On C^1 regularity for degenerate elliptic equations in the plane

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Abstract

We show that Lipschitz solutions u of $\operatorname{div} G(\nabla u) = 0$ in $B_1 \subset \mathbb{R}^2$ are C^1, for strictly monotone vector fields $G \in C^0(\mathbb{R}^2; \mathbb{R}^2)$ satisfying a mild ellipticity condition. If $G = \nabla F$ for a strictly convex function F, and $0 \leq \lambda(\xi) \leq \Lambda(\xi)$ are the two eigenvalues of $\nabla^2 F(\xi)$, our assumption is that the set $\{\lambda = 0\} \cap \{\Lambda = \infty\}$, where ellipticity degenerates *both* from below and from above, is finite. This extends results by De Silva and Savin (Duke Math. J. 151, No. 3, p.487-532, 2010), which assumed either that set empty, or the larger set $\{\lambda = 0\}$ finite. Our main new input is to transfer estimates in $\{\lambda > 0\}$ to estimates in $\{\Lambda < \infty\}$ by means of a conjugate equation. When G is not a gradient, the ellipticity assumption needs to be interpreted in a specific way, and we highlight the nontrivial effect of the antisymmetric part of ∇G.

1 Introduction

We consider solutions $u \colon B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ of the nonlinear equation

$$\operatorname{div} G(\nabla u) = 0 \qquad \text{in } B_1, \tag{1}$$

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where the vector field $G : \mathbb{R}^2 \to \mathbb{R}^2$ is strictly monotone, that is,

$$\langle G(\xi) - G(\zeta), \xi - \zeta\rangle > 0 \quad \forall \xi \neq \zeta \in \mathbb{R}^2.$$

Sufficient conditions on G ensuring local Lipschitz regularity of weak solutions can be found in [20] (see also the survey [33], the recent works [11, 8, 13, 10, 12], and counterexamples in [17, 5] when integrability properties of ∇u are not good enough). Here, as in [19, 9, 29, 35], we focus on conditions ensuring that Lipschitz solutions are C^1.

It is known since the works of Morrey and Nirenberg [37, 38] (see [22, Chapter 12]) that Lipschitz solutions of (1) are $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ (and smooth if G is smooth) if G is *uniformly elliptic*, that is,

$$\lambda \leq \nabla^s G \leq \Lambda \qquad \text{for some } 0 < \lambda < \Lambda < +\infty\,,$$

where $\nabla^s G = (\nabla G + \nabla G^T)/2$ denotes the symmetric gradient of G.

In this two-dimensional setting, the antisymmetric part of ∇G does not play any role in the uniform ellipticity condition, because the regularity can be inferred from a priori estimate for the equation $\mathrm{tr}(A\nabla^2 u) = 0$, with $A = \nabla^s G(\nabla u)$. In higher dimensions, the estimates of de Giorgi, Nash and Moser also provide $C^{1,\alpha}$ regularity for (1) under the uniform ellipticity assumption $\lambda \leq \nabla^s G \leq |\nabla G| \leq \Lambda$, which does require the antisymmetric part of ∇G to be bounded.

For a general strictly monotone G, ellipticity may degenerate from below: $\lambda = 0$; or above: $\Lambda = +\infty$. Specific types of degeneracy have been studied extensively, including the fundamental case of the p-Laplacian $G = \nabla|\cdot|^p$ for $1 < p < \infty$, but the general setting still raises several open questions, see [36] for a recent survey. Very degenerate equations, where the field G is not strictly monotone, have also attracted recent attention [39, 16, 15, 30].

1.1 The variational case $G = \nabla F$

One feature of the present work is to highlight the nontrivial effect of the antisymmetric part of ∇G, which played no role in the two-dimensional uniformly elliptic case.

We focus first on the case where this effect is absent, that is, ∇G is symmetric. Then we have $G = \nabla F$ for some strictly convex function F. Lipschitz solutions of

$$\mathrm{div}\, \nabla F(\nabla u) = 0 \text{ in } B_1\,, \tag{2}$$

are minimizers of the energy $\int F(\nabla u)\,dx$, hence the term *variational.*

Following [19], it is natural to separate regions where ellipticity degenerates: $\lambda = 0$, or becomes singular: $\Lambda = +\infty$. Accordingly we set, leaving regularity considerations and rigorous definitions aside for the moment,

$$\mathcal{D} = \left\{ \xi \in \mathbb{R}^2 \colon \text{`` } \nabla^2 F(\xi) \text{ has an eigenvalue equal to } 0 \text{ ''} \right\},$$
$$\mathcal{S} = \left\{ \xi \in \mathbb{R}^2 \colon \text{`` } \nabla^2 F(\xi) \text{ has an eigenvalue equal to } +\infty \text{ ''} \right\}.$$

In [19], de Silva and Savin show that M-Lipschitz solutions of (2), with F strictly convex, are C^1, provided one of the two following conditions is satisfied:

- $\overline{B}_M \cap \mathcal{D} \cap \mathcal{S}$ is empty: ellipticity is not lost simultaneously from below and above [19, Theorem 1.1];
- $\overline{B}_M \cap \mathcal{D}$ is finite: ellipticity is lost from below at most at a finite number of values [19, Theorem 1.2].

A remarkable feature of [19, Theorem 1.2] is that it requires no control from above on the ellipticity. It is natural to wonder about a counterpart involving only $\mathcal{S}$. As a consequence of our main result we obtain C^1 regularity under an even less restrictive condition, generalizing both [19, Theorem 1.1 & 1.2].

Theorem 1.1. *If $F \colon \mathbb{R}^2 \to \mathbb{R}$ is C^1 and strictly convex, and*

$$\overline{B}_M \cap \mathcal{D} \cap \mathcal{S} \text{ is finite,}$$

then any Lipschitz solution u of $\operatorname{div} \nabla F(\nabla u) = 0$ in B_1 with $|\nabla u| \leq M$ is C^1.

We describe next a family of examples where Theorem 1.1 applies, but the results of [19] do not. Consider $F(x, y) = f(x) + g(y)$ where $f, g \colon \mathbb{R} \to \mathbb{R}$ are C^1 strictly convex functions. Denote by $D_f, S_f \subset \mathbb{R}$ the closures of the sets where $f'' = 0$ or $+\infty$, and similarly for g. Since $D_f \cap S_f = D_g \cap S_g = \emptyset$, we have

$$\begin{aligned} \mathcal{D} &= (D_f \times \mathbb{R}) \cup (\mathbb{R} \times D_g), \\ \mathcal{S} &= (S_f \times \mathbb{R}) \cup (\mathbb{R} \times S_g), \\ \mathcal{D} \cap \mathcal{S} &= (D_f \times S_g) \cup (S_f \times D_g). \end{aligned}$$

Hence $\mathcal{D}$ is infinite as soon as D_f or D_g is non-empty, but the intersection $\mathcal{D} \cap \mathcal{S}$ is finite and non-empty as soon as $D_f \cup S_f$ and $D_g \cup S_g$ are finite and non-empty.

