The PBW Theorem

Thiago B. Garcia

thm (PBW). Given $\mathfrak{g}$, the map

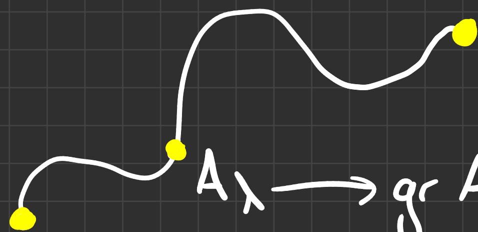
$$\begin{aligned} S(\mathfrak{g}) &\twoheadrightarrow \operatorname{gr} \mathcal{U}(\mathfrak{g}) \\ X &\longmapsto X + F^0\mathcal{U}(\mathfrak{g}) \end{aligned}$$

is an isomorphism of graded $\mathbb{C}$-algebras.

Braverman & Gaitsgory

ingredients: deformations + cohomology

a cartoonish picture

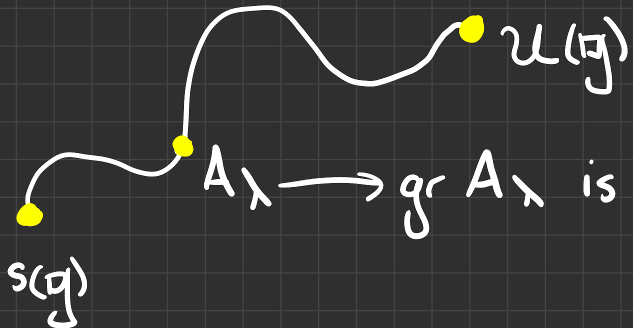


$\mathcal{U}(\mathfrak{g})$

$A_\lambda \longrightarrow \mathrm{gr}\, A_\lambda$ is constant!

$S(\mathfrak{g})$

a cartoonish picture



$A_\lambda \longrightarrow \mathrm{gr}\, A_\lambda$ is constant!

deformations!

$$A_\lambda := T\mathfrak{g} \Big/ (x\otimes y - y \otimes x - \lambda[x,y] : x,y\in\mathfrak{g})$$

def. A (flat graded) **deformation** A_t of a graded associative $\mathbb{C}$-algebra A is a $\mathbb{C}[t]$-algebra, which is free as a module, and whose fiber $A_0 = A_t/(t)$ at $t=0$ is isomorphic to A as a filtered $\mathbb{C}$-algebra.

$$\downarrow$$

$$A_\lambda := A_t/(t-\lambda)$$

def. A (flat graded) **i-level deformation** A_t of a graded associative $\mathbb{C}$-algebra A is a $\mathbb{C}[t]/(t^{i+1})$-algebra, which is free as a module, and whose fiber $A_0 = A_t/(t)$ at $t=0$ is isomorphic to A as a filtered $\mathbb{C}$-algebra.

$$\downarrow$$

$$A_\lambda := A_t/(t-\lambda)$$

def. A (flat graded) **i-level deformation** A_t of a graded associative $\mathbb{C}$-algebra A is a $\mathbb{C}[t]/(t^{i+1})$-algebra, which is free as a module, and whose fiber $A_0 = A_t/(t)$ at $t=0$ is isomorphic to A as a filtered $\mathbb{C}$-algebra.

$$A_\lambda := A_t/(t-\lambda)$$

reduction mod i

$$\begin{tikzcd}
 & & & \mathrm{Def}(A) \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d, dotted] \\
\mathrm{Def}_1(A) & \mathrm{Def}_2(A) \arrow[l] & \mathrm{Def}_3(A) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$$\mathrm{Def}(A) \cong \varprojlim_i \mathrm{Def}_i(A)$$

def. Given a graded algebra A and a graded A-bimodule M, we define the **bar complex**

$$0 \longrightarrow C^0(M) \xrightarrow{d} C^1(M) \xrightarrow{d} C^2(M) \xrightarrow{d} \cdots,$$

where

$$C^i(M)_n = \left\{ f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes i}, M) : \begin{array}{l} \deg f(a_1, \ldots, a_i) \\ = \deg a_1 + \cdots + \deg a_i + n \end{array} \right\}$$

and

$$df(a_1, \ldots, a_{i+1}) = a_1 f(a_2, \ldots, a_{i+1}) + (-1)^{i+1} f(a_1, \ldots, a_i)\, a_{i+1}$$
$$+ \sum_{k=1}^{i} (-1)^k f(a_1, \ldots, a_k a_{k+1}, \ldots, a_{i+1}).$$

$$HH^i(A, M) = H^i(C^\bullet(M)) = \frac{\ker(C^i(M) \longrightarrow C^{i+1}(M))}{\operatorname{im}(C^{i-1}(M) \longrightarrow C^i(M))}$$

Prop.

