ON THE EXISTENCE OF DEGENERATE SOLUTIONS OF THE TWO-DIMENSIONAL H-SYSTEM

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Abstract. We consider entire solutions $\omega \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ of the H-system

$$\Delta\omega = 2\omega_x \wedge \omega_y \,,$$

which we refer to as *bubbles*. Surprisingly, and contrary to conjectures raised in the literature, we find that bubbles with degree at least three can be degenerate: the linearized H-system around a bubble can admit solutions that are not tangent to the smooth family of bubbles. We then give a complete algebraic characterization of degenerate bubbles.

1. Introduction

In this short note we study critical points of the functional $\mathcal{E}\colon \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}\mathcal{L}^2 + \frac{2}{3}\int_{\mathbb{R}^2} \langle u, u_x \wedge u_y\rangle \,\mathrm{d}\mathcal{L}^2 \,, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^3$, $\mathcal{L}^2$ the 2-dimensional Lebesgue measure and u_x, u_y the partial derivatives of u with respect to x, y respectively. Both terms in (1.1) are *conformally invariant* and so, after identifying $\overline{\mathbb{R}^2} \cong \mathbb{S}^2$ through stereographic projection, the term

$$\mathcal{V}(u) := \frac{1}{3}\int_{\mathbb{R}^2} \langle u, u_x \wedge u_y\rangle \,\mathrm{d}\mathcal{L}^2 \tag{1.2}$$

corresponds to the signed algebraic volume of the region enclosed by $u(\mathbb{S}^2)$, whenever u is regular. For maps which are just in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^2;\mathbb{R}^3)$, one can define $\mathcal{V}(u)$ through its continuous extension [26].

The functional (1.1) is very classical, as it appears naturally in the study of constant mean curvature surfaces, see *e.g.* [25, Section III.5] and the references therein. To see this connection, we note that the first variation $\mathcal{E}'\colon \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \to \dot{H}^{-1}(\mathbb{R}^2;\mathbb{R}^3)$ is given by

$$\mathcal{E}'(u) = -\Delta u + 2u_x \wedge u_y \,, \tag{1.3}$$

and thus we arrive at the following:

Definition 1.1. A *bubble* is a map $\omega \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ such that $\mathcal{E}'(\omega) = 0$, *i.e.*,

$$\Delta\omega = 2\,\omega_x \wedge \omega_y \quad \text{in } \mathcal{D}'(\mathbb{R}^2)\,. \tag{1.4}$$

In other words, bubbles are simply entire finite-energy solutions of (1.4), which is known as the *H-system*. Any such solution is necessarily *weakly conformal*, *i.e.*, it satisfies

$$|\omega_x|^2 - |\omega_y|^2 = \langle \omega_x, \omega_y\rangle = 0 \quad \text{in } \mathbb{R}^2\,. \tag{1.5}$$

Combined, equations (1.4) and (1.5) assert that ω is a (possibly branched) weakly conformal parametrization of a closed surface with mean curvature identically 1. In fact, a classical theorem due to Hopf asserts that any such surface is a unit sphere. This can also be seen as a consequence of the following classification result, due to Brezis and Coron [1, Lemma A.1], which describes completely the collection of bubbles:

Theorem 1.2 (Classification of bubbles [1])**.** *Let $\omega \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ be a bubble. Then there exist complex polynomials $P, Q \in \mathbb{C}[z]$ and a vector $b \in \mathbb{R}^3$ such that*

$$\omega(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + b\,, \tag{1.6}$$

where $\pi\colon \mathbb{C} \to \mathbb{S}^2$ is the inverse stereographic projection, i.e.,

$$\pi(z) := \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)\,, \tag{1.7}$$

and where we identify $z = (x,y) = x+iy$. Moreover, if

$$P/Q \text{ is irreducible}, \quad k := \max\{\deg P, \deg Q\}, \tag{1.8}$$

then we have

$$\frac{1}{8\pi}\int_{\mathbb{R}^2} |\nabla\omega|^2 \,\mathrm{d}\mathcal{L}^2 = k\,.$$

We refer to $k \in \mathbb{N}$ as in Theorem 1.2 as the *degree* of ω, since it coincides with the topological degree of ω, viewed as a map between two-dimensional unit spheres.