Note that, in the case of the p-Laplacian, ellipticity is degenerate only at the origin: bad values of the gradient are small, which facilitates regularity theory. Theorem 1.1 goes well beyond that case, as did already the results of [19]. However, as in [19], our approach is purely two-dimensional. A counter-example in dimension 4 is given in [35], where it is also conjectured that there should be a counter-example in dimension 3.

1.2 The general case

Consider a general C^0 strictly monotone vector field $G\colon \mathbb{R}^2 \to \mathbb{R}^2$. In the uniformly elliptic case, the condition satisfied by G constrains only the symmetric part $\nabla^s G$ of the gradient, hence a natural generalization of the sets $\mathcal{D}$ and $\mathcal{S}$ could be the sets where $\nabla^s G$ has a zero or infinite eigenvalue. This turns out to be wrong for $\mathcal{S}$, as we explain next.

Note that G is strictly monotone, and therefore injective, so we may consider its inverse G^{-1}. Then, the (loose) definitions

$$\mathcal{D}(G) = \left\{\xi \in \mathbb{R}^2 \colon \text{“ } \nabla^s G(\xi) \text{ has an eigenvalue equal to 0 ”}\right\},$$
$$\mathcal{S}(G) = \left\{\xi \in \mathbb{R}^2 \colon \text{“ } \nabla^s (G^{-1})(G(\xi)) \text{ has an eigenvalue equal to 0 ”}\right\},$$

ensure the validity of an exact analog of Theorem 1.1 in the general strictly monotone setting. Before stating it, we give the rigorous versions:

$$\begin{aligned}
\mathcal{D}(G) &= \bigcap_{\lambda>0} \operatorname{clos}\left\{\xi \in \mathbb{R}^2 : \liminf_{|\zeta|\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|\zeta|^2} \le \lambda\right\} \\
\mathcal{S}(G) &= \bigcap_{\Lambda>0} \operatorname{clos}\left\{\xi \in \mathbb{R}^2 : \liminf_{|\zeta|\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|G(\xi+\zeta) - G(\xi)|^2} \le \frac{1}{\Lambda}\right\}.
\end{aligned} \tag{3}$$

Here, $\operatorname{clos} A$ denotes the topological closure of $A \subset \mathbb{R}^2$.

Remark 1.2. These definitions make sense without any regularity assumption on G. The quotient appearing in the definition of $\mathcal{S}(G)$ can be rewritten as

$$\frac{\langle G^{-1}(G(\xi)+\eta) - G^{-1}(G(\xi)), \eta\rangle}{|\eta|^2}, \quad \text{with } \eta = G(\xi+\zeta) - G(\xi),$$

which explains the previous loose definition. If G is C^1, then $\mathcal{D}(G)$ is the set where ∇G has a zero eigenvalue, and similarly for $\mathcal{S}(G)$ if G^{-1} is C^1. In general, such a pointwise description fails. For instance, the inclusion

$$D(G) := \operatorname{clos}\left\{\xi \in \mathbb{R}^2 \colon \liminf_{\zeta\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|\zeta|^2} = 0\right\} \subset \mathcal{D}(G),$$

might be strict: consider $G(x,y) = (g(x), y)$ with $g(x) = \int_0^x f$, where $f(t) = |t| + |\sin(1/t)|$ for $t \neq 0$, then $D(G) = \emptyset$ but $\mathcal{D}(G) = \{0\}$.

With these definitions, our main result is as follows.

Theorem 1.3. *If $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is C^0 and strictly monotone, and*

$$\overline{B}_M \cap \mathcal{D}(G) \cap \mathcal{S}(G) \text{ is finite,}$$

then any Lipschitz solution u of $\operatorname{div} G(\nabla u) = 0$ in B_1 with $|\nabla u| \leq M$ is C^1.

Note that, in the uniformly elliptic case $0 < \lambda \leq \nabla^s G \leq \Lambda$ we have $\mathcal{D}(G) = \emptyset$, so in that case Theorem 1.3 also requires no condition on the antisymmetric part of ∇G. Next we give some further explanations about the role of that antisymmetric part and the set $\mathcal{S}(G)$.

1.3 The role of the antisymmetric part of ∇G

In view of the uniformly elliptic case, the set

$$\widetilde{\mathcal{S}}(G) = \left\{\xi \in \mathbb{R}^2 \colon \text{“ } \nabla^s G(\xi) \text{ has an eigenvalue equal to } +\infty \text{ ”}\right\},$$

or rather its rigorous version

$$\tilde{\mathcal{S}}(G) = \bigcap_{\Lambda>0} \operatorname{clos}\left\{\xi \in \mathbb{R}^2 : \limsup_{|\zeta|\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|\zeta|^2} \geq \Lambda\right\}, \tag{4}$$

could have been a natural candidate to replace $\mathcal{S}$ in a generalization of Theorem 1.1 to the nonvariational setting. We demonstrate instead that the antisymmetric part of ∇G actually plays an important role in this degenerate setting. Before doing so, we observe the following elementary property to help compare $\mathcal{S}$ and $\tilde{\mathcal{S}}$.

Proposition 1.4. *For any continuous strictly monotone $G\colon \mathbb{R}^2 \to \mathbb{R}^2$, we have the inclusion $\widetilde{\mathcal{S}}(G) \subset \mathcal{S}(G)$, with equality if $G = \nabla F$.*

This implies in particular that Theorem 1.3 reduces exactly to Theorem 1.1 when $G = \nabla F$.

Note that the inclusion $\widetilde{\mathcal{S}}(G) \subset \mathcal{S}(G)$ can be strict. For instance, the strictly monotone field $G_0(x, y) = (x^3 - y, x + y)$ satisfies $\mathcal{S}(G_0) = \mathbb{R} \times \{0\}$ and $\widetilde{\mathcal{S}}(G_0) = \emptyset$. This elementary example is however not so interesting from the standpoint of equation (1), because its antisymmetric contribution is linear, and therefore disappears in (1).

Note also that the set $\mathcal{S}(G)$ can be non-empty even when G is uniformly elliptic: for any non-zero antisymmetric matrix M of operator norm $\|M\| \leq 1/2$, the field $G_M(\xi) = \xi + \ln(|\xi|)M\xi$ satisfies $1/2 \leq \nabla^s G_M \leq 3/2$, but the antisymmetric part of ∇G_M blows up at the origin, and $\mathcal{S}(G_M) = \{0\}$.

We provide here a more involved example which shows that Theorem 1.3 is false with $\mathcal{S}(G)$ replaced by $\widetilde{\mathcal{S}}(G)$.

