb{R}$ and R is monotonic with respect to z, such that the set $\{z \,|\, R(z, I(t)) = 0\}$ has a single element. Similarly if u is a strictly concave function with respect to z, then the set $\{z \,|\, u(t,z) = 0\}$ which corresponds to the set of maximum points of u consists of a single point. The strict concavity of u can be guaranteed under concavity assumptions on R and u_0 (see Section 6).

5.1 Main ingredients

The proof of Theorem 5.1, on the convergence of (u_ε), is based on the following results.

Theorem 5.4 *Assume* (30)–(33) *and* (35)–(36)*. Then,* $(I_\varepsilon)_\varepsilon$ *is locally uniformly bounded in* $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$*, and* I_ε *converges, along subsequences, in* $L^1_{\text{loc}}(\mathbb{R}^+)$ *and a.e. to* $I : \mathbb{R}^+ \to \mathbb{R}^+$*, which is non-decreasing and locally of bounded variation.*

Theorem 5.5 *Assume* (30)–(36)*. Then,*
(i) there exists positive constants C *and* ε_0*, such that we have the following uniform bounds on*

u_ε, *for* $\varepsilon \le \varepsilon_0$*:*

$$-A_1 - A_2|z|^2 - Ct \le u_\varepsilon(t,z) \le A_3 - A_4|z| + Ct. \tag{49}$$

(ii) The sequence $(u_\varepsilon)_{\varepsilon\le\varepsilon_0}$ *is locally uniformly Lipschitz with respect to* z *in* $(0,+\infty)\times\mathbb{R}^d$*.*
(iii) The sequence $(u_\varepsilon)_{\varepsilon\le\varepsilon_0}$ *is locally equicontinuous with respect to* t *in* $(0,+\infty)\times\mathbb{R}^d$*.*
(iv) If we assume that $(u_\varepsilon^0)_\varepsilon$ *is a sequence of locally uniformly Lipschitz functions, then* (u_ε) *is locally uniformly Lipschitz in* $[0,\infty)\times\mathbb{R}^d$*.*

5.2 Regularity estimates on I_ε: the proof of Theorem 5.4

We prove the result under additional assumptions (37)–(38).

We prove that for all $T>0$, there exists a positive constant C such that

$$\int_0^T |\frac{d}{dt}I_\varepsilon(s)ds| \le C.$$

This estimate together with (39) implies that $(I_\varepsilon)_\varepsilon$ is locally uniformly bounded in $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$, and hence it converges, along subsequences, in $L^1_{loc}(\mathbb{R}^+)$ and a.e. to a function $I:\mathbb{R}^+\to\mathbb{R}^+$ which is of bounded variation.

Using the computations in Section 4.2 we write

$$\frac{dI_\varepsilon}{dt} = \varepsilon\int_{\mathbb{R}^d} n_\varepsilon(t,z)\Delta\psi(z)dz + \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)R(z,I_\varepsilon(t))dz. \tag{50}$$

Let's define

$$J_\varepsilon(t) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)R(z,I_\varepsilon(t))dz. \tag{51}$$

We then differentiate J_ε:

$$\begin{aligned}\tfrac{d}{dt}J_\varepsilon(t) &= \textstyle\int_{\mathbb{R}^d} n_\varepsilon\Delta(\psi(z)R(z,I_\varepsilon(t)))dz + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)R^2(z,I_\varepsilon(t))dz\\ &\quad+\textstyle\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)\frac{\partial}{\partial I}R(z,I_\varepsilon(t))dz\,\frac{d}{dt}I_\varepsilon(t).\end{aligned}$$

We next use (50)–(51) to recover J_ε:

$$\begin{aligned}\tfrac{d}{dt}J_\varepsilon &= \textstyle\int_{\mathbb{R}^d} n_\varepsilon\Delta(\psi(z)R(z,I_\varepsilon(t)))dz + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)R^2(z,I_\varepsilon(t))dz\\ &\quad+\textstyle\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)\frac{\partial}{\partial I}R(z,I_\varepsilon(t))dz\,\big(J_\varepsilon - \varepsilon\int_{\mathbb{R}^d}n_\varepsilon(t,z)\Delta\psi(z)dz\big).\end{aligned}$$

Thanks to (39) and the regularity assumptions on R and ψ we obtain that

$$\frac{d}{dt}J_\varepsilon = O(1) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)R^2(z,I_\varepsilon(t))dz + \frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d}\psi(z)n_\varepsilon(t,z)\frac{\partial}{\partial I}R(z,I_\varepsilon(t))dz\Big)\,J_\varepsilon(t). \tag{52}$$

Furthermore, using assumption (30) together with (39) we deduce that

$$\int_{\mathbb{R}^d} \psi(z) n_\varepsilon(t,z) \frac{\partial}{\partial I} R(z, I_\varepsilon(t)) dz \leq -C_1.$$

We then multiply (52) by $\mathbb{1}_{J_\varepsilon<0}$ and use the fact that

$$(\mathbb{1}_{J_\varepsilon<0}) \frac{d}{dt} J_\varepsilon = \frac{d}{dt}(J_\varepsilon)_-, \qquad \text{a.e.,}$$

to obtain that

$$\frac{d}{dt}(J_\varepsilon)_- \leq C_2 - \frac{C_1}{\varepsilon}(J_\varepsilon)_-.$$

Note that here we have used the positivity of the term $\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \psi(z) n_\varepsilon(t,z) R^2(z, I_\varepsilon(t)) dz$. Using the Gronwall Lemma we then obtain that

$$(J_\varepsilon)_-(t) \leq \frac{\varepsilon C_2}{C_1} + (J_\varepsilon)_-(0) e^{-\frac{C_1 t}{\varepsilon}}.$$

Inserting this in (50) and using (33), (36) and (39), we obtain that as $\varepsilon \to 0$

$$\big(\frac{d}{dt} I_\varepsilon(t)\big)_- \leq o(1). \tag{53}$$

We then use (39) to conclude

$$\begin{aligned}
\int_0^T |\tfrac{d}{dt} I_\varepsilon(t)| dt &= \int_0^T \tfrac{d}{dt} I_\varepsilon(t) dt + 2\int_0^T (\tfrac{d}{dt} I_\varepsilon(t))_- dt \\
&= I_\varepsilon(T) - I_\varepsilon(0) + 2\int_0^T (\tfrac{d}{dt} I_\varepsilon(t))_- dt \\
&\leq I'_M + o(1) T.
\end{aligned}$$

As explained above, this inequality implies that $(I_\varepsilon)_\varepsilon$ is locally uniformly bounded in $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$, and hence it converges, along subsequences, in $L^1_{\text{loc}}(\mathbb{R}^+)$ and a.e. to a function $I : \mathbb{R}^+ \to \mathbb{R}^+$ which is of bounded variation. Finally, the function I is nondecreasing thanks to (53).

5.3 Regularity estimates on u_ε: the proof of Theorem 5.5

In this section, we provide the main ingredients for the proof of Theorem 5.5, assuming additionally (37)–(38).

5.3.1 Locally uniform bounds on u_ε

The proof of the bounds on u_ε is equivalent with

$$\exp\Big(-\frac{A_1+A_2|z|^2+Ct}{\varepsilon}\Big)\leq n_\varepsilon(t,z)\leq \exp\Big(\frac{A_3-A_4|z|+Ct}{\varepsilon}\Big).$$

We prove this using the comparison principle, the assumption (34) on the initial condition and the fact that, thanks to the assumption (37),

$$-K_5 n_\varepsilon \leq \varepsilon\partial_t n_\varepsilon - \varepsilon^2\Delta n_\varepsilon = n_\varepsilon R(z,I_\varepsilon)\leq K_5 n_\varepsilon,$$

5.3.2 Lipschitz estimates

Note from (49) that, for all $T>0$, there exists a constant $D(T)$ such that

$$u_\varepsilon(t,z)\leq D(T),\qquad \text{for all } (t,z)\in[0,T]\times\mathbb{R}^d.$$

We define, for all $(t,z)\in[0,T]\times\mathbb{R}^d$,

$$v_\varepsilon(t,z)=\sqrt{2D(T)-u_\varepsilon(t,z)}. \tag{54}$$

To prove the locally uniform Lipschitz bound on u_ε, in $(0,T]\times\mathbb{R}^d$, we will prove that, for all $0<t_0<T$,

$$|\nabla v_\varepsilon|(t,z)\leq C(T)(1+\frac{1}{\sqrt{t_0}}),\qquad \text{for all } t_0\leq t\leq T \text{ and } z\in\mathbb{R}^d. \tag{55}$$