Theorem 1.2 shows that the collection of bubbles can be seen as a disjoint union of the smooth, finite-dimensional manifolds

$$\mathcal{M}_k := \left\{(P,Q,b) \in \mathbb{C}[z]\times\mathbb{C}[z]\times\mathbb{R}^3 : P, Q \text{ are non-zero}\,, P \text{ is monic and (1.8) holds}\right\}\,. \tag{1.9}$$

An important problem is to understand the behavior of $\mathcal{E}$ near its critical points. To be more concrete, for any bubble $\omega \in \mathcal{M}_k$, we regard the second variation of $\mathcal{E}$ at ω as a linear operator $\mathcal{E}''(\omega)\colon \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \to \dot{H}^{-1}(\mathbb{R}^2;\mathbb{R}^3)$. In view of (1.3) it is explicitly given by

$$\mathcal{E}''(\omega)[u] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{E}'(\omega+tu) = -\Delta u + 2(\omega_x\wedge u_y + u_x\wedge\omega_y)\,. \tag{1.10}$$

Variations tangent to $\mathcal{M}_k$ at ω generate elements in the kernel of $\mathcal{E}''(\omega)$, so that

$$\dim\ker\mathcal{E}''(\omega) \geq \dim\mathcal{M}_k = 4k+5\,.$$

This leads us to the following standard concept:

Definition 1.3. A bubble ω with degree k is said to be *degenerate* if $\dim\ker\mathcal{E}''(\omega) > 4k+5$, and it is said to be *non-degenerate* otherwise.

As each $\mathcal{M}_k$ is smooth, non-degeneracy is equivalent to the usual notion of integrability, see *e.g.* [22, Section 3.13]. Due to the conformal invariance of $\mathcal{E}$, we can and will regard each bubble as a map $\omega\colon \mathbb{S}^2 \to \mathbb{R}^3$.

In [14, Lemma 5.5] it was shown that bubbles with degree one are non-degenerate, see further [5, Appendix] and [20, Section 3], while in [23, Theorem 1.1] it was shown that the *standard k-bubble*, corresponding to the choice $P(z) = z^k$ and $Q(z) = 1$ in (1.6), is non-degenerate as well. These works also raise the conjecture that all bubbles should be non-degenerate, cf. [5, page 190] and [23, page 4]. Surprisingly, in this note we observe that although this is generically the case, for $k \geq 3$ there is an *exceptional set of degenerate bubbles* in $\mathcal{M}_k$:

Theorem 1.4 (Characterization of degenerate bubbles)**.** *Let $\omega\colon \mathbb{S}^2 \to \mathbb{R}^3$ be a bubble as in* (1.6) *whose set of branch points is given by*

$$\{|\nabla\omega| = 0\} =: \{p_1, \dots, p_n\}.$$

Let z be a conformal coordinate on $\mathbb{S}^2$ with respect to which none of the branch points is ∞. Then ω is degenerate if and only if there exists a non-zero polynomial $R \in \mathbb{C}[z]$ with $\deg R \leq n-4$ such that the meromorphic function $h\colon \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) := \frac{R(z)}{(z-p_1)\dots(z-p_n)}$$

solves the algebraic system of equations

$$\mathrm{Res}_{p_j}\left(\frac{h}{(P/Q)'}\right) = 0 \quad \text{for } j \in \{1, \dots, n\}\,.$$

We note that Theorem 1.4 implies that the set of degenerate bubbles of degree k has (real) codimension at least two in $\mathcal{M}_k$, see the comment after [9, Theorem 2]. The above characterization follows from a correspondence between nontrivial elements in the kernel of $\mathcal{E}''[\omega]$ and solutions of a well-studied Schrödinger equation on $\mathbb{S}^2$, *cf.* [7, 8, 9, 10, 16, 19, 21]. As immediate consequences of Theorem 1.4 we obtain the following:

Corollary 1.5 (Examples of non-degenerate bubbles)**.** *Let ω be a degree k bubble as in* (1.6)*. If ω is degenerate then it has at least 4 branch points. In particular:*

(i) *if $k \leq 2$ then ω is non-degenerate;*
(ii) *the standard k-bubble corresponding to*

$$P(z) = z^k, \quad Q(z) = 1\,,$$

is non-degenerate for all $k \in \mathbb{N}$.

Other non-degenerate examples can be inferred from [19], see *e.g.* [19, Corollary 15].

Corollary 1.6 (Examples of degenerate bubbles)**.** *Let ω be a degree k bubble as in* (1.6)*. If $k = 3$ then ω is degenerate if and only if*

$$P(z) = z^3 + 2, \quad Q(z) = z\,,$$

up to a Möbius transformation of the sphere.