Theorem 1.5. *There exists a C^0 strictly monotone vector field $G\colon \mathbb{R}^2 \to \mathbb{R}^2$, with $\mathcal{D}(G) \cap \tilde{\mathcal{S}}(G)$ finite, and a Lipschitz solution of* (1) *which is not C^1.*

This example arises from connections with the Aviles-Giga energy and degenerate differential inclusions [31, 27, 28], which partially motivated the present work. The bad set $\mathcal{D} \cap \mathcal{S}$ is infinite, and the non-C^1 solution u satisfies $\nabla u \in \mathcal{D} \cap \mathcal{S}$ almost everywhere. In view of this and of Theorem 1.3, it seems natural to conjecture that, for quite general strictly monotone vector fields G and any Lipschitz solution u of (1), the function $x \mapsto \mathrm{dist}(\nabla u(x), \mathcal{D} \cap \mathcal{S})$ is continuous. Results in that spirit are proved in [39, 16, 15, 30] in different contexts.

In fact, our proof of Theorem 1.3 does imply this continuity, under the assumption that the complement of any small enough neighborhood of $\mathcal{D} \cap \mathcal{S}$ is connected (cf Remark 2.7). But in the example of Theorem 1.5 the complement of $\mathcal{D} \cap \mathcal{S}$ is not connected: additional arguments would be needed to prove the above conjecture.

1.4 Nonlinear Beltrami equations

In two dimensions, Minty's correspondence between monotone and 1-Lipschitz vector fields [34] induces a correspondence between equations of the form (1) and nonlinear autonomous Beltrami equations

$$f_{\bar{z}} = H(f_z), \tag{5}$$

where $f_z = (\partial_x f - i\partial_y f)/2$, $f_{\bar{z}} = (\partial_x f + i\partial_y f)/2$, and $H\colon \mathbb{C} \to \mathbb{C}$ is *strictly 1-Lipschitz*, that is,

$$|H(\xi) - H(\zeta)| < |\xi - \zeta| \qquad \forall \xi \neq \zeta \in \mathbb{C}\,.$$

This connection is described and exploited for instance in [26, Chapter 15], [7, Chapter 16], or [4]. In another instance of degenerate elliptic setting, Minty's correspondence also has applications to the regularity of Monge-Ampère equations in the plane [1, 18].

The nonlinear Beltrami equation (5) is uniformly elliptic if the function H is k-Lipschitz for some $k \in (0,1)$, and the 1-Lipschitz case allows for degenerate ellipticity. In that setting, we have the following transposition of Theorem 1.3.

Theorem 1.6. *Let $H : \mathbb{C} \to \mathbb{C}$ be strictly 1-Lipschitz and define*

$$\Gamma_\pm = \bigcap_{\lambda > 0} \operatorname{clos} \left\{ \xi \in \mathbb{C}\colon \liminf_{\zeta \to 0} \frac{1 - |L_H(\xi,\zeta)|^2}{|1 \pm L_H(\xi,\zeta)|^2} \leq \lambda \right\},$$

where $L_H(\xi,\zeta) = \dfrac{H(\xi+\zeta) - H(\xi)}{\bar{\zeta}}$.

If $\Gamma_+ \cap \Gamma_-$ is locally finite, then any Lipschitz solution $f\colon B_1 \to \mathbb{C}$ of (5) is C^1.

A complete version, with the explicit role of the Lipschitz constant M in Theorem 1.3, will be given in Theorem 4.3. Many other types of regularizing effects of nonlinear Beltrami equations have been studied, for instance in [6, 3, 2, 4, 32, 5].

A few comments about the bad set $\Gamma_+ \cap \Gamma_-$ are in order. As expected, it contains points $\xi \in \mathbb{C}$ at which the local Lipschitz constant

$$\mathrm{Lip}(H;\xi) = \limsup_{\zeta \to 0} |L_H(\xi,\zeta)|$$

is equal to 1. However it takes into account more precise information, not only about the modulus of finite differences of H, but also about their angle. For instance, if H is differentiable at a point ξ_0, with local Lipschitz constant equal to 1, but its differential is the conjugation operator $\zeta \mapsto \bar{\zeta}$ (or its opposite), then $\xi_0 \notin \Gamma_-$ (or $\xi_0 \notin \Gamma_+$). This angular effect, and the particular role played by the conjugation operator, was already observed in [23] for linear Beltrami equations with varying coefficient, and studied further in the context of quasilinear elliptic equations of the form $\mathrm{div}(A(z,u)\nabla u) = 0$, see the survey [24].

Remark 1.7. A basic interpretation of the distinguished role of the conjugation operator is provided by the following elementary fact about degenerate linear Beltrami equations

$$f_{\bar{z}} = L[f_z],$$

where $L\colon \mathbb{C} \to \mathbb{C}$ is a $\mathbb{R}$-linear operator of operator norm $\|L\| = 1$. If $\pm L$ is the conjugation operator, then $\mathfrak{Re}\, f$ or $\mathfrak{Im}\, f$ must be constant, and this is the only operator of norm 1 with that property (see Proposition B.1). In other words, at the linear level, the conjugation operator has a smoothing effect on one of the two components of f. The regularization of both components in Theorem 1.6 is due to further nonlinear effects, making use of the fact that H is strictly 1-Lipschitz.

1.5 Ideas of the proof of Theorem 1.3

The strategy combines the ideas of [19] with a duality argument which allows to symmetrize the roles of $\mathcal{D}$ and $\mathcal{S}$.

The main tool in the proof of [19, Theorem 1.2] is a localization lemma [19, Lemma 3.1] stating the following, for $\xi_0 \notin \mathcal{D}$ and $\rho > 0$ such that $B_{4\rho}(\xi_0) \cap \mathcal{D} = \emptyset$. If the image of ∇u stays outside $B_\rho(\xi_0)$, then, at a smaller scale, it must localize either completely inside $B_{4\rho}(\xi_0)$ or completely outside $B_{3\rho}(\xi_0)$. The first case already provides control on the oscillations of ∇u, and in the second case one can iterate this lemma: if the second case keeps occuring, one eventually concludes that ∇u localizes inside a small neighborhood of $\mathcal{D}$. When $\mathcal{D}$ is finite, this forces small oscillations.

The core idea in the proof of [19, Lemma 3.1] is Lebesgue's observation that even though H^1 functions in the plane fail to be continuous, continuity can be recovered if they satisfy a maximum and minimum principle (see e.g. [26, § 7.3-5]). This idea is applied to functions of $|\nabla u - \xi_0|$. This requires an H^1 bound, and a maximum/minimum principle for such functions. While the latter is somewhat general, the H^1 bound follows from a Cacciopoli-type inequality which heavily uses the fact that $\xi_0 \notin \mathcal{D}$, and this is why [19, Theorem 1.2] is constrained to $\mathcal{D}$.

If, instead, we assume only that $\xi_0 \notin \mathcal{S}$, there is no obvious reason why the H^1 bound should be valid. A simple, but key, observation is that there is a dual vector field G^* such that $\eta_0 = iG(\xi_0) \notin \mathcal{D}(G^*)$, and with the property that $iG(\nabla u) = \nabla v$ for a solution v of $\operatorname{div} G^*(\nabla v) = 0$. This duality is already presented and exploited in [7, § 16.4]. Since it exchanges the roles of $\mathcal{D}$ and $\mathcal{S}$, one can apply the argument of [19, Lemma 1.3] to localize the image of ∇v, and come back to ∇u thanks to the strict monotony of G. With this new localization lemma for $\xi_0 \notin \mathcal{S}$, one can then follow the strategy of [19, Theorem 1.2], as described above, with $\mathcal{D}$ replaced by $\mathcal{D} \cap \mathcal{S}$.