We replace (54) in (44), neglecting the subscript ε here for the simplification of notations, we obtain the following equation on v in $[0,T]\times\mathbb{R}^d$:

$$\partial_t v-\varepsilon\Delta v-\left[\frac{\varepsilon}{v}-2v\right]|\nabla v|^2=-\frac{R(z,I)}{2v}\,. \tag{56}$$

Define $p=\nabla v$. We differentiate (56) with respect to z_i to obtain that

$$\partial_t p_i-\varepsilon\Delta p_i-2\left[\frac{\varepsilon}{v}-2v\right]\nabla v\cdot\nabla p_i+\left[\frac{\varepsilon}{v^2}+2\right]|\nabla v|^2 p_i=-\frac{\frac{\partial}{\partial z_i}R(z,I)}{2v}+\frac{R(z,I)}{2v^2}p_i\,.$$

We multiply the equation by p_i and sum over i to obtain that

$$\partial_t\frac{|p|^2}{2}-\varepsilon\sum_i(\Delta p_i)p_i-2\left[\frac{\varepsilon}{v}-2v\right]p\cdot\nabla\frac{|p|^2}{2}+\left[\frac{\varepsilon}{v^2}+2\right]|p|^4=-\frac{1}{2v}\nabla_z R\cdot p+\frac{1}{2v^2}R(z,I)|p|^2.$$

We then divide the above equation by $|p|$ to find that

$$\partial_t|p| - \varepsilon\Delta|p| - 2\left[\frac{\varepsilon}{v} - 2v\right] p\cdot\nabla|p| + \left[\frac{\varepsilon}{v^2} + 2\right]|p|^3 \leq -\frac{1}{2v}\nabla_z R\cdot\frac{p}{|p|} + \frac{1}{2v^2}R(z,I)|p|. \qquad (57)$$

Here we have used the fact that

$$-\varepsilon|p|\Delta|p| \leq -\varepsilon\sum_i(\Delta p_i)p_i.$$

To prove this, we first compute $\sum_i(\Delta p_i)p_i$:

$$\begin{aligned}\textstyle\sum_i(\Delta p_i)p_i &= \textstyle\sum_i \Delta\frac{p_i^2}{2} - \sum|\nabla p_i|^2\\ &= \Delta\tfrac{|p|^2}{2} - \textstyle\sum|\nabla p_i|^2\\ &= |p|\Delta|p| + |\nabla|p||^2 - \textstyle\sum_i|\nabla p_i|^2.\end{aligned}$$

We also have

$$|\nabla|p||^2 = \textstyle\sum_i\frac{|p\cdot\partial_{z_i}p|^2}{|p|^2} \leq \sum_i|\partial_{z_i}p|^2 = \sum_{i,j}|\partial_{z_i}p_j|^2 = \sum_j|\nabla p_j|^2.$$

We deduce that

$$\sum_i(\Delta p_i)p_i \leq |p|\Delta|p|.$$

We now go back to inequality (57). We note that from the assumption (37) we have that R and ∇R are bounded. Moreover, using (54), we find that

$$\frac{1}{v} \leq \frac{1}{\sqrt{D(T)}}.$$

We deduce that, for some positive constant $C(T)$,

$$\partial_t|p| - \varepsilon\Delta|p| - 2\left[\frac{\varepsilon}{v} - 2v\right] p\cdot\nabla|p| + 2|p|^3 \leq C(T)(1+|p|).$$

As a consequence, and thanks to (49), there exists positive constants ε_0, C_1 and $C_2(T)$ and $\theta(T)$ such that, for all $\varepsilon\leq\varepsilon_0$ and $(t,z)\in[0,T]\times\mathbb{R}^d$,

$$\partial_t|p| - \varepsilon\Delta|p| - [C_1|z| + C_2]\,|p|\,|\nabla|p|| + 2(|p|-\theta(T))^3 \leq 0. \qquad (58)$$

Define the function

$$q(t,z) = q(t) = \frac{1}{2\sqrt{t}} + \theta + 1.$$

One can verify that q is a strict supersolution to (58). We would like to prove that

$$|p|(t,z) \leq q(t). \tag{59}$$

Let t_0 be the first time such that $\sup_{z\in\mathbb{R}^d} |p|(t,z) - q(t) = 0$. Note that t_0 cannot be equal to 0 since $q(0) = +\infty$. Let's suppose that $|p|(t_0,\cdot) - q(t_0)$ attains its maximum at an interior point z_0 of $\mathbb{R}^d$. We then have

$$0 \leq \partial_t(|p| - q)(t_0, z_0), \qquad 0 \leq -\Delta(|p| - q)(t_0, z_0), \quad \text{and} \quad |p|\,|\nabla|p|| = q\,|\nabla q|\,(t_0, z_0).$$

Combining the above properties with the facts that $|p|$ and z are respectively sub and strict supersolution of (57), we obtain that

$$2(|p|(t_0,z_0) - \theta)^3 - 2(q(t_0,z_0) - \theta)^3 < 0.$$

It follows that

$$|p|(t_0,z_0) < q(t_0,z_0),$$

which is in contradiction with the choice of (t_0,z_0).

Note however that the maximum of $|p|(t_0,\cdot) - q(t_0)$ may not be attained at interior point of $\mathbb{R}^d$. To overcome this difficulty one can modify the supersolution q by adding a barrier function forcing the maximum point of $|p| - q$ to be attained at an interior point [BMP09]. This argument is left as an exercise below.

Exercise 5.6 *For $(t,z) \in [0,T] \times B_L(0)$ and C a positive constant, define*

$$w(t,z) = \frac{1}{2\sqrt{t}} + \frac{CL^2}{L^2 - |z|^2} + \theta.$$

(i) Show that for C large enough w is a strict supersolution of (58).
(ii) Prove that $|p| \leq w(t,z)$ for C large enough and consequently (59) *by letting $L \to \infty$.*

Exercise 5.7 *Let's suppose that $(u_{\varepsilon,0})$ is locally uniformly Lipschitz with respect to z. Prove that (u_ε) is is locally uniformly Lipschitz with respect to z in $[0,+\infty) \times \mathbb{R}^d$.*
Hint: Follow the arguments above replacing $w(t,z)$ by

$$w(t,z) = \frac{1}{2\sqrt{t + C_0}} + \frac{CL^2}{L^2 - |z|^2} + \theta.$$

5.3.3 Equi-continuity in time

From the uniform bounds and the Lipschitz bound with respect to z, we can also deduce uniform continuity in time i.e. for all $\eta > 0$, there exists $\theta > 0$ such that for all $(t, s, z) \in [0, T] \times [0, T] \times \mathrm{B}(0, \frac{R}{2})$, such that $0 < t - s < \theta$, and for all $\varepsilon < \varepsilon_0$ we have:

$$|u_\varepsilon(t, z) - u_\varepsilon(s, z)| \leq 2\eta.$$

We prove this with the same method as that of Lemma 9.1 in [BBL02] (see also [BBAL09] for another proof of this claim). We prove that for any $\eta > 0$, we can find positive constants A, B large enough such that, for any $z \in \mathrm{B}(0, \frac{R}{2})$, $s \in [0, T]$ and for every $\varepsilon < \varepsilon_0$,

$$u_\varepsilon(t, y) - u_\varepsilon(s, z) \leq \eta + A|z - y|^2 + B(t - s), \quad \text{for all } (t, y) \in [s, T] \times \mathrm{B}(0, R), \tag{60}$$

and

$$u_\varepsilon(t, y) - u_\varepsilon(s, z) \geq -\eta - A|z - y|^2 - B(t - s), \quad \text{for all } (t, y) \in [s, T] \times \mathrm{B}(0, R). \tag{61}$$

We prove inequality (60), the proof of (61) is analogous. We fix (s, z) in $[0, T[\times \mathrm{B}(0, \frac{R}{2})$. Define

$$\xi(t, y) = u_\varepsilon(s, z) + \eta + A|y - z|^2 + B(t - s), \qquad (t, y) \in [s, T[\times \mathrm{B}(0, R),$$

where A and B are constants to be determined. We prove that, for A and B large enough, ξ is a super-solution to (44) on $[s, T] \times \mathrm{B}(0, R)$ and $\xi(t, y) > u_\varepsilon(t, y)$ for $(t, y) \in \{s\} \times \mathrm{B}(0, R) \cup [s, T] \times \partial\mathrm{B}(0, R)$.