An interesting question that can subsequently be pursued is to prove optimal Łojasiewicz inequalities for degenerate bubbles. There are only a few examples of variational problems where this is achieved [11, 12]. We also refer the reader to [18] for a related result, where Łojasiewicz inequalities near the simplest possible bubble tree in a surface with positive genus were obtained. Such inequalities determine in a precise way the leading order behavior of $\mathcal{E}$ near a bubble, and this information is useful in arguments involving blow-up or perturbative analysis, see *e.g.* [1, 2, 3, 4, 5, 14, 15, 23, 24].

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2. Characterization of degenerate bubbles

As explained in the introduction, the non-degeneracy of bubbles (cf. Definition 1.1) is characterized through the study of the kernel of the linearized operator $\mathcal{E}''(\omega)\colon \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \to \dot{H}^{-1}(\mathbb{R}^2;\mathbb{R}^3)$ defined in (1.10). In other words, we are interested in classifying the space of solutions $u \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ to

$$\Delta u = 2(\omega_x \wedge u_y + u_x \wedge \omega_y) \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \tag{2.1}$$

for an arbitrary bubble ω. We note that, by elliptic regularity, any such solution is necessarily smooth.

Given a bubble ω as in (1.6), we can assume without loss of generality that $b = 0$, since $\mathcal{E}$ is invariant under translations in the target. Then, due to the conformal invariance of $\mathcal{E}$, we can regard bubbles as maps $\omega\colon \mathbb{S}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$. More explicitly, given a bubble ω, we will write

$$\omega(z) = \pi(\varphi(z)), \qquad \varphi(z) := \frac{P(z)}{Q(z)}, \qquad \deg P \leq \deg Q, \tag{2.2}$$

where π is as in (1.7) and $P, Q \in \mathbb{C}[z]$ for some conformal coordinate z on $\mathbb{S}^2$; the assumption on the degrees of P, Q can always be achieved by choosing a suitable coordinate z.

We find it convenient to use complex notation, and henceforth we will write

$$\partial_z := (\partial_x - i\partial_y)/2, \quad \partial_{\bar{z}} := (\partial_x + i\partial_y)/2$$

for the usual Wirtinger derivatives. Let the Euclidean inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^3$ be extended as a complex-bilinear form on $\mathbb{C}^3$. Then the fact (1.5) that each bubble is weakly conformal is expressed concisely as

$$\langle \omega_z, \omega_z \rangle = 0 \text{ in } \mathbb{C}, \tag{2.3}$$

and the linearization of this equation leads us, as in [13, Section 2], to the following:

Definition 2.1. Given a bubble $\omega\colon \mathbb{S}^2 \to \mathbb{S}^2$ and a vector field $v \in \dot{H}^1(\mathbb{S}^2;\mathbb{R}^3)$, we say that v is a *conformal Jacobi field along* ω if $\langle \omega_z, v_z \rangle = 0$.

We can also express equations (1.4) and (2.1) respectively as

$$\omega_{z\bar{z}} = i\,\omega_{\bar{z}} \wedge \omega_z, \tag{2.4}$$

$$u_{z\bar{z}} = i(u_{\bar{z}} \wedge \omega_z + \omega_{\bar{z}} \wedge u_z), \tag{2.5}$$

in complex notation. We then have the following:

Lemma 2.2. *Let $u\colon \mathbb{S}^2 \to \mathbb{R}^3$ solve* (2.5) *for a bubble ω. Then u is a conformal Jacobi field.*

Proof. We use (2.4) and (2.5) to compute

$$\begin{aligned}\partial_{\bar{z}}\langle \omega_z, u_z \rangle &= i\left(\langle \omega_{\bar{z}} \wedge \omega_z, u_z \rangle + \langle \omega_z, u_{\bar{z}} \wedge \omega_z + \omega_{\bar{z}} \wedge u_z \rangle\right) \\ &= i\left(\langle \omega_{\bar{z}} \wedge \omega_z, u_z \rangle + \langle \omega_z, \omega_{\bar{z}} \wedge u_z \rangle\right) = 0,\end{aligned}$$

where the last equality is simply an algebraic identity resulting from $\langle a, b \wedge c \rangle = \det(a|b|c)$, where $(a|b|c)$ denotes the 3×3 matrix with column vectors a, b, c (in this order). Thus $\langle \omega_z, u_z \rangle$ is integrable and holomorphic, hence by Liouville's theorem it is zero. □

Note that each bubble ω as in (2.2) is a *harmonic map* into $\mathbb{S}^2$, *i.e.*,

$$\Delta \omega + |\nabla \omega|^2 \omega = 0 \quad \iff \quad \omega_{z\bar{z}} + |\omega_z|^2 \omega = 0, \tag{2.6}$$

as can be checked directly by differentiating $|\omega| = 1$, using the weak conformality (1.5) of ω. Since the target is a 2-sphere, ω is an integrable harmonic map [13]; we note that this integrability does not hold for higher-dimensional spheres [17]. The following proposition is essentially a consequence of this integrability result.