All these arguments are performed at the level of *a priori* estimates, that is, assuming that u is smooth, and we combine them with an approximation argument in order to conclude.

1.6 Plan of the paper

In Section 2, we prove the a priori estimates described above. In Section 3, we perform the approximation argument to prove Theorem 1.3. In Section 4, we transpose it to Beltrami equations, proving Theorem 1.6. In Section 5, we prove Proposition 1.4. In Section 6 we construct the example of Theorem 1.5. Several technical results are gathered in the Appendices.

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1.7 Notations

$\langle \cdot,\cdot\rangle$ & $\vert\cdot\vert$	the scalar product & associated euclidean norm on $\mathbb{R}^2$
$B_r(x)$	the ball centered at x with radius r
B_r	the ball centered at 0 with radius r
i	the counterclockwise rotation of angle $\pi/2$
$\operatorname{int} A$	the interior of $A \subset \mathbb{R}^2$
$\operatorname{clos} A$	the closure of $A \subset \mathbb{R}^2$
∂A	the boundary of $A \subset \mathbb{R}^2$
$\operatorname{diam} A$	the diameter of $A \subset \mathbb{R}^2$
$\mathbb{S}^1$	the unit circle $\{x \in \mathbb{R}^2 \colon \vert x\vert = 1\}$
$a \otimes b$	the matrix with entries $(a\otimes b)_{ij} = a_i b_j$
D^ζ	the finite difference operator $D^\zeta G(\xi) = G(\xi+\zeta) - G(\xi)$
$\nabla^s G$	the symmetric part $(\nabla G + \nabla G^T)/2$ of ∇G

2 A priori estimate

In this section we prove an explicit a priori estimate on the oscillation of ∇u, in terms of:

- the *modulus of monotony* $\omega_G \colon (0,\infty) \to (0,\infty)$, given by

$$\omega_G(t) = \inf_{|\xi-\zeta|>t} \langle G(\xi) - G(\zeta), \xi - \zeta\rangle,$$

- and the open sets $O_\lambda(G)$, $V_\Lambda(G)$ given, as in [19], by

$$O_\lambda(G) = \operatorname{int}\left\{\xi \in \mathbb{R}^2 \colon \liminf_{|\zeta|\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|\zeta|^2} \geq \lambda\right\}, \qquad (6)$$

$$V_\Lambda(G) = \operatorname{int}\left\{\xi \in \mathbb{R}^2 \colon \liminf_{|\zeta|\to 0} \frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|G(\xi+\zeta) - G(\xi)|^2} \geq \frac{1}{\Lambda}\right\}, \qquad (7)$$

  for $\lambda, \Lambda > 0$, where $\operatorname{int} A$ denotes the topological interior of $A \subset \mathbb{R}^2$.

The relevance of these sets with respect to Theorem 1.3 is that we have

$$\mathcal{D}(G) = \mathbb{R}^2 \setminus \bigcup_{\lambda>0} O_\lambda(G), \qquad \mathcal{S}(G) = \mathbb{R}^2 \setminus \bigcup_{\Lambda>0} V_\Lambda(G).$$

If G is C^1, they coincide with

$$O_\lambda(G) = \operatorname{int}\left\{\xi \in \mathbb{R}^2 \colon \langle \nabla^s G(\xi)\zeta, \zeta\rangle \geq \lambda|\zeta|^2,\ \forall \zeta \in \mathbb{R}^2\right\},$$
$$V_\Lambda(G) = \operatorname{int}\left\{\xi \in \mathbb{R}^2 \colon \Lambda\langle \nabla^s G(\xi)\zeta, \zeta\rangle \geq |\nabla G(\xi)\zeta|^2,\ \forall \zeta \in \mathbb{R}^2\right\}.$$

The main result of this section is the following quantitative version of Theorem 1.3 for a priori smooth solutions, and a *strongly monotone* vector field G, that is, there exists $C > 0$ such that

$$C\langle G(\xi) - G(\zeta), \xi - \zeta\rangle \geq |\xi - \zeta|^2 + |G(\xi) - G(\zeta)|^2, \tag{8}$$

for all $\xi, \zeta \in \mathbb{R}^2$. (This is equivalent to G being both uniformly elliptic and globally Lipschitz.)

Proposition 2.1. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ smooth and strongly monotone. Let $r > 0$ and assume that there exist $\lambda, \Lambda, M > 0$ and $\xi_1, \ldots, \xi_N \in \mathbb{R}^2$ with $|\xi_i - \xi_j| \geq 4r$ for $i \neq j$, and such that*

$$\overline{B}_{2M} \subset V_\Lambda(G) \cup O_\lambda(G) \cup \bigcup_{j=1}^N B_r(\xi_j).$$

Then any smooth solution u of $\operatorname{div}(G(\nabla u)) = 0$ in B_1 with $|\nabla u| \leq M$ satisfies

$$\operatorname{diam}(\nabla u(B_\delta)) \leq r,$$

where $\delta > 0$ depends on

- *a Lebesgue number $\eta \in (0, r)$ of the above open covering: any ball $B_\eta(\xi)$ with $|\xi| \leq 2M$ must be contained in $V_\Lambda(G)$, $O_\lambda(G)$ or $B_r(\xi_j)$ for some $j \in \{1, \ldots N\}$;*
- *the gradient bound M and the ellipticity constants λ, Λ;*
- *the integrals $\int_{B_1} |\nabla u|^2\, dx$ and $\int_{B_1} |G(\nabla u)|^2\, dx$;*
- *the modulus of monotony ω_G via any $c > 0$ such that $\omega_G(t)/t \geq c$ for all $t \in [\eta/4, M + \eta]$.*

As explained in the introduction, the proof of this a priori estimate follows the strategy in [19, Theorem 1.2] and relies on two localization lemmas:

- one dealing with values of ∇u away from $\mathcal{D}$, that is, in O_λ, already proved in [19, Lemma 3.1];
- a counterpart dealing with values of ∇u away from $\mathcal{S}$, that is, in V_Λ, which is our main new contribution.

We start by proving this new localization lemma before proceeding to the proof of Proposition 2.1.

2.1 The localization lemma in V_Λ

This subsection is devoted to the proof of the following, where $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is still assumed smooth and strongly monotone.