According to section 5.3.1, u_ε are locally uniformly bounded, so we can take A a constant such that for all $\varepsilon < \varepsilon_0$,

$$A \geq \frac{8 \parallel u_\varepsilon \parallel_{L^\infty([0,T] \times \mathrm{B}(0,R))}}{R^2}.$$

With this choice, $\xi(t, y) > u_\varepsilon(t, y)$ on $[0, T] \times \partial\mathrm{B}(0, R)$, for all η, B and $z \in \mathrm{B}(0, \frac{R}{2})$. Next we prove that, for A large enough, $\xi(s, y) > u_\varepsilon(s, y)$ for all $y \in \mathrm{B}(0, R)$. We argue by contradiction. Assume that there exists $\eta > 0$ such that for all constants A there exists $y_{A,\varepsilon} \in \mathrm{B}(0, R)$ such that

$$u_\varepsilon(s, y_{A,\varepsilon}) - u_\varepsilon(s, z) > \eta + A|y_{A,\varepsilon} - z|^2. \tag{62}$$

It follows that

$$|y_{A,\varepsilon} - z| \leq \sqrt{\frac{2M}{A}},$$

where M is a uniform upper bound for $\| u_\varepsilon \|_{L^\infty([0,T]\times \mathrm{B}(0,R))}$. Now let $A \to \infty$. Then for all ε, $|y_{A,\varepsilon} - z| \to 0$. According to Section 5.3.2, u_ε are uniformly continuous on space. Thus there exists $h > 0$ such that if $|y_{A,\varepsilon} - z| \leq h$ then $|u_\varepsilon(s, y_{A,\varepsilon}) - u_\varepsilon(s,z)| < \frac{\eta}{2}$, for all ε. This is in contradiction with (62). Therefore $\xi(s,y) > u_\varepsilon(s,y)$ for all $y \in \mathrm{B}(0,R)$. Finally, noting that R is bounded we deduce that for B large enough, ξ is a super-solution to (44) in $[s,T] \times \mathrm{B}(0,R)$. Since u_ε is a solution of (44) we have

$$u_\varepsilon(t,y) \leq \xi(t,y) = u_\varepsilon(s,z) + \eta + A|y-z|^2 + B(t-s) \quad \text{for all} \quad (t,y) \in [s,T] \times \mathrm{B}(0,R).$$

Thus (60) is satisfied for $t \geq s$. We can prove (61) for $t \geq s$ analogously. Then we put $z = y$ and we conclude taking $\theta < \frac{\eta}{B}$.

5.4 Convergence to the Hamilton-Jacobi equation with constraint: the proof of Theorem 5.1

Thanks to the Arzela-Ascoli Theorem, using the regularity estimates obtained in Theorem 5.5, as $\varepsilon \to 0$ and along subsequences, $(u_\varepsilon)_\varepsilon$ converges locally uniformly to $u \in \mathrm{C}([0,\infty) \times \mathbb{R}^d)$. Moreover, thanks to Theorem 5.4, $(I_\varepsilon)_\varepsilon$ converges a.e. to $I \in \mathrm{BV}(\mathbb{R}^+)$. In order to complete the proof of Theorem 5.1, it remains to prove that u is a viscosity solution of (45) and to derive the constraint $\max\limits_{z\in\mathbb{R}^d} u = 0$.

(i) u **is a viscosity solution of** (45)**.** Here, we show that u is a viscosity sub-solution of (45) assuming that $(I_\varepsilon)_\varepsilon$ converges locally uniformly to I, being a continuous function. The proof of the super-solution property is analogous. The interested reader is referred to [PB08] for the proof of the result in the case where the convergence of $(I_\varepsilon)_\varepsilon$ is not uniform, and I is only of bounded variation, using the notion of viscosity solutions for discontinuous Hamilton-Jacobi equations [Bar94].

We fix $T > 0$ and assume that $u - \varphi$ attains a strict maximum point at $(\bar{t}, \bar{z}) \in (0,T] \times \mathbb{R}^d$, with $\varphi \in \mathrm{C}^1((0,T] \times \mathbb{R}^d)$. Let $(u_{\varepsilon_K})_{\varepsilon_K}$ be a subsequence which converges locally uniformly to u, with $\varepsilon_K \to 0$ as $K \to +\infty$. Thanks to Lemma 3.10, there exists a sequence (t_K, z_K) such that u_{ε_K} attains a maximum at (t_K, z_K) and that (t_K, z_K) converges to $(\bar{t}, \bar{z})$ as $K \to +\infty$. Evaluating (44) at (t_K, z_K) we obtain

$$\partial_t u_{\varepsilon_K}(t_K, z_K) - \varepsilon_K \Delta u_{\varepsilon_K}(t_K, z_K) = |\nabla u_{\varepsilon_K}|^2(t_K, z_K) + R(z_K, I_{\varepsilon_K}(t_K)).$$

Moreover, since $(t_K, z_K) \in (0,T] \times \mathbb{R}^d$ is a maximum point of u_{ε_K}, we deduce that

$$\partial_t \varphi(t_K, z_K) \leq \partial_t u_{\varepsilon_K}(t_K, z_K), \quad \nabla \varphi(t_K, z_K) = \nabla u_{\varepsilon_K}(t_K, z_K),$$

$$-\Delta\varphi(t_K, z_K) \leq -\Delta u_{\varepsilon_K}(t_K, z_K).$$

Combining the properties above we find that

$$\partial_t\varphi(t_K, z_K) - \varepsilon_K \Delta\varphi(t_K, z_K) \leq |\nabla\varphi|^2(t_K, z_K) + R(z_K, I_{\varepsilon_K}(t_K)).$$

Finally, we pass to the limit $K \to \infty$ and use the uniform convergence of I_{ε_K} to I to obtain

$$\partial_t\varphi(\bar{t}, \bar{z}) \leq |\nabla\varphi|^2(\bar{t}, \bar{z}) + R(\bar{z}, I(\bar{t})),$$

which means that u is a viscosity sub-solution of (45) at $(\bar{t}, \bar{z})$.

(ii) $\max\limits_{z\in\mathbb{R}^d} \mathbf{u} = \mathbf{0}$. Assume that for some (t, z) we have $0 < a \leq u(t, z)$. Since u is continuous, we have $u(t, y) \geq \frac{a}{2}$ on $\mathrm{B}(z, r)$, for some $r > 0$. This implies that, for any $y \in \mathrm{B}(z, r)$, $n_\varepsilon(t, y) \to \infty$, as $\varepsilon \to 0$ and hence $I_\varepsilon(t) \to \infty$ as $\varepsilon \to 0$. This is in contradiction with (39).

To prove that $\max\limits_{z\in\mathbb{R}^d} u(t, z) = 0$, it is then sufficient to show that $\lim\limits_{\varepsilon\to 0} n_\varepsilon(t, z) \neq 0$, for some $z \in \mathbb{R}^d$. On the one hand, using (49) we obtain that for M large enough

$$\lim_{\varepsilon\to 0} \int_{|z|>M} n_\varepsilon(t, z) dz \leq \lim_{\varepsilon\to 0} \int_{|z|>M} e^{\frac{A_3 - A_4|z| + Ct}{\varepsilon}} dz = 0. \tag{63}$$

Combining the inequality above and (39) we deduce that

$$\lim_{\varepsilon\to 0} \int_{|z|\leq M} n_\varepsilon(t, z) dz \geq \frac{I'_m}{\psi_M}.$$

On the other hand, if $u(t, z) < 0$ for all $|z| < M$, then $\lim\limits_{\varepsilon\to 0} \int_{|z|\leq M} n_\varepsilon(t, z) dz = 0$. This is in contradiction with (63). We conclude that $\max\limits_{z\in\mathbb{R}^d} u(t, z) = 0, \quad \forall t > 0$.

5.5 The inclusion properties: the proof of Theorem 5.3

(i) From Theorem 5.4 we deduce that $(n_\varepsilon)_{\varepsilon\leq\varepsilon_0(T)}$ is uniformly bounded in $L^\infty([0, T]; L^1(\mathbb{R}^d)$. It follows that, as $\varepsilon \to 0$ along subsequences, (n_ε) converges in $L^\infty(w * [0, T]; \mathcal{M}_1(\mathbb{R}^d))$ to a measure n such that

$$\int \psi(z) n(t, z) dz = I(t) \quad \text{a.e..}$$

We next prove the inclusion properties:

(ii) $\operatorname{supp} n(t, \cdot) \subset \{z \,|\, u(t, z) = 0\}$. Recall that $n_\varepsilon = \frac{1}{(2\pi\varepsilon)^{d/2}} \exp(\frac{u_\varepsilon}{\varepsilon})$, such that u_ε converges locally uniformly to $u \in \mathrm{C}([0, \infty) \times \mathbb{R}^d)$, with $u \leq 0$. Let's suppose that $(t_0, z_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ is such that $u(t_0, z_0) = -a < 0$. Then, there exists $\varepsilon_1 > 0$ and $r > 0$ such that, for all $\varepsilon \leq \varepsilon_1$ and

$|z - z_0| \le r$, we have $u_\varepsilon(t_0, z) \le -\frac{a}{2}$. This implies that

$$\int_{|z-z_0|\le r} n_\varepsilon(t,z)dz \to 0,$$

and consequently $z_0 \notin \operatorname{supp} n(t,\cdot)$.