Proposition 2.3 (Decomposition)**.** *Let $\omega\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a degree k bubble and decompose maps $u \in \ker \mathcal{E}''(\omega)$ according to*

$$\ker \mathcal{E}''(\omega) = T(\omega) \oplus N(\omega)\,\omega\,, \qquad u = [u - \langle u, \omega\rangle \omega] + \langle u, \omega\rangle\omega\,,$$

where this decomposition is orthogonal pointwise with respect to the inner product of $\mathbb{R}^3$. The elements of $T(\omega)$ are generated by infinitesimal variations in the coefficients of φ in (2.2)*, so that* $\dim T(\omega) = 4k + 2$*, and*

$$N(\omega) := \left\{ f \in C^\infty(\mathbb{S}^2; \mathbb{R}) : \Delta f + |\nabla\omega|^2 f = 0 \right\}\,. \tag{2.7}$$

Proof. Let us note that the equation defining $N(\omega)$ can be written in complex notation as

$$f_{z\bar{z}} + |\omega_z|^2 f = 0\,. \tag{2.8}$$

We now observe that $f \in N(\omega)$ if and only if the map $f\omega$ solves (2.5), since

$$\begin{aligned}
&(f\omega)_{z\bar{z}} - i\big((f\omega)_{\bar{z}} \wedge \omega_z + \omega_{\bar{z}} \wedge (f\omega)_z\big)\\
&= f_{z\bar{z}}\,\omega - i f\omega_{\bar{z}} \wedge \omega_z + f_z\omega_{\bar{z}} + f_{\bar{z}}\omega_z - i\left(f_{\bar{z}}\omega \wedge \omega_z + f_z\omega_{\bar{z}} \wedge \omega\right)\\
&= \left[f_{z\bar{z}} + |\omega_z|^2 f\right]\omega\,,
\end{aligned}$$

where in the second line we used (2.4) and in the last line we used the identities

$$\omega_z = i\,\omega \wedge \omega_z\,,\ \omega_{\bar{z}} = i\,\omega_{\bar{z}} \wedge \omega\,,\ \text{and } i\,\omega_z \wedge \omega_{\bar{z}} = |\omega_z|^2\omega\,, \tag{2.9}$$

which follow from the fact that $\left(\frac{\sqrt{2}\omega_x}{|\nabla\omega|}, \frac{\sqrt{2}\omega_y}{|\nabla\omega|}, -\omega\right)$ is a positively oriented orthonormal frame of $\mathbb{R}^3$. In view of (2.5), $N(\omega)\omega \subset \ker \mathcal{E}''(\omega)$, and reversely, for any $u \in \ker \mathcal{E}''(\omega)$ using (2.4), (2.5) and (2.9) we have $\langle u, \omega\rangle \in N(\omega)$, and so from the above, $\langle u, \omega\rangle\omega \in \ker \mathcal{E}''(\omega)$. Note that, by elliptic regularity, any $f \in \dot{H}^1(\mathbb{S}^2; \mathbb{R})$ solving (2.8) is actually smooth.

It remains to prove the characterization of $T(\omega)$. For this, we first note that since $|\omega|^2 = 1$ we have $\langle \omega_z, \omega\rangle = 0$. Thus, by (2.3), for any $f \in \dot{H}^1(\mathbb{S}^2; \mathbb{R})$ the vector field $f\omega$ is a conformal Jacobi field along ω. By Lemma 2.2, any $u \in \ker \mathcal{E}''(\omega)$ is also a conformal Jacobi field along ω, and hence by linearity the same also holds for the map $\widetilde{u} := u - \langle u, \omega\rangle\omega$. Moreover, note that $\widetilde{u}(z) \in T_{\omega(z)}\mathbb{S}^2$. Let us assume that the polynomials P, Q in (2.2) satisfy $\det P \le \deg Q$, the other case being entirely analogous. Applying [13, Lemma 3], we deduce that $\widetilde{u}$ is integrable, *i.e.*, there are polynomials $A, B \in \mathbb{C}[z]$ such that

$$\widetilde{u} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi\left(\frac{P + tA}{Q + tB}\right), \tag{2.10}$$

where $\deg(A) \le \deg(P)$ and $\deg(B) < \det(Q)$. Conversely, given polynomials A, B satisfying these conditions, we can define a map $\widetilde{u} \in T(\omega)$ through (2.10). This completes the characterization and thus $\dim T(\omega) = 4k + 2$. □