Lemma 2.2. *Let u a solution of $\operatorname{div}(G(\nabla u)) = 0$ in B_1 and assume that*

$$\nabla u(B_1) \cap B_\rho(\xi_0) = \emptyset \quad \textit{and} \quad B_{4\rho}(\xi_0) \subset V_\Lambda(G)\,,$$

for some $\Lambda, \rho > 0$ and $\xi_0 \in \mathbb{R}^2$. Then we have

$$\textit{either } \nabla u(B_\delta) \subset B_{4\rho}(\xi_0), \quad \textit{or } \nabla u(B_\delta) \cap B_{3\rho}(\xi_0) = \emptyset\,,$$

for some $\delta > 0$ depending on Λ, ρ, $\|\nabla u\|_{L^2(B_1)}$, and any $c > 0$ such that $\omega_G(t)/t \geq c$ for $t \in [\rho, \|\nabla u\|_\infty + 3\rho]$.

As described in the introduction, the proof of Lemma 2.2 relies on a well-known duality [7, § 16.4]. We recall here the basic properties that we will use. Note that G is invertible, as follows e.g. from the Minty-Browder theorem [14, Theorem 9.14-1], its inverse G^{-1} is smooth thanks to the inverse function theorem, and the strong monotony (8) of G implies that G^{-1} is strongly monotone. At the core of the proof of Lemma 2.2 are the two following elementary but crucial properties.

Lemma 2.3. *The dual vector field $G^*(\xi) = iG^{-1}(-i\xi)$ is strongly monotone and satisfies:*

1. *For any smooth solution u of $\operatorname{div} G(\nabla u) = 0$ in B_1, there exists a smooth function v such that $G(\nabla u) = -i\nabla v$ and*

   $$\operatorname{div} G^*(\nabla v) = 0 \quad \textit{in } B_1\,.$$

2. *For any* $\Lambda > 0$ *and* $\xi \in \mathbb{R}^2$ *we have*

$$\xi \in V_\Lambda(G) \quad \Longleftrightarrow \quad iG(\xi) \in O_{1/\Lambda}(G^*).$$

Proof. The strong monotony of G^{-1} implies that of G^*. For the first property, the existence of v such that $G(\nabla u) = -i\nabla v$ follows from Poincaré's lemma and the fact that $iG(\nabla u)$ is curl-free. Then we see that

$$G^*(\nabla v) = G^*(iG(\nabla u)) = iG^{-1}(G(\nabla u)) = i\nabla u,$$

is divergence-free. The second property follows from the identity

$$\frac{\langle G(\xi+\zeta) - G(\xi), \zeta\rangle}{|G(\xi+\zeta) - G(\xi)|^2} = \frac{\langle G^*(\eta+\sigma) - G^*(\eta), \sigma\rangle}{|\sigma|^2},$$

for any $\zeta \in \mathbb{R}^2 \setminus \{0\}$ and $\eta = iG(\xi)$, $\sigma = iG(\xi+\zeta) - iG(\xi)$, and the fact that $\zeta = G^{-1}(G(\xi) - i\sigma) - \xi \to 0$ if and only if $\sigma \to 0$. $\square$

The duality provided by Lemma 2.3 enables us to prove an H^1 estimate for $G(\nabla u)$ in the preimage $(\nabla u)^{-1}(V_\Lambda)$, by reducing it to the following estimate in the preimage of O_λ.

Lemma 2.4. *Let* u *a smooth solution of* $\operatorname{div}(G(\nabla u)) = 0$ *in* B_1. *Then we have*

$$\int_{B_{1/2}\cap(\nabla u)^{-1}(O_\lambda)} \left|\nabla^2 u\right|^2 dx \le C\,, \tag{9}$$

where $C = \frac{c_0}{\lambda^2}\|G(\nabla u)\|^2_{L^2(B_1)}$ *for a universal constant* $c_0 > 0$.

Proof of Lemma 2.4. This is proved in [19, Proposition 3.3] in the variational case $G = \nabla F$. We provide here the minor changes needed to deal with a general strongly monotone G.

Multiplying the equation $\operatorname{div}(\nabla G(\nabla u)\nabla u_k) = 0$ (where the subscript k denotes differentiation with respect to x_k) by $\xi^2 G^k(\nabla u)$ with ξ a smooth cut-off function satisfying $\mathbf{1}_{B_{1/2}} \le \xi \le \mathbf{1}_{B_1}$, and performing the same manipulations as in [19, Proposition 3.3], using in particular that the matrix $\nabla G(\nabla u)\nabla^2 u$ is trace-free, we obtain

$$\begin{aligned} 2\int_{B_1} \xi^2 \det(\nabla G(\nabla u))|\det(\nabla^2 u)|\,dx &= 2\int_{B_1} G^i(\nabla u)G^k(\nabla u)\partial_k(\xi_i\xi)\,dx \\ &\le c_0 \int_{B_1} |G(\nabla u)|^2\,dx. \end{aligned}$$

For any $A \in \mathbb{R}^{2\times 2}$ and A^s its symmetric part, we have

$$\det(A^s) = a_{11}a_{22} - \left(\frac{a_{12}+a_{21}}{2}\right)^2 \le a_{11}a_{22} - a_{12}a_{21} = \det(A),$$

so we infer

$$2\int_{B_1} \xi^2 \det(\nabla^s G(\nabla u))|\det(\nabla^2 u)|\,dx \le c_0 \int_{B_1} |G(\nabla u)|^2\,dx. \tag{10}$$

Then, using that the matrix

$$A = (\nabla^s G(\nabla u))^{\frac{1}{2}}\,\nabla^2 u\,(\nabla^s G(\nabla u))^{\frac{1}{2}}\,,$$

is symmetric and trace-free we obtain, arguing as in [19, Proposition 3.3], the inequality

$$\det(\nabla^s G(\nabla u))\left|\det \nabla^2 u\right| \ge \frac{\lambda^2}{2}\left|\nabla^2 u\right|^2 \mathbf{1}_{(\nabla u)^{-1}(O_\lambda(G))},$$

which, plugged back into (10), concludes the proof. □

We combine the estimate of Lemma 2.4 and the duality of Lemma 2.3 to obtain an estimate in the preimage of $(\nabla u)^{-1}(V_\Lambda)$.

Lemma 2.5. *Let u a smooth solution of* $\operatorname{div}(G(\nabla u)) = 0$ *in B_1. Then we have*

$$\int_{B_{1/2}\cap(\nabla u)^{-1}(V_\Lambda(G))} \left|\nabla\left(G(\nabla u)\right)\right|^2 dx \le C$$

Where $C = c_0\Lambda^2\|\nabla u\|^2_{L^2(B_1)}$ for a universal constant $c_0 > 0$.

Proof. By Lemma 2.3 we have $iG(\nabla u) = \nabla v$, where v satisfies

$$\operatorname{div}(G^*(\nabla v)) = 0 \quad \text{in } B_1, \quad (\nabla v)^{-1}\left(O_{\frac{1}{\Lambda}}(G^*)\right) = (\nabla u)^{-1}\left(V_\Lambda(G)\right)\,,$$

hence Lemma 2.5 follows from Lemma 2.4 applied to v and G^*. □

We are ready to prove Lemma 2.2:

Proof of Lemma 2.2. As remarked in [30, Proposition 3.6], we have the maximum/minimum principle

$$\partial(\nabla u(B_r)) \subset \nabla u(\partial B_r), \tag{11}$$

which follows from [25, Theorem II] since $\det(\nabla^2 u)$ does not change sign (as a consequence of $\nabla^s G(\nabla u)\nabla^2 u$ being trace-free).