(iii) $\{z \,|\, u(t,z) = 0\} \subset \{z \,|\, R(z, I(t)) = 0\}$. Let (t_0, z_0) be such that $u(t_0, z_0) = 0$. This means that (t_0, z_0) is a maximum point of u. If u is differentiable at this point, we will have

$$\partial_t u(t_0, z_0) = 0, \qquad \nabla u(t_0, z_0) = 0.$$

These properties combined by (46) would imply that $R(z_0, I(t_0)) = 0$ at a continuity point of I.

It then reminds to prove that u is differentiable at (t_0, z_0). One can indeed prove that any viscosity solution u to the Hamilton-Jacobi equation (46) is semi-convex [PB08]. Any semi-convex function is differentiable at its maximum points. It follows that u is differentiable at (t_0, z_0).

5.6 Examples

5.6.1 Example of a homogeneous fitness function

Let's consider the following function for the growth rate:

$$R(z, I) = r - \kappa I,$$

and the following initial condition

$$n_{\varepsilon,0} = \frac{\rho_0}{\sqrt{4\pi\varepsilon}} \exp\big(\frac{-z^2}{4\varepsilon}\big).$$

In this case study, the corresponding Hamilton-Jacobi equation is written

$$\begin{cases} \frac{\partial}{\partial t}u(t,z) - |\frac{\partial}{\partial z}u|^2(t,z) = r - \kappa I, \\ \max_z u(t,z) = 0, \\ u(0,z) = -\frac{z^2}{4}. \end{cases}$$

To solve this equation we will introduce a new function to avoid the dependence of the equation on the function I:

$$v(t,z) = u(t,z) + \kappa \int_0^t I(s)ds.$$

The function v solves

$$\begin{cases} \frac{\partial}{\partial t}v(t,z) - |\frac{\partial}{\partial z}v|^2(t,z) = r, \\ v(0,z) = -\frac{z^2}{4}. \end{cases}$$

We can solve the equation above using the method of characteristics. Let's define

$$H(p,z) = -|p|^2 - r.$$

Then, the characteristic curves satisfy

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_z H(\mathbf{p}(s), \mathbf{z}(s) = 0, \\ \dot{\mathbf{v}}(s) = D_p H(\mathbf{p}(s), \mathbf{z}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{z}(s)) = -|\mathbf{p}(s)|^2 + r, \\ \dot{\mathbf{z}}(s) = D_p H(\mathbf{p}(s), \mathbf{z}(s)) = -2\mathbf{p}(s). \end{cases}$$

Let's fix a point (t,z). One can solve the equations above to find in a unique way a characteristic curve $\mathbf{z} : [0,t] \to \mathbb{R}^d$, such that $\mathbf{z}(t) = z$. This characteristic curve is given by

$$\mathbf{z}(s) = \frac{1+s}{1+t}z.$$

Moreover, the value function $\mathbf{v}(s)$ is given by

$$\mathbf{v}(s) = -\frac{z^2}{4}\frac{1+s}{(1+t)^2} + rs.$$

Evaluating this at time t, we deduce that

$$v(t,z) = \mathbf{v}(t) = -\frac{z^2}{4(1+t)} + rt.$$

We then re-write this result in terms of u:

$$u(t,z) = -\frac{z^2}{4(1+t)} + rt - \kappa \int_0^t I(s)ds. \tag{64}$$

It remains to identify the function $I(t)$. To this end, we will use the constraint

$$\max_{z \in \mathbb{R}^d} u(t,z) = 0.$$

Note from (64) that the maximum of $u(t,\cdot)$ is always attained at the point $z=0$. Therefore, we have

$$\max_z u(t,z) = rt - \kappa \int_0^t I(s)ds.$$

This implies that for all $t \in \mathbb{R}^+$,

$$rt = \kappa \int_0^t I(s)ds,$$

and consequently

$$u(t,z) = -\frac{z^2}{4(1+t)},$$

$$I(t) = \frac{r}{\kappa}, \quad \text{at all the continuity points of } I.$$

Finally, we note that this implies that

$$n(t,z) = \rho(t)\delta(z), \qquad \rho(t) = \frac{r}{\kappa\psi(0)}, \quad \text{a.e. } t.$$

With a homogeneous fitness function, there is indeed no reason for the dominant trait in the population to evolve and the phenotypic distribution remains concentrated on $z=0$ because of the initial state of the population. Furthermore, as t goes to infinity the phenotypic density becomes more and more flat since

$$n_\varepsilon(t,z) \approx \frac{C}{(2\pi\varepsilon)^{d/2}} \exp\big(-\frac{z^2}{4(1+t)\varepsilon}\big).$$

5.6.2 Example of a linear fitness function

Let's consider the following function for the growth rate:

$$R(z,I) = r + z - I, \tag{65}$$

and the following initial condition

$$n_{\varepsilon,0} = \frac{\rho_0}{\sqrt{\pi\varepsilon}} \exp\big(-\frac{z^2}{\varepsilon}\big).$$

In this case study, the corresponding Hamilton-Jacobi equation is written

$$\begin{cases} \frac{\partial}{\partial t}u(t,z) - |\frac{\partial}{\partial z}u|^2(t,z) = r + z - I, \\ \max_z u(t,z) = 0, \\ u(0,z) = -z^2. \end{cases}$$

As above, to solve this equation we will introduce a new function to avoid the dependence of the equation on the function I:

$$v(t,z) = u(t,z) + \int_0^t I(s)ds.$$

The function v solves

$$\begin{cases} \frac{\partial}{\partial t} v(t,z) - |\frac{\partial}{\partial z} v|^2(t,z) = r + z, \\ v(0,z) = -z^2. \end{cases}$$

It is possible to solve this equation using the method of characteristics and this is let as an exercise to the reader. Alternatively, we can look directly for solutions of the following form:

$$v(t,z) = -A(t)(z - B(t))^2 + C(t).$$

Replacing this in the equation on v we obtain

$$\begin{cases} A'(t) = -4A(t)^2, \\ B'(t)A(t) = \frac{1}{2}, \\ C'(t) = r + B(t), \\ A(0) = 1, \quad B(0) = 0, \quad C(0) = 0. \end{cases}$$

We can solve the equations above to obtain

$$\begin{cases} A(t) = \frac{1}{1+4t}, \\ B(t) = \frac{t}{2} + t^2, \\ C(t) = rt + \frac{t^2}{4} + \frac{t^3}{3}. \end{cases}$$

Therefore, the function v is given by

$$v(t,z) = -\frac{1}{1+4t}(z - \frac{1}{2}t - t^2)^2 + rt + \frac{t^2}{4} + \frac{t^3}{3},$$

and hence

$$u(t,z) = -\frac{1}{1+4t}(z - \frac{1}{2}t - t^2)^2 + rt + \frac{t^2}{4} + \frac{t^3}{3} - \int_0^t I(s)ds.$$

Similarly to the previous subsection, using the constraint $\max_z u = 0$ we obtain that

$$u(t,z) = -\frac{1}{1+4t}(z - \frac{t}{2} - t^2)^2, \qquad r + \frac{t}{2} + t^2 = I(t) \quad \text{a.e. } t.$$

We can also identify the limit phenotypic density

$$n(t,z) = \rho(t)\delta(z - \frac{t}{2} - t^2), \qquad \rho(t) = \frac{r + \frac{t}{2} + t^2}{\psi(\frac{t}{2} + t^2)} \quad \text{a.e. } t.$$

The dominant trait $\bar{z}(t) = \frac{t}{2} + t^2$ evolves indeed increasingly to reach higher values of the fitness function. As a consequence, the function $I(t)$ increases with time.

Example of discontinuous $I(t)$.