Corollary 2.4. *We have* $\dim N(\omega) \ge 3$ *with equality if and only if ω is non-degenerate.*

Proof. By (2.6), the linear space

$$L(\omega) := \left\{ \langle \xi, \omega\rangle : \xi \in \mathbb{R}^3 \right\} \tag{2.11}$$

is a subspace of $N(\omega)$, and thus $\dim N(\omega) \geq 3$. By Proposition 2.3, equality holds if and only if $\dim \ker \mathcal{E}''(\omega) = 4k+5$, *i.e.*, if and only if ω is non-degenerate. $\square$

We refer to $L(\omega)$ as the space of *trivial* solutions to the Schrödinger equation (2.8). Since ω is conformal, (2.8) can be rewritten as

$$\Delta_{g_\omega} \widetilde{f} + 2\widetilde{f} = 0\,, \text{ where } g_\omega := \omega^* g_0 \text{ and } \widetilde{f} := f \circ \pi^{-1} \in \dot{H}^1(\mathbb{S}^2;\mathbb{R})\,.$$

Here g_ω is the pullback metric of the standard round metric g_0 on $\mathbb{S}^2$ induced by $\omega\colon \mathbb{S}^2 \to \mathbb{S}^2$; the metric g_ω in general has *conical singularities* at the branch points of ω. Equation (2.8) has received a lot of attention in the last decades, and the space of solutions is completely understood, see *e.g.* [7, 8, 9, 10, 16, 19, 21]. Surprisingly, for non-generic ω there are non-trivial solutions to (2.8), *i.e.* $\dim N(\omega) > 3$; the space of such ω has codimension larger than one [10, Proof of Theorem A]. In general, $\dim N(\omega) - 3$ is *even*, since $N(\omega)/L(\omega)$ is a complex vector space [19, Proposition 18], and so we can write $\dim N(\omega) = 3 + 2d$ for $d \in \mathbb{N}_0$. The $2d$ extra directions can be interpreted as arising from smooth one-parameter families of harmonic maps with values in higher dimensional spheres [8, Theorem A].

The following result provides a concrete algebraic characterization of those bubbles ω for which there are nontrivial elements in $N(\omega)$.

Theorem 2.5 ([10, 19])**.** *Let ω be a bubble as in* (2.2)*, let $\{p_1,\dots,p_n\} \subset \mathbb{S}^2$ be its set of branch points and z a conformal coordinate on $\mathbb{S}^2$ with respect to which none of the branch points is ∞. There is a linear bijection between the space $N(\omega)/L(\omega)$ of non-trivial solutions to* (2.8) *and the space of non-zero polynomials $R \in \mathbb{C}[z]$ with $\deg R \leq n-4$ and such that*

$$\operatorname{Res}_{p_j}\left(h/\varphi'\right) = 0 \text{ for all } j \in \{1,\dots,n\}\,, \text{ where } h(z) := \frac{R(z)}{(z-p_1)\dots(z-p_n)}\,. \tag{2.12}$$

The proof of Theorem 2.5 in the above references is carried out in a more general context. In order to keep the exposition mostly self-contained, we include here a direct proof of their result in our setting, following essentially the arguments in [19].