We denote $M = \|\nabla u\|_\infty$ and prove Lemma 2.2 by contradiction: assume that $\delta \in (0, 1/2]$ is such that $\nabla u(B_\delta)$ intersects both $B_{3\rho}(\xi_0)$ and $\overline{B}_M \setminus B_{4\rho}(\xi_0)$.

Since $\nabla u(B_1) \cap B_\rho(\xi_0) = \emptyset$, this implies that, for any $r \in (\delta, 1)$ the boundary of $\nabla u(B_r)$ intersects both $B_{3\rho}(\xi_0)$ and $\overline{B}_M \setminus B_{4\rho}(\xi_0)$. Indeed, $\nabla u(B_r)$ contains a point ζ in $B_{3\rho}(\xi_0)$ and does not contain ξ_0, so on the segment $[\xi_0, \zeta] \subset B_{3\rho}(\xi_0)$ there must be a point belonging to the boundary of $\nabla u(B_r)$. Similarly, $\nabla u(B_r)$ contains a point ζ in $\overline{B}_M \setminus B_{4\rho}(\xi_0)$, and, since $\nabla u(B_r) \subset \overline{B}_M$, on the half line $\{\zeta + t(\zeta - \xi_0)\}_{t \geq 0} \subset \mathbb{R}^2 \setminus B_{4\rho}(\xi_0)$ there must be a point belonging to the boundary of $\nabla u(B_r)$ (and then automatically also to $\overline{B}_M$).

Thanks to the maximum/minimum principle (11) we deduce that $\nabla u(\partial B_r)$ intersects both $B_{3\rho}(\xi_0)$ and $\overline{B}_M \setminus B_{4\rho}(\xi_0)$. Define the sets

$$\overline{\Sigma} = G\left(\overline{B}_M \setminus B_{4\rho}(\xi_0)\right) \quad \underline{\Sigma} = G\left(B_{3\rho}(\xi_0)\right),$$

and note that, by definition of the modulus of monotony ω_G, we have

$$\operatorname{dist}(\overline{\Sigma}, \underline{\Sigma}) \geq \eta := \inf_{\rho \leq t \leq M + 3\rho} \frac{\omega_G(t)}{t} > 0.$$

Let $\mathcal{R} \in C^1(\mathbb{R}^2)$ such that $|\nabla \mathcal{R}| \leq \frac{4}{\eta} \mathbf{1}_{\mathbb{R}^2 \setminus (\overline{\Sigma} \cup \underline{\Sigma})}$ and

$$\mathcal{R}(\xi) = \begin{cases} 0 & \text{if } \xi \in \overline{\Sigma}, \\ 1 & \text{if } \xi \in \underline{\Sigma}. \end{cases}$$

Recall that $\nabla u(\partial B_r)$ intersects both $B_{3\rho}(\xi_0)$ and $\overline{B}_M \setminus B_{4\rho}(\xi_0)$. This implies that $\mathcal{R}(G(\nabla u)))$ takes both the value 0 and the value 1 on ∂B_r, and therefore, by the mean value theorem,

$$1 \leq \int_{\partial B_r} |\nabla (\mathcal{R}(G(\nabla u)))| \, ds \leq \sqrt{2\pi r} \left(\int_{\partial B_r} |\nabla (\mathcal{R}(G(\nabla u)))|^2 \, ds \right)^{\frac{1}{2}}$$

Dividing by $\sqrt{r}$, squaring and and integrating it on $[\delta, 1/2]$ we find

$$\log\left(\frac{1}{2\delta}\right) \leq 2\pi \int_{B_{1/2}} |\nabla\left(\mathcal{R}(G(\nabla u))\right)|^2 \, dx$$
$$\leq \frac{32\pi}{\eta^2} \int_{B_{1/2} \cap (\nabla u)^{-1}(B_{4\rho}(\xi_0))} |\nabla(G(\nabla u))|^2 \, dx.$$

The last inequality follows from the chain rule, the fact that $|\nabla \mathcal{R}| = 0$ on $\overline{\Sigma} = G(\overline{B}_M \setminus B_{4\rho}(\xi_0))$, and the inequality $|\nabla \mathcal{R}| \leq 4/\eta$. Recalling that $B_{4\rho}(\xi_0) \subset V_\Lambda(G)$, we can use Lemma 2.5 to deduce

$$\log\left(\frac{1}{2\delta}\right) \leq \frac{32\pi}{\eta^2} \int_{B_{1/2} \cap (\nabla u)^{-1}(V_\Lambda(G))} |\nabla(G(\nabla u))|^2 \, dx \leq \frac{32\pi C}{\eta^2}.$$

For $\delta < \exp(-32\pi C/\eta^2)/2$ this is impossible, and the conclusion of Lemma 2.2 is therefore verified. □

2.2 Proof of the a priori estimate

Before proving Proposition 2.1, we recall the localization lemma [19, Lemma 3.1] near O_λ.

Lemma 2.6. *Let u a solution of $\operatorname{div} G(\nabla u) = 0$ in B_1 and assume that*

$$\nabla u(B_1) \cap B_\rho(\xi_0) = \emptyset \quad \textit{and} \quad B_{4\rho}(\xi_0) \subset O_\lambda(G)\,,$$

for some $\lambda, \rho > 0$ and $\xi_0 \in \mathbb{R}^2$. Then we have

$$\textit{either } \nabla u(B_\delta) \subset B_{4\rho}(\xi_0), \quad \textit{or } \nabla u(B_\delta) \cap B_{3\rho}(\xi_0) = \emptyset\,,$$

for some $\delta > 0$ depending on λ, ρ, and $\|G(\nabla u)\|_{L^2(B_1)}$.

In [19] this is proved in the variational setting $G = \nabla F$, but the only step that needs minor adaptation is the H^1 estimate [19, Proposition 3.3], which we have adapted here in Lemma 2.4. Then the proof of Lemma 2.6 is completed using arguments similar to Lemma 2.2, based on the maximum/minimum principle (11) and the estimate of oscillations on the circles ∂B_r.