(i) The set of iso. classes of 1^{st} level deformations of A is canonically identified with $HH^2(A,A)_{-1}$.

$$\{1^{st}\text{ level }A\text{-defs}\}/_{\cong} \rightsquigarrow HH^2(A,A)_{-1}$$

(ii) If $A_t^{(i)}$ is an i-level deformation of A, the obstruction to its continuation to the $(i+1)^{st}$ level lies in $HH^3(A,A)_{-i-1}$.

(iii) The set of iso. classes of continuations of $A_t^{(i)}$ has the natural structure of a $HH^2(A,A)_{-i-1}$-homogeneous space.

$$\{\text{continuations of }A_t^{(i)}\}/_{\cong} \leftrightsquigarrow HH^2(A,A)_{-i-1}$$

big thm. Let $A = S(\mathfrak{g})$. There is a quasi-iso.

$$\begin{tikzcd}
0 \arrow[r] & C^0(M) \arrow[r, "d"] \arrow[d, "i"'] & C^1(M) \arrow[r, "d"] \arrow[d, "i"'] & C^2(M) \arrow[r, "d"] \arrow[d, "i"'] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^0 \mathfrak{g}, M) \arrow[r, "0"'] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^1 \mathfrak{g}, M) \arrow[r, "0"'] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^2 \mathfrak{g}, M) \arrow[r, "0"'] & \cdots
\end{tikzcd}$$

where $i f(a_1, \ldots, a_i) = \sum_{\sigma \in S^i} (-1)^{|\sigma|} f(a_{\sigma(1)}, \ldots, a_{\sigma(i)})$.

big thm. Let $A = S(\mathfrak{g})$. There is a quasi-iso.

$$\begin{tikzcd} 0 \arrow[r] & C^0(M) \arrow[r, "d"] \arrow[d, "i"'] & C^1(M) \arrow[r, "d"] \arrow[d, "i"'] & C^2(M) \arrow[r, "d"] \arrow[d, "i"'] & \cdots \\ 0 \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^0 \mathfrak{g}, M) \arrow[r, "0"'] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^1 \mathfrak{g}, M) \arrow[r, "0"'] & \mathrm{Hom}_{\mathbb{C}}(\Lambda^2 \mathfrak{g}, M) \arrow[r, "0"'] & \cdots \end{tikzcd}$$

where $i f(a_1, \ldots, a_i) = \sum_{\sigma \in S^i} (-1)^{|\sigma|} f(a_{\sigma(1)}, \ldots, a_{\sigma(i)})$.

$$\Omega(a,b,c) = \sum_{k=1}^{i} f_k(f_{i-k+1}(a,b),c) - f_k(a, f_{i-k+1}(b,c))$$

$$\rightsquigarrow$$

$$i\Omega(X,Y,Z) = if_i(if_1(X,Y),Z) + if_i(if_1(Y,Z),X) + if_i(if_1(Z,X),Y)$$

the proof

$\vdots$

$\downarrow$

4th

$\downarrow$

3rd

$\downarrow$

2nd

$\downarrow$

1st

the proof

$$\begin{array}{c} \vdots \\ \downarrow \\ 4^{th} \\ \downarrow \\ 3^{rd} \\ \downarrow \\ 2^{nd} \\ \downarrow \\ 1^{st} \end{array}$$

always extends because $HH^3(S(\mathfrak{g}), S(\mathfrak{g}))_{-i} = 0$ for $i \geq 3$

the proof

$\vdots$

$\downarrow$

4^{th} → always extends because $HH^3(S(\mathfrak{g}), S(\mathfrak{g}))_{-i-1} = 0$ for $i \geq 3$

$\downarrow$

3^{rd}

$\downarrow$

2^{nd}

$\downarrow$ → Jacobi identity!