The main idea behind Theorem 2.5 is to use the *Gauss parametrization* [6]. A surface $S \subset \mathbb{R}^3$ with invertible Gauss map ω is locally parametrized by an immersion

$$X = f\omega + \nabla_{g_\omega} f = f\omega + \frac{1}{|\omega_z|^2}\left(f_z \omega_{\bar z} + f_{\bar z}\omega_z\right), \tag{2.13}$$

where $f = \langle X, \omega\rangle$ is the *support function* of S. Thus, in the setting of Theorem 2.5, we have a local correspondence between functions $f\colon \mathbb{S}^2 \setminus \{p_1,\dots,p_n\} \to \mathbb{R}$ and punctured surfaces with generalized Gauss map ω. The next lemma gives a geometric interpretation of the condition $f \in N(\omega)$ (recall (2.7)) in terms of the corresponding immersion defined through (2.13):

Lemma 2.6. *Let ω be a bubble with branch points $\{p_1,\dots,p_n\} \subset \mathbb{S}^2$, let $f \in C^\infty(\mathbb{S}^2;\mathbb{R})$ and let $X\colon \mathbb{S}^2 \setminus \{p_1,\dots,p_n\} \to \mathbb{R}^3$ be defined as in* (2.13)*. Then*

$$f \in N(\omega) \quad \Longleftrightarrow \quad X \textit{ is minimal}\,, \textit{ i.e., it is weakly conformal and harmonic}\,.$$

Proof. Using (2.3) and $|\omega| = 1$, after differentiation we deduce that

$$\omega_{zz} = \left[\log(|\omega_z|^2)\right]_z \omega_z\,. \tag{2.14}$$

An elementary but lengthy calculation, using (2.14), then shows that

$$\left[\log(|\omega_z|^2)\right]_{z\bar z} + |\omega_z|^2 = 0\,. \tag{2.15}$$

Note also that $\partial_z(\frac{1}{|\omega_z|^2}) = -\frac{\log(|\omega_z|^2)_z}{|\omega_z|^2}$; this, combined with (2.6) and (2.15), yields

$$X_z = \left(f_{z\bar z} + |\omega_z|^2 f\right)\frac{\omega_z}{|\omega_z|^2} + h_f\frac{\omega_{\bar z}}{|\omega_z|^2}\,, \tag{2.16}$$

where we set

$$h_f := f_{zz} - \left[\log(|\omega_z|^2)\right]_z f_z\,. \tag{2.17}$$

We can then further compute

$$\langle X_z, X_z\rangle = 2\frac{h_f}{|\omega_z|^2}\left(f_{z\bar z} + |\omega_z|^2 f\right)\,. \tag{2.18}$$

The lemma follows from the above identities using elementary calculations. If $f \in N(\omega)$ then clearly X is weakly conformal by (2.18) and one can also check that $X_{z\bar z} = 0$, thus X is minimal. Conversely, if X is minimal then $f \in N(\omega)$: this can be deduced from (2.16) since

$$\langle X_{z\bar z}, \omega\rangle = -\left(f_{z\bar z} + |\omega_z|^2 f\right)\,,$$

so if X is harmonic then $f \in N(\omega)$. □

It will be useful to convert between ω and φ through the stereographic projection, as in (2.2). Using the fact that φ is meromorphic, one can verify that

$$\frac{\omega_{\bar z}}{|\omega_z|^2} = \frac{1}{\varphi'}\left(\frac{1-\varphi^2}{2}, \frac{i(1+\varphi^2)}{2}, \varphi\right)\,. \tag{2.19}$$

The next lemma is a simple consequence of the above calculations.

Lemma 2.7. *For any conformal coordinate z on $\mathbb{S}^2$ and $f \in N(\omega)$, let $h_f\colon \mathbb{S}^2\backslash\{p_1,\dots,p_n\} \to \mathbb{R}$ be defined as in* (2.17)*. Then:*

(i) *h_f is meromorphic, i.e., $\partial_{\bar z} h_f = 0$ on $\mathbb{S}^2 \setminus \{p_1,\dots,p_n\}$, and its poles are simple;*
(ii) *the meromorphic function h_f/φ', where $\varphi = P/Q$, has zero residue at each pole p_j.*

Proof. For (i) we note that, since $f \in N(\omega)$, from (2.16) and (2.19) we obtain

$$X_z = h_f\frac{\omega_{\bar z}}{|\omega_z|^2} = \frac{h_f}{\varphi'}\left(\frac{1-\varphi^2}{2}, \frac{i(1+\varphi^2)}{2}, \varphi\right)\,. \tag{2.20}$$

Thus $\partial_{\bar z} h_f = 0$, since φ is meromorphic and $X_{z\bar z} = 0$ according to Lemma 2.6. Moreover, the poles of h_f are simple, since in (2.17) f is smooth and $|\omega_z|$ vanishes to finite order at each branch point. Claim (ii) follows again from (2.20), which implies that $h_f/\varphi'\,\mathrm{d}z = (X^1 - iX^2)_z\,\mathrm{d}z$: the latter differential has zero integral along any loop, and hence zero residue at each pole. □