In the examples above, the functions $I(t)$ was continuous. From the estimates obtained in Section 5.1 we know that I is at least of bounded variation. One could wonder whether we can show stronger regularity on the function $I(t)$. Can we show for instance that I is a continuous function ? Here, we provide a counterexample, given in [PB08], which shows that the function $I(t)$ may be discontinuous. To this end, we consider the same fitness function as in (65). However, we consider a different initial condition:

$$n_{\varepsilon,0} = \frac{\rho_0}{(\pi\varepsilon)^{d/2}} \exp\big(\frac{u_0(z)}{\varepsilon}\big),$$

with

$$u_0(z) = \max(-z^2, -(z-\alpha)^2 - \delta).$$

Then, one can prove that the viscosity solution to the Hamilton-Jacobi equation is given by

$$u(t,z) = \begin{cases} \max\big(-\frac{1}{1+4t}(z - \frac{t}{2} - t^2)^2, -\frac{1}{1+4t}(z - \frac{t}{2} - t^2 - \alpha)^2 - \delta + \alpha t\big), & \text{for } t \le \frac{\delta}{\alpha}, \\ \max\big(-\frac{1}{1+4t}(z - \frac{t}{2} - t^2)^2 + \delta - \alpha t, -\frac{1}{1+4t}(z - \frac{t}{2} - t^2 - \alpha)^2\big), & \text{for } t > \frac{\delta}{\alpha}. \end{cases}$$

Moreover, the competition function I is given by

$$I(t) = \begin{cases} r + \frac{t}{2} + t^2, & \text{for } t \le \frac{\delta}{\alpha}, \\ r + \frac{t}{2} + t^2 + \alpha, & \text{for } t > \frac{\delta}{\alpha}. \end{cases}$$

The function I has indeed a jump at $t_0 = \frac{\delta}{\alpha}$. This discontinuity is indeed due to the jump of the dominant trait $\bar{z}(t)$ from $\bar{z}(t_0^-) = \frac{t_0}{2} + t_0^2$ to $\bar{z}(t_0^+) = \frac{t_0}{2} + t_0^2 + \alpha$.

5.6.3 Example of a quadratic fitness function

Let's consider the following fitness function:

$$R(z,I) = r_{\max} - sz^2 - \kappa I.$$

Then, the corresponding Hamilton-Jacobi equation is written

$$\begin{cases} \frac{\partial}{\partial t}u(t,z) - |\frac{\partial}{\partial z}u|^2(t,z) = r_{\max} - sz^2 - \kappa I, \\ \max_z u(t,z) = 0, \\ u(0,z) = u_0(z). \end{cases} \tag{66}$$

Exercise 5.8 *Let $u_0 = -a(z-\theta_0)^2$. Identify the solution (u, I) to the Hamilton-Jacobi equation with constraint* (66)*.*

Here, we show how to identify the limit (u, I), assuming arbitrary initial condition. In the case of quadratic fitness function above, one can use an alternative way to identify the phenotypic density using a method based on cumulated generating functions [**?**] which provides an explicit solution for arbitrary ε. This method works particularly well when one considers a quadratic stabilizing selection. The Hamilton-Jacobi approach has the advantage to apply to arbitrary forms of growth rates $R(z, I)$ and with possible heterogeneities.

Let's consider that the maximum of u is attained at a point $\overline{z}$. We provide an analytic formula for the dominant trait $\overline{z}(t)$:

$$\overline{z}(t) = \frac{2e^{2\sqrt{s}t}}{1+e^{4\sqrt{s}t}} \arg\max_C \{u_0(C) - C^2\frac{\sqrt{s}}{2}\tanh\big(2\sqrt{s}t\big)\}.$$

Note however that depending on the initial condition this may not be defined in a unique way. Notice also that since

$$R(\overline{z}(t), I(t)) = 0 \quad \text{a.e. t},$$

one can then express the value of I in terms of $\overline{z}(t)$, if the point $\overline{z}(t)$ is unique:

$$I(t) = \frac{1}{\kappa}\big(r_{\max} - s\overline{z}^2(t)\big).$$

Computation of $\overline{z}(t)$:

We next show how to identify analytically $\overline{z}(t)$. Note that the function u solves

$$\frac{\partial}{\partial t}u(t,z) - |\frac{\partial}{\partial z}u|^2(t,z) = -sz^2 + s\overline{z}^2(t).$$

The viscosity solution of the above equation is indeed given by the following representation formula:

$$u(t,z) = \sup_{\substack{\gamma\in W^{1,\infty}([0,t]) \\ \gamma(t)=z}} u_0(\gamma(0)) - \int_0^t \big(\frac{|\dot\gamma|^2}{4}(\tau) + s\gamma^2(\tau) - s\overline{z}^2(\tau)\big)d\tau.$$

The maximizing trajectory satisfies the following Euler-Lagrange equation:

$$\ddot{\gamma}(\tau) = 4s\gamma(\tau).$$

As a consequence $\gamma(\tau)$ can be written as follows

$$\gamma(\tau) = Ae^{2\sqrt{s}\,\tau} + (C-A)e^{-2\sqrt{s}\,\tau}, \qquad \text{with } A \text{ and } C \text{ some constants.}$$

We deduce that

$$\begin{aligned} u(t,z) &= \sup_{\substack{A,C\in\mathbb{R} \\ Ae^{2\sqrt{s}\,t}+(C-A)e^{-2\sqrt{s}\,t}=z}} u_0(C) - s\int_0^t \big(2A^2e^{4\sqrt{s}\,\tau} + 2(C-A)^2e^{-4\sqrt{s}\,\tau} - \overline{z}^2(\tau)\big)d\tau \\ &= \sup_{\substack{A,C\in\mathbb{R} \\ Ae^{2\sqrt{s}\,t}+(C-A)e^{-2\sqrt{s}\,t}=z}} u_0(C) - \tfrac{\sqrt{s}}{2}\big(A^2(e^{4\sqrt{s}\,t}-1) + (C-A)^2(1-e^{-4\sqrt{s}\,t})\big) + s\int_0^t \overline{z}^2(\tau)d\tau. \end{aligned}$$

We are interested in identifying the point $\overline{z}(t)$ which corresponds to the maximum point of $u(t,\cdot)$. Let's define

$$F(A,C) = u_0(C) - \frac{\sqrt{s}}{2}\big(A^2(e^{4\sqrt{s}\,t}-1) + (C-A)^2(1-e^{-4\sqrt{s}\,t})\big).$$

If the maximum of F is taken at some point (A_m, C_m). Then,

$$\overline{z}(t) = A_m e^{2\sqrt{s}\,t} + (C_m - A_m)e^{-2\sqrt{s}\,t},$$

is a maximum point of $u(t,\cdot)$. Note that the maximum point A_m can be expressed in terms of C_m:

$$A_m = C_m \frac{1-e^{-4\sqrt{s}t}}{e^{4\sqrt{s}t} - e^{-4\sqrt{s}t}} = C_m \frac{1}{1+e^{4\sqrt{s}t}}.$$

We deduce that

$$\begin{aligned} C_m &= \arg\max_C u_0(C) - C^2\tfrac{\sqrt{s}}{2}\Big(\tfrac{e^{4\sqrt{s}\,t}-1}{(1+e^{4\sqrt{s}t})^2} + \tfrac{e^{8\sqrt{s}\,t}-e^{4\sqrt{s}\,t}}{(1+e^{4\sqrt{s}t})^2}\Big) \\ &= \arg\max_C u_0(C) - C^2\tfrac{\sqrt{s}}{2}\tanh\big(2\sqrt{s}t\big). \end{aligned}$$

This implies that

$$\overline{z}(t) = \frac{2e^{2\sqrt{s}t}}{1+e^{4\sqrt{s}t}} \arg\max_C \{u_0(C) - C^2\frac{\sqrt{s}}{2}\tanh\big(2\sqrt{s}t\big)\}.$$

6 The concave framework

In this section we assume additionally that $R(\cdot, I)$ and $u_{\varepsilon,0}(\cdot)$ are strictly concave functions. We assume indeed that

$$-2\underline{L}_1 \leq D^2 u_{\varepsilon,0}(z) \leq -2\overline{L}_1 < 0,$$

$$-2\underline{K}_1 \leq D^2 R(z, I) \leq -2\overline{K}_1 < 0.$$

We will show that such concavity assumptions lead to many nice properties. Under these assumptions there exists a unique viscosity solution to (45) which is indeed smooth and classical. Moreover, in this case the solution u remains strictly concave and hence has a unique maximum point. This property implies that the limit phenotypic density is a single Dirac mass.

We prove indeed the following Theorem.