Proof of Proposition 2.1. Let u be a smooth solution of $\operatorname{div} G(\nabla u) = 0$ in B_1 with $|\nabla u| \le M$. By assumption we have

$$\overline{B}_{2M} \subset V_\Lambda(G) \cup O_\lambda(G) \cup \bigcup_{j=1}^{N} B_r(\xi_j)\,,$$

and we fix a Lebesgue number $\eta \in (0, r)$ of this open covering, with the property that any ball $B_\eta(\xi)$ centered at $\xi \in \overline{B}_{2M}$ is contained in $V_\Lambda(G)$, $O_\lambda(G)$, or $B_r(\xi_j)$ for some $j \in \{1, \dots, N\}$. We set $\rho = \eta/4$. The ball $\overline{B}_{2M}$ can be covered by a finite number of balls of radius ρ. Removing the balls that are contained in one of the $B_r(\xi_j)$, we are left with a covering

$$\overline{B}_{2M} \setminus \bigcup_{j=1}^{N} B_r(\xi_j) \subset \bigcup_{k=1}^{K} B_\rho^k,$$

with $K \le cM^2/\eta^2$ for some universal constant $c > 0$, and the property that each ball $B_{4\rho}^k$ satisfies

$$B_{4\rho}^k \subset V_\Lambda(G) \quad \text{or} \quad B_{4\rho}^k \subset O_\lambda(G).$$

Since $\nabla u(B_1) \subset \overline{B_M}$, there exists a ball $B_{4\rho}^k \subset B_{2M} \setminus \overline{B_M}$ such that :

$$\nabla u(B_1) \cap B_\rho^k = \emptyset$$

Since $B_{4\rho}^k \subset V_\Lambda(G)$ or $B_{4\rho}^k \subset O_\lambda(G)$, we can apply Lemma 2.2 or Lemma 2.6 to ensure the existence of some $\delta > 0$ such that

$$\text{either } \nabla u(B_\delta) \subset B_{4\rho}^k \quad \text{or } \nabla u(B_\delta) \cap B_{3\rho}^k = \emptyset.$$

If the first case occurs, then we are done since $4\rho = \eta < r$. If the second case occurs, we infer that $\nabla u(B_\delta) \cap B_\rho^j = \emptyset$ for all neighboring balls B_ρ^j such that $B_\rho^j \cap B_\rho^k \neq \emptyset$. Then we can apply again Lemma 2.2 or Lemma 2.2 to the rescaled function $\delta^{-1} u(\delta \cdot)$ and these neighboring balls B_ρ^j.

We iterate this argument: if at some step we reach the first case, we are done. Otherwise, since $\overline{B}_{2M} \setminus \bigcup_{j=1}^N B_r(\xi_j)$ is connected, we eventually cover it with the neighboring balls added at each step, and deduce that $\nabla u(B_{\delta'}) \subset \bigcup_{j=1}^N B_r(\xi_j)$. Here $\delta' = \delta^K$ for δ as in Lemma 2.2 and Lemma 2.6. By connectedness, $\nabla u(B_{\delta'})$ is contained in one of the balls $B_r(\xi_j)$, and this concludes the proof. $\square$

Remark 2.7. If, in the assumptions of Proposition 2.1, the union of the balls $B_r(\xi_j)$ is replaced by any open subset $U \subset \mathbb{R}^2$ such that $\overline{B}_{2M} \setminus U$ is connected, then the same proof shows, for any $r > 0$, the existence of $\delta > 0$ such that either $\operatorname{diam}(\nabla u(B_\delta)) < r$ or $\nabla u(B_\delta) \subset U$.

3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 using the a priori estimate of Proposition 2.1 and an approximation argument. This is quite standard, see e.g. [16, 30], but some details seem needed to make sure it applies in our situation. For the reader's convenience we recall here the statement of Theorem 1.3 .

Theorem 3.1. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ a continuous strictly monotone vector field such that $\mathcal{S}(G) \cap \mathcal{D}(G) \cap \overline{B_M}$ is a finite set. Then any M-Lipschitz solution u of $\operatorname{div} G(\nabla u) = 0$ in B_1 is C^1.*

Proof. Let u and G as in the Theorem. Modifying G ouside $\overline{B}_M$ does not change the equation satisfied by u. Thanks to Lemma A.1 we can therefore assume that

$$\mathcal{D}(G) \cap \mathcal{S}(G) \subset \overline{B}_M,$$

and apply Lemma A.2, to obtain smooth strongly monotone vector fields G_ϵ such that $\omega_{G_\epsilon} \geq \omega_G$ and

$$\begin{aligned} &B_{2\epsilon}(\xi) \subset O_\lambda(G) \;\Rightarrow\; \xi \in O_\lambda(G_\epsilon)\,, \\ &B_{2\epsilon}(\xi) \subset V_\Lambda(G) \;\Rightarrow\; \xi \in V_{\Lambda+\epsilon}(G_\epsilon)\,. \end{aligned} \tag{12}$$

Thanks to Lemma A.4, the smooth solutions u_ϵ of

$$\operatorname{div} G_\epsilon(\nabla u_\epsilon) = 0 \text{ in } B_1, \quad u_\epsilon = u \text{ on } \partial B_1,$$

satisfy $|\nabla u_\epsilon| \leq \widetilde{M}$ in $B_{1/2}$ for some $\widetilde{M} \geq M$, and converge to u in $H^1(B_1)$.

Rescaling $B_{1/2}$ to B_1, we assume that $|\nabla u_\epsilon| \leq \widetilde{M}$ in B_1 and apply Proposition 2.1 to u_ϵ. For any given $r > 0$, we will check that the radius $\delta > 0$ obtained that way, such that $\operatorname{diam}(\nabla u_\epsilon(B_\delta)) < r$, does not depend on ϵ. Passing to the limit will then prove continuity of ∇u at 0 (and, translating and rescaling, at any point $x \in B_1$).

Denote

$$\mathcal{D}(G) \cap \mathcal{S}(G) = \{\xi_1, \ldots, \xi_N\} \subset \overline{B}_{\widetilde{M}}.$$

Let $r > 0$. Since $\mathcal{D}$ is the complement of $\bigcup_{\lambda>0} O_\lambda$ and $\mathcal{S}$ the complement of $\bigcup_{\Lambda>0} V_\Lambda$, we can find $\lambda, \Lambda > 0$ such that

$$\overline{B}_{2\widetilde{M}} \subset O_\lambda(G) \cup V_\Lambda(G) \cup \bigcup_{j=1}^{N} B_r(\xi_j).$$

We may also fix a Lebesgue number $\eta \in (0, r/2)$ such that any ball $B_{4\eta}(\xi)$ with $|\xi| \leq 2\widetilde{M}$ must be contained in $O_\lambda(G)$ or $V_\Lambda(G)$ or one of the balls $B_r(\xi_j)$ for some $j = 1, \ldots, N$. Thanks to the properties (12) of G_ϵ, for any $0 < \epsilon < \min(\eta, \Lambda)$ we have

$$\overline{B}_{2\widetilde{M}} \subset O_\lambda(G_\epsilon) \cup V_{2\Lambda}(G_\epsilon) \cup \bigcup_{j=1}^{N} B_r(\xi_j),$$

and η has the Lebesgue number property that any ball $B_\eta(\xi)$ with $|\xi| \leq 2\widetilde{M}$ must be contained in $O_\lambda(G_\epsilon)$ or $V_\Lambda(G_\epsilon)$ or one of the balls $B_r(\xi_j)$ for some $j = 1, \ldots, N$. Moreover, if $c > 0$ is such that $\omega_G(t)/t \geq c$ for all $t \in [\eta/4, \widetilde{M} + \eta]$, then we have $\omega_{G_\epsilon}(t)/t \geq \omega_G(t)/t \geq c$ for all $t \in [\eta/4, \widetilde{M} + \eta]$. Applying Proposition 2.1 we obtain therefore a radius $\delta > 0$, independent of ϵ, such that $\operatorname{diam}(\nabla u_\epsilon(B_\delta)) \leq r$. □

4 Nonlinear Beltrami equations

In this section we describe how to transform Theorem 1.3 about degenerate elliptic equations $\operatorname{div} G(\nabla u) = 0$ into Theorem 1.6 about degenerate Beltrami equations $f_{\bar{z}} = H(f_z)$. This relies on Minty's correspondence [34] and is described thoroughly in [7, § 16]. For the readers' convenience, we recall here and sketch the proof of the basic features that we are going to use.