Corollary 2.8. *For $f \in N(\omega)$ and any choice of conformal coordinate z with respect to which none of the branch points is ∞, there exists $R \in \mathbb{C}[z]$ of degree at most $n-4$ such that*

$$h_f(z) = \frac{R(z)}{(z-p_1)\cdots(z-p_n)}\,. \tag{2.21}$$

Proof. Applying a translation in the complex plane if necessary, we may without restriction fix a conformal coordinate w in which none of the poles of φ is 0 or ∞. By Lemma 2.7(i), there exists an entire function g and $a_1,\dots,a_n \in \mathbb{C}$ such that

$$h_f(w) = g(w) + \sum_{j=1}^{n}\frac{a_j}{w-p_j}\,.$$

If $w = w(z)$ is a conformal change of variables then the meromorphic quadratic differential $h_f(w)(\mathrm{d}w)^2$ pulls back to $h_f(w(z))(\mathrm{d}w/\mathrm{d}z)^2(\mathrm{d}z)^2$. Thus, when $w(z) = 1/z$, we obtain a new meromorphic function

$$h_f(w(z))\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 = \frac{1}{z^4}g(1/z) + \frac{1}{z^4}\sum_{j=1}^n \frac{a_j}{1/z - p_j} = \frac{1}{z^4}g(1/z) + \frac{1}{z^3}\frac{\widetilde{R}(z)}{\prod_{j=1}^n (z - 1/p_j)}\,,$$

for some $\widetilde{R} \in \mathbb{C}[z]$ with $\deg \widetilde{R} \leq n-1$. In these new coordinates, Lemma 2.7(i) implies that the entire function g is zero and that $\widetilde{R}(z) = z^3 R(z)$ for some $R \in \mathbb{C}[z]$ with $\deg R \leq n-4$. □

Proof of Theorem 2.5. By Lemma 2.7 and Corollary 2.8, for each $f \in N(\omega)$ one can associate a function $h_f \colon \mathbb{S}^2 \setminus \{p_1, \dots, p_n\} \to \mathbb{R}$ satisfying (2.12). It is easy to see from (2.14) and (2.17) that if $f \in L(\omega)$ then $h_f = 0$, *i.e.*, the corresponding polynomial R in (2.21) is zero.

Conversely, given a function $h \colon \mathbb{S}^2 \setminus \{p_1, \dots, p_n\} \to \mathbb{R}$ satisfying (2.12), one constructs an immersion $X_h \colon \mathbb{S}^2 \setminus \{p_1, \dots, p_n\} \to \mathbb{R}^3$ such that

$$\partial_z X_h = h\frac{\omega_{\bar{z}}}{|\omega_z|^2},$$

cf. (2.20). In fact, we construct X_h by

$$X_h = \mathrm{Re}\int \partial_z X_h \, \mathrm{d}z\,, \tag{2.22}$$

where the path starts from a fixed point on $\mathbb{S}^2 \setminus \{p_1, \dots, p_n\}$; the no residue condition in (2.12) guarantees that X_h is independent of the choice of path. Moreover, X_h is unique up to addition of a constant vector in $\mathbb{R}^3$. We claim that $\langle X_h, \omega\rangle \in N(\omega)$. Once this is shown the theorem follows, since the linear maps

$$N(\omega) \ni f \mapsto f\omega + \frac{1}{|\omega_z|^2}(f_z\omega_{\bar{z}} + f_{\bar{z}}\omega_z)\,, \qquad X \mapsto \langle X, \omega\rangle \in N(\omega)$$

are inverse to each other, and hence

$$N(\omega)/L(\omega) \ni f \mapsto h_f, \qquad h \mapsto \langle X_h, \omega\rangle \in N(\omega)/L(\omega)$$

are also inverses, since adding a constant to X_h amounts to adding a trivial solution to f.