Theorem 6.1 *[LMP11, MR16] In the concave framework, there exists a unique viscosity solution u to* (45)*. This solution is indeed smooth and a classical solution. Moreover, $u(t, \cdot)$ is a strictly concave function and hence, for all $t \geq 0$, there exists a unique point $\overline{z}(t)$ such that*

$$\max_z u(t, z) = u(t, \overline{z}(t)) = 0,$$

which implies that

$$n(t, z) = \rho(t)\delta(z - \overline{z}(t)).$$

Finally, the following equation describes the dynamics of the dominant trait $\overline{z}(t)$:

$$\dot{\overline{z}}(t) = (-D^2 u(t, \overline{z}(t)))^{-1} \nabla R(\overline{z}(t), I(t)). \tag{67}$$

6.1 Assumptions

- **Assumptions on $R(z, I)$.** We choose R to be smooth, and we suppose that there is $I_M > 0$ such that

$$\max_{z \in \mathbb{R}^d} R(z, I_M) = 0, \tag{68}$$

$$-\underline{K}_0 - \underline{K}_1 |z|^2 \leq R(z, I) \leq \overline{K}_0 - \overline{K}_1 |z|^2, \qquad \text{for } \ 0 \leq I \leq I_M, \tag{69}$$

$$-2\underline{K}_1 \leq D^2 R(z, I) \leq -2\overline{K}_1 < 0 \text{ as symmetric matrices}, \tag{70}$$

$$-\underline{K}_2 \leq \frac{\partial R}{\partial I} \leq -\overline{K}_2, \tag{71}$$

$$|\frac{\partial^2 R}{\partial I \partial z_i}(z, I)| + |\frac{\partial^3 R}{\partial I \partial z_i \partial z_j}(z, I)| \leq K_3, \qquad \text{for } 0 \leq I \leq I_M, \text{ and } i,\, j = 1, 2, \cdots, d, \tag{72}$$

$$\|D^3 R(\cdot, I)\|_{L^\infty(\mathbb{R}^d)} \leq K_4, \qquad \text{for } 0 \leq I \leq I_M. \tag{73}$$

- **Assumptions on $u_0(.)$ and I_0.** We assume the existence of positive constants $\underline{L}_0, \overline{L}_0, \underline{L}_1, \overline{L}_1$ such that

$$-\underline{L}_0 - \underline{L}_1|z|^2 \leq u_0(z) \leq \overline{L}_0 - \overline{L}_1|z|^2, \tag{74}$$

$$-2\underline{L}_1 \leq D^2 u_0 \leq -2\overline{L}_1. \tag{75}$$

Note that this implies

$$|Du_0(z)| \leq L_2(1+|z|), \tag{76}$$

for a large constant $L_2 > 0$. We also need that, for a positive constant L_3,

$$\|D^3 u_0\|_{L^\infty(\mathbb{R}^d)} \leq L_3. \tag{77}$$

Finally we assume that

$$\max_z u_0(z) = u_0(\overline{z}_0) = 0, \qquad R(\overline{z}_0, I_0) = 0. \tag{78}$$

In the following section we will study an unconstrained Hamilton-Jacobi equation where we replace $R(z,I)$ by $R(t,z)$. To prove our results on this unconstrained problem we assume same type of regularity and concavity assumptions on R that we state below:

- **Assumptions on $R(t,z)$.** We choose R to be continuous in t and to have first and second derivatives with respect to z, that are continuous both with respect to t and z. We suppose that

$$-\underline{K}_0 - \underline{K}_1|z|^2 \leq R(t,z) \leq \overline{K}_0 - \overline{K}_1|z|^2, \qquad \text{for } t \in \mathbb{R}^+, \tag{79}$$

$$-2\underline{K}_1 \leq D^2 R(t,z) \leq -2\overline{K}_1 < 0 \text{ as symmetric matrices}, \tag{80}$$

$$\|D^3 R(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq K_4, \qquad \text{for } t \in \mathbb{R}^+. \tag{81}$$

6.2 The concavity assumptions lead to smoothness, uniqueness and strict concavity of the solution

We will prove the Theorem 6.1 in several steps.

We first study the unconstrained Hamilton-Jacobi equation

$$\begin{cases} u_t = |\nabla u|^2 + R(t,z) \quad (t>0, z \in \mathbb{R}^d), \\ u(0,z) = u_0(z), \end{cases} \tag{82}$$

and prove any viscosity solution of such a Hamilton-Jacobi equation, under concavity assumptions, stated in the preceding subsection, is indeed smooth, a classical solution and strictly concave.

Theorem 6.2 (The Cauchy problem) *Assume* (74)–(77) *and* (79)–(81). *Equation* (82) *has a unique viscosity solution u that is bounded from above. Moreover, it is a classical solution:* $u \in L^\infty_{\text{loc}}\big(\mathbb{R}^+; W^{3,\infty}_{\text{loc}}(\mathbb{R}^d)\big) \cap C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, $\nabla u \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, $-\max(2\,\underline{L}_1, \sqrt{\underline{K}_1}) \leq D^2 u \leq -\min(2\overline{L}_1, \sqrt{\overline{K}_1})$ *and* $\|D^3 u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq L_4(T)$ *where $L_4(T)$ is a positive constant depending on* $\underline{L}_1$, $\underline{K}_1$, K_4, L_3 *and* T.

We next prove the uniqueness property for the original problem.

Theorem 6.3 (The uniqueness result) *Assume* (68)–(78). *The Hamilton-Jacobi equation with constraint* (46) *has a unique solution* (u, I). *Moreover we have*

$$(u, I) \in L^\infty_{\text{loc}}\big(\mathbb{R}^+; W^{3,\infty}_{\text{loc}}(\mathbb{R}^d)\big) \cap C^1(\mathbb{R}^+ \times \mathbb{R}^d) \times C^1(\mathbb{R}^+) \quad \textit{and} \quad \nabla u \in C^1(\mathbb{R}^+ \times \mathbb{R}^d).$$

The combination of the theorems above imply that there exists a unique viscosity solution to (46) and that such viscosity solution is indeed smooth, a classical solution and strictly concave.

The main ingredients in the proof of Theorem 6.2 are the variational formulation of the viscosity solution (22) and the concavity assumptions. Theorem 6.2 implies that the solution of (46) is classical and smooth. This leads to the equivalence of the Hamilton-Jacobi equation with constraint with the following ODE-PDE system:

$$\begin{cases} R\left(\overline{z}(t), I(t)\right) = 0, & \text{for } t \in \mathbb{R}^+, \\ \dot{\overline{z}}(t) = \left(-D^2 u\big(t, \overline{z}(t)\big)\right)^{-1} \nabla R\big(\overline{z}(t), I(t)\big), & \text{for } t \in \mathbb{R}^+, \\ \partial_t u = |\nabla u|^2 + R(z, I), & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \tag{83}$$

with initial conditions

$$\begin{gathered} I(0) = I_0, \qquad u(0, \cdot) = u_0(\cdot), \qquad \overline{z}(0) = \overline{z}_0, \\ \text{such that} \quad \max_z u_0(z) = u_0(\overline{z}_0) = 0 \quad \text{and} \quad R(\overline{z}_0, I_0) = 0. \end{gathered} \tag{84}$$

We have indeed

Theorem 6.4 *Solving the constrained problem* (46) *is equivalent to solving the initial value ODE-PDE problem* (83)-(84).

Note that the second equation in (83) corresponds to the differential equation (67) satisfied by the dominant trait $\overline{z}(t)$ which is called the canonical equation.

This equivalence is the main ingredient to prove Theorem 6.3. Note that (83) is really a differential system because the assumptions on R imply that $I(t)$ can implicitly be expressed in terms of $\bar{z}(t)$. And it is slightly nonstandard because $\bar{z}$ solves an ODE whose nonlinearity depends on u.

6.3 The Cauchy problem

In this section, we provide the main elements to prove Theorem 6.2.

A first main ingredient is that the only viscosity solution to (82), is given by the following variational problem:

$$u(t,z) = \sup_{\gamma \in \mathcal{A}(t,z)} F([\gamma]), \tag{85}$$

with

$$F([\gamma]) = u_0(\gamma(0)) + \int_0^t \left(-\frac{|\dot{\gamma}|^2}{4}(s) + R(s,\gamma(s)) \right) ds,$$

$$\mathcal{A}(t,z) := \left\{ (s,\gamma(s)) \in [0,t] \times \mathbb{R}^d,\ \gamma \in W^{1,2}([0,t];\mathbb{R}^d),\ \gamma(t) = z \right\}.$$

Note that by the density of C^1 functions in $W^{1,2}$, one would obtain the same quantity if we take the supremum above in a set $\widetilde{\mathcal{A}}(t,z)$ where we replace the set $W^{1,2}([0,t];\mathbb{R}^d)$ by $C^1([0,t];\mathbb{R}^d)$.

The above function $u(t,x)$ is a viscosity solution to (46). We will prove, in addition, that it is classical and strictly concave. For more details on the uniqueness of such viscosity solution and the regularity results see [MR16].