1^{st} → take f_1 such that $\partial f_1 = [\,,\,]$

the proof

$\vdots$

$\downarrow$ → always extends because $HH^3(S(\mathfrak{g}), S(\mathfrak{g}))_{-i-1} = 0$ for $i \geqslant 3$

4th

$\downarrow$

3rd

$\downarrow$

2nd → take f_2 with $\partial f_2 = 0$ (i.e. f_2 symmetric)

$\downarrow$ → Jacobi identity!

1st → take f_1 such that $\partial f_1 = [\,,\,]$

the proof

take f_i symmetric for $i > 3$

always extends because $HH^3(S(\mathfrak{g}), S(\mathfrak{g}))_{-i-1} = 0$ for $i \geq 3$

4th

3rd — take f_3 symmetric

2nd — take f_2 with $\partial f_2 = 0$ (i.e. f_2 symmetric)

Jacobi identity!

1st — take f_1 such that $\partial f_1 = [\,,\,]$

the proof

take f_i symmetric for $i > 3$

always extends because $HH^3(S(\mathfrak{g}), S(\mathfrak{g}))_{-i-1} = 0$ for $i \geq 3$

4th

3rd → take f_3 symmetric

2nd → take f_2 with $\partial f_2 = 0$ (i.e. f_2 symmetric)

→ Jacobi identity!

1st → take f_1 such that $\partial f_1 = [\,,\,]$

We get a deformation $A_t = A \oplus At \oplus \cdots \oplus At^i \oplus \cdots$ with

$$a * b = ab + f_1(a,b)t + \cdots + f_i(a,b)t^i + \cdots$$

$$
\begin{tikzcd}
x \arrow[r, mapsto] & x + 0t + \cdots + (t-1) \\
\mathfrak{g} \arrow[r] \arrow[d] & A_1 \\
\mathcal{U}(\mathfrak{g}) \arrow[ur, dotted, "f"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
S(\mathfrak{g}) \arrow[r, "g"] & \operatorname{gr} A_1 \\
x \arrow[r, mapsto] & (x + 0t + \cdots + (t-1)) + F^0 A_1
\end{tikzcd}
$$

$$X \longmapsto X + 0t + \cdots + (t-1)$$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d] & A_1 \\
\mathcal{U}(\mathfrak{g}) \arrow[ur, dotted, "f"'] &
\end{tikzcd}
$$

$$S(\mathfrak{g}) \xrightarrow{g} \operatorname{gr} A_1$$

$$X \longmapsto (X + 0t + \cdots + (t-1)) + F^0 A_1$$

$$S(\mathfrak{g}) \longrightarrow \operatorname{gr} \mathcal{U}(\mathfrak{g}) \xrightarrow{\operatorname{gr} f} \operatorname{gr} A_1 \xrightarrow{g^{-1}} S(\mathfrak{g})$$

$$X \longmapsto X + F^0 \mathcal{U}(\mathfrak{g}) \longmapsto (X + 0t + \cdots + (t-1)) + F^0 A_1 \longmapsto X$$

$$x \longmapsto x + 0t + \cdots + (t-1)$$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, hook] & A_1 \\
\mathcal{U}(\mathfrak{g}) \arrow[ur, dotted, "f"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
S(\mathfrak{g}) \arrow[r, "g"] & \operatorname{gr} A_1
\end{tikzcd}$$

$$x \longmapsto (x + 0t + \cdots + (t-1)) + F^0 A_1$$

$$\begin{tikzcd}
S(\mathfrak{g}) \arrow[r] \arrow[rrr, bend left, "\mathrm{id}"] & \operatorname{gr} \mathcal{U}(\mathfrak{g}) \arrow[r, "\operatorname{gr} f"] & \operatorname{gr} A_1 \arrow[r, "g^{-1}"] & S(\mathfrak{g})
\end{tikzcd}$$

$$x \longmapsto x + F^0\mathcal{U}(\mathfrak{g}) \longmapsto (x + 0t + \cdots + (t-1)) + F^0 A_1 \longmapsto x$$

$\blacksquare$

Thanks!

https://linux.ime.usp.br/~pablo