Thus, to complete the proof, it remains to show that $\langle X_h, \omega\rangle \in N(\omega)$. To be precise, note that $\langle X_h, \omega\rangle$ is only defined in $\mathbb{S}^2 \setminus \{p_1, \dots, p_n\}$ and that, in this punctured sphere, it is a solution of the Schrödinger equation (2.8), since

$$\begin{aligned}\partial_{z\bar{z}}\langle X_h, \omega\rangle + |\omega_z|^2\langle X_h, \omega\rangle &= \partial_{\bar{z}}\left(\langle \partial_z X_h, \omega\rangle + \langle X_h, \omega_z\rangle\right) + |\omega_z|^2\langle X_h, \omega\rangle \\ &= \langle X_h, \omega_{z\bar{z}}\rangle + \langle X_h, |\omega_z|^2\omega\rangle = 0\,.\end{aligned}$$

The last equality follows from (2.6) and to pass to the second line we used that

$$\langle \partial_z X_h, \omega\rangle = \frac{h}{|\omega_z|^2}\langle \omega_{\bar{z}}, \omega\rangle = 0\,,$$

since $|\omega| = 1$, and that, since X_h is a real vector field,

$$\langle \partial_{\bar{z}} X_h, \omega_z\rangle = \langle \partial_{\bar{z}}\overline{X_h}, \omega_z\rangle = \langle \overline{\partial_z X_h}, \omega_z\rangle = \frac{h}{|\omega_z|^2}\langle \omega_z, \omega_z\rangle = 0\,,$$

cf. (2.3). Thus from elliptic regularity theory we deduce that $\langle X_h, \omega\rangle$ can be extended to a function in $N(\omega)$ once we show that $\langle X_h, \omega\rangle \in L^\infty(\mathbb{S}^2 \setminus \{p_1, \dots, p_n\})$.

To prove the boundedness of $\langle X_h, \omega\rangle$, fix p_j for some $j = 1, \dots, n$, and choose a local conformal coordinate z in a neighborhood of p_j with $z(p_j) = 0$. Up to a rotation in $\mathbb{R}^3$, we can suppose that $\varphi(z) = z^{m_j}$ in this neighborhood, where $m_j > 1$, and so, from (2.19), we see that

$$\partial_z X_h = \frac{h}{m_j}\left(\frac{z^{1-m_j} - z^{1+m_j}}{2}, \frac{i(z^{1-m_j} + z^{1+m_j})}{2}, 2z\right). \tag{2.23}$$

In these coordinates we may write h as $h(z) = \frac{c_0}{z} + \sum_{\ell=0}^{\infty} c_\ell z^\ell$, for some $(c_\ell)_{\ell\in\mathbb{N}} \in \mathbb{C}$, so that (2.22) and (2.23) imply that in a neighbourhood of 0,

$$X_h(z) = \frac{1}{2m_j(1-m_j)}\Big(c_0\mathrm{Re}(z^{1-m_j}) + o(z^{1-m_j}), c_0\mathrm{Im}(z^{1-m_j}) + o(z^{1-m_j}), c_0(1-m_j)2\mathrm{Re}(z) + o(z)\Big),$$

where $\lim_{z\to 0}\frac{o(z)}{z} = 0$. By (2.2) and (1.7) the bubble ω can be expressed in these coordinates as

$$\omega(z) = \left(\frac{2z^{m_j}}{|z|^{2m_j}+1}, \frac{|z|^{2m_j}-1}{|z|^{2m_j}+1}\right),$$

hence the fact that $\langle X_h, \omega\rangle$ is bounded near 0 follows from the last two formulas. □

Proof of Theorem 1.4. The result follows by combining Corollary 2.4 with Theorem 2.5. □

Proof of Corollary 1.5. The main claim follows from Theorem 1.4, since if ω is degenerate then there must exist a corresponding non-zero polynomial R with $\deg R \le n-4$, where ω has n branch points, say $p_1, \dots, p_n$. Hence if ω is degenerate we must have $n \ge 4$. This also immediately implies (ii). To prove (i) note that if $m_j > 1$ is the multiplicity of ω at p_j (*i.e.* if $m_j - 1$ is the algebraic multiplicity of p_j as a zero of $|\nabla\omega|$), then by the Riemann–Hurwitz formula we have

$$2(k-1) = \sum_{j=1}^{n}(m_j - 1), \tag{2.24}$$

where k is the degree of ω. Thus ω can have at least 4 branch points only if $k \ge 3$. □

Proof of Corollary 1.6. Note that, by (2.24), a degree 3 bubble can only have at most 4 different branch points, so by Corollary 1.5 if ω is degenerate then indeed it must have exactly 4 different branch points $p_1, \dots, p_4$. By Theorem 1.4, we see that ω is degenerate if and only if

$$\mathrm{Res}_{p_j}\left(1/\varphi'\right) = 0 \quad \text{for } j = 1, 2, 3, 4,$$

where $\varphi = P/Q$ is as in (2.2). Elementary computations then yield the conclusion, see [19, page 171] for further details. □

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