The function u is a classical solution. Let us suppose that $(\gamma_n)_{1\leq n}$, with $\gamma_n \in W^{1,2}([0,t];\mathbb{R}^d)$ and $\gamma_n(t) = z$, is such that $F(\gamma_n) \to u(t,z)$ as $n \to \infty$. Since R and u_0 are bounded from above, we obtain that, for n large enough and some constant C

$$\int_0^t |\dot{\gamma}_n|^2(s)ds < C.$$

Consequently, from $\gamma_n(t) = z$ we deduce, modifying the constant C if necessary, that

$$\|\gamma_n\|_{W^{1,2}[0,t]} < C.$$

It follows that, there exists $\overline{\gamma} \in W^{1,2}([0,t];\mathbb{R}^d)$, such that as $n \to \infty$, $\gamma_n \to \overline{\gamma}$ strongly in

$C([0,t];\mathbb{R}^d)$ and weakly in $W^{1,2}([0,t];\mathbb{R}^d)$. We deduce that, as $n\to\infty$,

$$u_0(\gamma_n(0)) \to u_0(\overline{\gamma}(0)), \quad \int_0^t R(s,\gamma_n(s))ds \to \int_0^t R(s,\overline{\gamma}(s))ds, \quad \int_0^t |\dot{\overline{\gamma}}|^2(s)ds \le \liminf_{n\to\infty} \int_0^t |\dot{\gamma}_n|^2(s)ds.$$

We deduce that

$$u(t,z) \ge \ u_0(\overline{\gamma}(0)) + \int_0^t \left(-\frac{|\dot{\overline{\gamma}}|^2}{4}(s) + R(s,\overline{\gamma}(s))\right) ds.$$

Since $\overline{\gamma}\in W^{1,2}([0,t];\mathbb{R}^d)$ and using (85) we conclude that

$$u(t,z) = \ u_0(\overline{\gamma}(0)) + \int_0^t \left(-\frac{|\dot{\overline{\gamma}}|^2}{4}(s) + R(s,\overline{\gamma}(s))\right) ds. \tag{86}$$

We claim that such a trajectory is unique, which implies that $u\in C^1(\mathbb{R}^+\times\mathbb{R}^d)$ is indeed a classical solution of (82).

We note indeed that such trajectory $\overline{\gamma}$ satisfies the following Euler-Lagrange equation

$$\begin{cases} \ddot{\overline{\gamma}}(s) = -2\nabla R(s,\overline{\gamma}(s)), \\ \dot{\overline{\gamma}}(0) = -2\nabla u_0(\overline{\gamma}(0)), \\ \overline{\gamma}(t) = z. \end{cases} \tag{87}$$

To justify such an Euler-Lagrange equation, let's consider an arbitrary trajectory $y\in W^{1,2}([0,t];\mathbb{R}^d)$ such that $y(t)=0$ and define $f:\mathbb{R}\to\mathbb{R}$ as below

$$f(\tau) = u_0(\overline{\gamma}(0)+\tau y(0)) + \int_0^t \left(-\frac{|\dot{\overline{\gamma}}+\tau\dot{y}|^2}{4}(s) + R\big(s,\overline{\gamma}(s)+\tau y(s)\big)\right) ds.$$

The function f has a maximum at $\tau=0$. Therefore its derivative vanishes at $\tau=0$, that is

$$f'(0) = \nabla u_0\big(\overline{\gamma}(0)\big)\cdot y(0) + \int_0^t \left(-\frac{\dot{\overline{\gamma}}(s)\cdot\dot{y}(s)}{2}(s) + \nabla R\big(s,\overline{\gamma}(s)\big)\cdot y(s)\right) ds = 0.$$

Since this equality holds for any function $y\in W^{1,2}([0,t];\mathbb{R}^d)$ such that $y(t)=0$, we deduce that $\overline{\gamma}$ is a weak solution to (87). Furthermore, from the elliptic regularity we deduce that $\overline{\gamma}$ is indeed a classical solution of (87).

We then prove that the solution to (87) is unique. To this end, let's assume that γ_1 and γ_2 are two distinct solutions to (87). We then compute

$$\ddot{\gamma}_1(s) - \ddot{\gamma}_2(s) = -2\nabla R(s,\gamma_1(s)) + 2\nabla R(s,\gamma_2(s)).$$

We then multiply the equality above by $\gamma_1(s)-\gamma_2(s)$ and integrate with respect to s to obtain

$$\int_0^t \big(\ddot\gamma_1(s)-\ddot\gamma_2(s)\big)\big(\gamma_1(s)-\gamma_2(s)\big)ds = -2\int_0^t \big(\nabla R(s,\gamma_1(s))-\nabla R(s,\gamma_2(s))\big)\big(\gamma_1(s)-\gamma_2(s)\big)ds$$

We next integrate by parts in the first term to find

$$\begin{aligned} &-\textstyle\int_0^t \big(\dot\gamma_1(s)-\dot\gamma_2(s)\big)^2 ds + \Big[\big(\dot\gamma_1(s)-\dot\gamma_2(s)\big)\big(\gamma_1(s)-\gamma_2(s)\big)\Big]_0^t \\ &\quad = -2\textstyle\int_0^t \big(\nabla R(s,\gamma_1(s))-\nabla R(s,\gamma_2(s))\big)\big(\gamma_1(s)-\gamma_2(s)\big)ds. \end{aligned}$$

Using the boundary condition in (87) we deduce that

$$\begin{aligned} &-\textstyle\int_0^t \big(\dot\gamma_1(s)-\dot\gamma_2(s)\big)^2 ds + 2\big(\nabla u_0(\gamma_1(0))-\nabla u_0(\gamma_2(0))\big)\big(\gamma_1(0)-\gamma_2(0)\big) \\ &\quad = -2\textstyle\int_0^t \big(\nabla R(s,\gamma_1(s))-\nabla R(s,\gamma_2(s))\big)\big(\gamma_1(s)-\gamma_2(s)\big)ds. \end{aligned}$$

Next, from the concavity conditions on R and u_0 we obtain that, for some positive constant C,

$$\int_0^t \big(\dot\gamma_1(s)-\dot\gamma_2(s)\big)^2 ds + C\int_0^t \big(\gamma_1(s)-\gamma_2(s)\big)^2 ds + C\big(\gamma_1(0)-\gamma_2(0)\big)^2 \le 0.$$

We deduce that $\gamma_1\equiv\gamma_2$ and hence the solution is unique.

Strict concavity. We will prove that $u(t,z)$ is uniformly strictly concave, namely that $D^2u \le -2\lambda I$ in the sense of symmetric matrices, for $\lambda = \min\big(\overline{L}_1, \frac{\sqrt{K_1}}{2}\big)$.
To this end, we show that, for all $\sigma\in[0,1]$ and $(z,y)\in\mathbb{R}^d\times\mathbb{R}^d$:

$$\sigma u(t,z) + (1-\sigma)u(t,y) + \lambda\sigma(1-\sigma)|z-y|^2 \le u(t,\sigma z+(1-\sigma)y). \tag{88}$$

Let γ_z and γ_y be optimal trajectories, solving (87), with $\gamma_z(t)=z$ and $\gamma_y(t)=y$. Note from the choice of γ_z and γ_y that we have

$$u(t,z) = u_0(\gamma_z(0)) + \int_0^t \left(-\frac{|\dot\gamma_z(s)|^2}{4} + R(s,\gamma_z(s))\right) ds,$$

$$u(t,y) = u_0(\gamma_y(0)) + \int_0^t \left(-\frac{|\dot\gamma_y(s)|^2}{4} + R(s,\gamma_y(s))\right) ds,$$

and

$$\begin{aligned} u(t,\sigma z+(1-\sigma)y) \ge\ & u_0(\sigma\gamma_z(0)+(1-\sigma)\gamma_y(0)) \\ & + \int_0^t \left(-\frac{|\sigma\dot\gamma_z+(1-\sigma)\dot\gamma_y(s)|^2}{4} + R(s,\sigma\gamma_z(s)+(1-\sigma)\gamma_y(s))\right) ds. \end{aligned}$$

Furthermore, from the concavity assumptions on R and u_0 we have

$$\sigma u_0(t,z) + (1-\sigma)u_0(t,y) + \overline{L}_1\sigma(1-\sigma)|\gamma_z(0)-\gamma_y(0)|^2 \le u_0(t,\sigma z+(1-\sigma)y),$$

and

$$\sigma\int_0^t R(s,\gamma_z(s))ds + (1-\sigma)\int_0^t R(s,\gamma_y(s))ds + \overline{K}_1\sigma(1-\sigma)\int_0^t |\gamma_z(s)-\gamma_y(s)|^2 ds$$
$$\le \int_0^t R(s,\sigma\gamma_z(s)+(1-\sigma)\gamma_y(s))ds.$$

Moreover, from the strict concavity of $\mu \mapsto -|\mu|^2$, we obtain that

$$\sigma\int_0^t -\frac{|\dot\gamma_z(s)|^2}{4}ds + (1-\sigma)\int_0^t -\frac{|\dot\gamma_y(s)|^2}{4}ds + \sigma(1-\sigma)\int_0^t \frac{|\dot\gamma_z(s)-\dot\gamma_y(s)|^2}{4}ds$$
$$\le \int_0^t -\frac{|\sigma\dot\gamma_z + (1-\sigma)\dot\gamma_y(s)|^2}{4}ds.$$

