# GdR "Analyse Fonctionnelle et Harmonique et Probabilités" Mini-courses

John E. McCarthy, Alexei Poltoratski

Toulouse, October 12 to 14, 2016

## 1 Schedule

**Wednesday, October 12**

13h: Lunch

14h15: Alexei Poltoratski, lecture 1
15h45: Coffee break
16h15: John E. McCarthy, lecture 1

**Thursday, October 13**

09h15: Alexei Poltoratski, lecture 2
10h45: Coffee break
11h15: John E. McCarthy, lecture 2

13h: Lunch

14h15: Alexei Poltoratski, lecture 3
15h45: Coffee break
16h15: John E. McCarthy, lecture 3

**Friday, October 14**

09h15: Alexei Poltoratski, lecture 4
10h45: Coffee break
11h15: John E. McCarthy, lecture 4

13h: Lunch

# 2 Abstracts

**John E. McCarthy: Non-commutative function theory and its applications to operator theory**

Non-commutative function theory, as developed in the book [5], is the study of functions whose input is a $d$-tuple of $n$-by-$n$ matrices and whose output is an $n$-by-$n$ matrix, with the idea that it should somehow be independent of $n$. Basic examples are non-commutative polynomials, like

$$p(x, y) = x^3 - 3x^2y + 4xyx - 5yx^2 + 6xy - yx + I. \tag{1}$$

More complicated examples can be thought of as limits of sequences of non-commutative polynomials, much as holomorphic functions can be thought of as limits of sequences of polynomials (locally).

We shall discuss:

(i) The motivation for studying non-commutative functions.

(ii) A realization formula for non-commutative functions that are bounded on polynomial polyhedra (we call these free n.c. functions).

(iii) A characterization of functions that are locally limits of non-commutative polynomials.

(iv) An implicit function theorem, which can be used to show that generically the matrix solutions to $p(x, y) = 0$ commute (where $p$ is as in (1)).

(v) How to evaluate free n.c. functions on $d$-tuples of operators, and how to use them to get a non-commutative functional calculus.

The talks are based on joint work with Jim Agler, which appears in [1, 3, 2].

# References


[1] J. Agler and J.E. McCarthy. Global holomorphic functions in several non-commuting variables. *Canad. J. Math.*, 67(2):241–285, 2015.

[2] J. Agler and J.E. McCarthy. Non-commutative holomorphic functions on operator domains. *European J. Math*, 1(4):731–745, 2015.

[3] J. Agler and J.E. McCarthy. The implicit function theorem and free algebraic sets. *Trans. Amer. Math. Soc.*, 368(5):3157–3175, 2016.

[4] J. Agler, J.E. McCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math.*, 176(3):1783–1826, 2012.

[5] Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov. *Foundations of free noncommutative function theory.* AMS, Providence, 2014.

[6] K. Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38:177–216, 1934.

**Alexei Poltoratski: Toeplitz Order in the Area of Uncertainty**

The area of Uncertainty Principle in Harmonic Analysis (UP) was founded in the 1920s by Norbert Wiener. It stems from a simple rule stating that "A function (measure, distribution) and its Fourier transform cannot be small simultaneously". For various mathematical meanings of smallness this statement leads to deep and important problems of analysis. At present the area of UP spans across many fields including questions on completeness, sampling and uniqueness in function spaces, multiple classical and abstract versions of the moment problem, estimates of singular integrals and spectral problems for differential equations.

The idea to apply Toeplitz operators to study Riesz bases in model spaces of analytic functions first appeared in the seminal work by Khruschev, Nikolski and Pavlov in the 1980s. In our papers with N. Makarov we tried to extend the Toeplitz approach to other problems of UP. In the last 10 years such methods led to solutions of some of the classical problems and revealed hidden connections between distant areas of analysis and mathematical physics.

In my course I will try to survey the basics of the Toeplitz approach and discuss recent developments. The new part of the course is the point of view based on partial order of the set of inner functions induced by Toeplitz operators. The study of the Toeplitz order includes several well-known open problems of analysis and points to new possibilities for further research.