We deduce that

$$\begin{aligned} & u(t,\sigma z+(1-\sigma)y)) \ge \sigma u(t,z) + (1-\sigma)u(t,y) \\ + \quad & \sigma(1-\sigma)\left(\int_0^t (\frac{1}{4}|\dot\gamma_z(s)-\dot\gamma_y(s)|^2 + \overline{K}_1|\gamma_z(s)-\gamma_y(s)|^2)ds + \overline{L}_1|\gamma_z(0)-\gamma_y(0)|^2\right) \end{aligned} \tag{89}$$

Next we have

$$\frac{\sqrt{\overline{K}_1}}{2}\int_0^t \frac{d}{ds}|\gamma_z(s)-\gamma_y(s)|^2 ds \le \overline{K}_1\int_0^t |\gamma_z(s)-\gamma_y(s)|^2 ds + \int_0^t \frac{|\dot\gamma_z-\dot\gamma_y|^2}{4}(s)ds.$$

Writing

$$|z-y|^2 = |\gamma_z(t)-\gamma_y(t)|^2 = |\gamma_z(0)-\gamma_y(0)|^2 + \int_0^t \frac{d}{ds}|\gamma_z(t)-\gamma_y(t)|^2 ds$$

we find

$$|z-y|^2 \le |\gamma_z(0)-\gamma_y(0)|^2 + 2\sqrt{\overline{K}_1}\int_0^t |\gamma_z(s)-\gamma_y(s)|^2 ds + \frac{1}{2\sqrt{\overline{K}_1}}\int_0^t |\dot\gamma_z-\dot\gamma_y|^2(s)ds.$$

Combing the above line with (89), we obtain (88) for $\lambda = \min\big(\overline{L}_1, \frac{\sqrt{\overline{K}_1}}{2}\big)$.

6.4 Equivalence with the ODE-PDE problem; the proof of Theorem 6.4

In this subsection we prove that the constrained Hamilton-Jacobi problem implies (83). For the proof of the converse argument see [MR16].

Let (u, I) be a solution of (46) with initial datum (u_0, I_0), the function I being continuous, and u a solution of the Hamilton-Jacobi equation in the sense of (85). Theorem 6.2 is applicable, and yields a solution $u(t, z)$ which has at least three locally bounded spatial derivatives, locally uniformly in time. Moreover, the $D^2 u$ is bounded uniformly in time and in z, and finally the function $u(t, .)$ is strictly concave. This allows a lot.

First note that the last equation in (83) immediately holds thanks to (46). Here we derive respectively the second and the first equation in (83). Note that the first equation in (83) was obtained in the previous section, in the limiting procedure. Here we provide an independent proof which does not rely on the viscous problem.

- There is, at each time, a unique $\overline{z}(t)$ maximising $u(t, .)$ over $\mathbb{R}^d$. Thus, because $\nabla u \in C^1(\mathbb{R} + \times \mathbb{R}^d)$, the trivial identity

$$\nabla u(t, \overline{z}(t)) = 0 \tag{90}$$

  can (use differential quotients) be differentiated with respect to t, to yield that (i) $\overline{z}(t) \in C^1(\mathbb{R}^+)$ and (ii) the (a priori less trivial) identity

$$\partial_t(\nabla u)(t, \overline{z}(t)) + D^2 u(t, \overline{z}(t)). \frac{d\overline{z}}{dt}(t) = 0. \tag{91}$$

- The function $u(t, z)$ has enough regularity so that we may take the gradient of (46) with respect to z, and evaluate the result at $z = \overline{z}(t)$. Because of (90) we have $D^2 u(t, \overline{z}(t)).\nabla u(t, \overline{z}(t)) = 0$ and, because of (91), we have

$$- D^2 u(t, \overline{z}(t)). \frac{d\overline{z}}{dt}(t) = \nabla R(\overline{z}(t), I(t)). \tag{92}$$

  Therefore, the second equation in (83) is proved.

- The last item to take into account is the constraint

$$u(t, \overline{z}(t)) = 0,$$

  which we may (still with the use of differential quotients) differentiate with respect to time, in order to yield

$$\partial_t u(t,\overline{z}(t)) + \nabla u(t,\overline{z}(t)).\frac{d\overline{z}}{dt}(t) = 0,$$

thus entailing

$$\partial_t u(t,\overline{z}(t)) = 0.$$

This yields, from (46),

$$R(\overline{z}(t), I(t)) = 0. \tag{93}$$

Gathering (93), (46) and (92) shows that the constrained problem implies (83).

6.5 Proof of the uniqueness result

We fix $T > 0$. To prove that (46) has a unique solution (u, I) in $[0,T] \times \mathbb{R}^d$, it is enough to prove that there exists a unique solution to (83)–(84).

We provide the main elements to prove such uniqueness result. We prove this using the Banach fixed point Theorem in a small interval and then iterate. To this end, we introduce the mapping $\Phi : X(\cdot) \to Y(\cdot)$ as follows:

$$X(\cdot) \xrightarrow[R(X,I)=0]{} I_X(\cdot) \xrightarrow[\partial_t u = |\nabla u|^2 + R(x, I_X)]{} u_X$$

$$\xrightarrow[\dot{Y}(t) = \left(-D^2 u_X\left(t, X(t)\right)\right)^{-1} \nabla R\left(X(t), I_X(t)\right)]{} Y(\cdot).$$

Define

$$\mathcal{A} = \left\{ x(\cdot) \in \mathrm{C}\Big([0,\delta]; B\big(\overline{x}_0, r_\delta\big)\Big) \;\Big|\; x(0) = \overline{x}_0 \right\}.$$

For δ and r_δ well-chosen constants, $\Phi : \mathcal{A} \to \mathcal{A}$ is well-defined and we have the following lemma:

Lemma 6.5 *The mapping Φ is a strict contraction from $\mathcal{A}$ into itself.*

This allows to use the Banach fixed point Theorem for $\Phi : \mathcal{A} \to \mathcal{A}$.

To be able to iterate to obtain uniqueness in the whole domain we use a monotony property stated in the following lemma.

Lemma 6.6 *Let (x, I, u) be the unique solution of the differential system (83)–(84) in $[0,\tau]$. Then, $I(t)$ is increasing with respect to t in $[0,\tau]$.*

6.6 Moments of the phenotypic distribution and the interpretation of the canonical equation

In the concave framework, and under regularity assumptions, one can indeed show that

$$u_\varepsilon(t,z) = u(t,z) + \varepsilon v(t,z) + o(\varepsilon), \qquad I_\varepsilon(t) = I(t) + \varepsilon J(t) + o(\varepsilon).$$

These asymptotic expansions lead to the following approximation of the phenotypic density:

$$n_\varepsilon(t,z) \approx \frac{1}{(2\pi\varepsilon)^{d/2}} \exp\big(\frac{u(t,z)+\varepsilon v(t,z)}{\varepsilon}\big).$$

It also allows to provide analytic approximations of the moments of the phenotypic distribution. In particular, we can compute the covariance matrix of the phenotypic distribution as follows

$$\mathbf{V}_{\varepsilon,t} = \varepsilon(-D^2u)^{-1}(t,\overline{z}(t)) + o(\varepsilon),$$

where $\mathbf{V}_{\varepsilon,t} = (v_{i,j}(t))$ with $v_{i,j} = \int z_i z_j \frac{n_\varepsilon(t,z)}{\rho_\varepsilon(t)} dx - (\int z_i \frac{n_\varepsilon(t,z)}{\rho_\varepsilon(t)} dz)(\int z_j \frac{n_\varepsilon(t,z)}{\rho_\varepsilon(t)} dz)$. Note that here we find again the term $(-D^2u)^{-1}(t,\overline{z}(t))$ which also appeared in equation (67). This property allows to provide a biological interpretation of this equation and in particular to compare it to the so-called canonical equation in Adaptive Dynamics [Die04, DL96] or to Lande's equation in Quantitative Genetics [Lus37, Lan79, LA83]. In these equations, which are very related equations under different formalisms, the change in the dominant/average trait is given by the product of the gradient of the fitness and a term that scales the rate of evolutionary change (proportional to mutational variance or genetic variance respectively in adaptive dynamics and quantitative genetics). In (67), the dynamics of the dominant trait $\dot{\overline{z}}(t)$ is also given by the product of the gradient of the fitness $\nabla_z R$ and the term $\big(-D^2u\big(t,\overline{z}(t)\big)\big)^{-1}$ which, when multiplied by ε, approximates well the phenotypic covariance matrix (note that here we do not consider any environmental contribution in the phenotype, therefore the phenotypic variance is equal to the genetic variance). In this way, (67) may be seen as a generalization of the canonical equation or Lande's equation to a case where the mutations are not assumed to be very rare (on the contrary to adaptive dynamics) and such that the evolution of the genetic variance is included in the dynamics (the phenotypic density is not assumed to be of Gaussian type with fixed variance).

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