Proposition 4.1. *Let $H \colon \mathbb{C} \to \mathbb{C}$ a strictly 1-Lipschitz function, that is,*

$$|H(\xi) - H(\zeta)| < |\xi - \zeta| \quad \forall \xi \neq \zeta \in \mathbb{C}.$$

Then:

1. *One may modify H outside any arbitrary compact in order to ensure*

$$\lim_{|z|\to\infty}\left(|z| \pm \langle \overline{H}(z), \frac{z}{|z|}\rangle\right) = +\infty\,, \tag{13}$$

which we assume from now on.

2. *The maps $F, F_*\colon \mathbb{C}\to\mathbb{C}$ given by*

$$F(z) = \frac{H(z)+\bar{z}}{2}, \quad F_*(z) = \frac{H(z)-\bar{z}}{2i}, \tag{14}$$

are homeomorphisms.

3. *The maps G, G^* given by*

$$G = -iF_*\circ F^{-1}, \qquad G^* = iF\circ F_*^{-1}, \tag{15}$$

are continuous, strictly monotone vector fields, and for any complex function $f\colon B_1\to\mathbb{C}$, we have the implication

$$f_{\bar{z}} = H(f_z) \quad \Longrightarrow \quad \operatorname{div} G(\frac{1}{2}\nabla u) = \operatorname{div} G^*(\frac{1}{2}\nabla v) = 0\ , \tag{16}$$

where $u = \mathfrak{Re}\, f$ and $v = \mathfrak{Im}\, f$.

4. *Under this correspondence, the sets $\mathcal{D}, \mathcal{S}$ (3) are transformed as*

$$\begin{aligned}&\mathcal{D}(G) = F(\Gamma_+), \quad \mathcal{S}(G) = F(\Gamma_-),\\ &\mathcal{D}(G^*) = F_*(\Gamma_-), \quad \mathcal{S}(G^*) = F_*(\Gamma_+).\end{aligned} \tag{17}$$

where $\Gamma_\pm$ are as in Theorem 1.6.

Proof of Proposition 4.1. 1. For any $R>0$ one may pick a smooth function $\chi\colon [0,\infty)\to[0,1]$ such that $\chi\equiv 1$ on $[0,R]$, $-1\le r\chi'(r)\le 0$ for all $r\ge 0$, and $\chi(r)\to 0$ as $r\to+\infty$. Then the map $\Phi\colon\mathbb{C}\to\mathbb{C}$ given by $\Phi(z)=\chi(|z|)z$ equals the identity in B_R, is 1-Lipschitz because its differential at $z=re^{i\theta}$ is symmetric with eigenvalues $\chi(r)$ and $\chi(r)+r\chi'(r)$, and $\Phi(z)\to 0$ as $|z|\to+\infty$. Thus $\widetilde{H} = H\circ\Phi$ equals H in B_R, is strictly 1-Lipschitz, and $H(z)\to H(0)$ as $|z|\to+\infty$, which implies (13).

2. To check that F, F_* are homeomorphisms, one can remark that $\overline{F}$ and $i\overline{F}_*$ are strictly monotone and continuous, and thanks to (13) they are coercive:

$$\lim_{|z|\to+\infty} \frac{\langle \overline{F}(z), z\rangle}{|z|} = \lim_{|z|\to+\infty} \frac{\langle i\overline{F}_*(z), z\rangle}{|z|} = +\infty\,.$$

Hence the Minty-Browder theorem [14, Theorem 9.14-1] ensures that they are invertible. Continuity of their inverses is also a consequence of the coercivity: if a sequence (z_k) is such that $F(z_k) \to \xi$, then coercivity forces (z_k) to be bounded, and by continuity of F any converging subsequence must converge to $F^{-1}(\xi)$.

3. Note that G, G^* are dual to each other in the sense of Lemma 2.3, that is, $-iG^*(iG(\xi)) = iG(-iG^*(\xi)) = \xi$ for all $\xi \in \mathbb{C}$.

Continuity of G, G^* follows from the previous item. For any $\xi \in \mathbb{C}$ and $\zeta \neq 0$, letting $\eta = F(\xi)$ and $\sigma = F(\xi+\zeta) - F(\xi)$, we have

$$\langle G(\eta+\sigma) - G(\eta), \sigma\rangle = |\zeta|^2 - |H(\xi+\zeta) - H(\xi)|^2 > 0,$$

so G is strictly monotone, and similarly for G^*. The implication (16) follows by rewriting $f_{\bar z} = H(f_z)$, as

$$2u_{\bar z} = H(f_z) + \overline{f_z} \text{ and } 2iv_{\bar z} = H(f_z) - \overline{f_z},$$

that is,

$$u_{\bar z} = F(f_z) \text{ and } v_{\bar z} = F_*(f_z),$$

or equivalently

$$G(u_{\bar z}) = -iv_{\bar z} \text{ and } G^*(v_{\bar z}) = iu_{\bar z},$$

which are divergence free. More details can be found e.g. in [4, Theorem 5].

4. For any $\xi \in \mathbb{C}$ and $\zeta \neq 0$, letting $\eta = F(\xi)$ and $\sigma = F(\xi+\zeta) - F(\xi)$, we have the identities

$$\frac{\langle G(\eta+\sigma) - G(\eta), \sigma\rangle}{|\sigma|^2} = \frac{1 - |L_H(\xi,\zeta)|^2}{|1 + L_H(\xi,\zeta)|^2}\,,$$
$$\frac{\langle G(\eta+\sigma) - G(\eta), \sigma\rangle}{|G(\eta+\sigma) - G(\eta)|^2} = \frac{1 - |L_H(\xi,\zeta)|^2}{|1 - L_H(\xi,\zeta)|^2}\,,$$

where

$$L_H(\xi,\zeta) = \frac{H(\xi+\zeta)-H(\xi)}{\bar{\zeta}}.$$

Recall moreover that F is a homeomorphism and $\sigma = F(\xi+\zeta)-F(\xi) \to 0$ if and only if $\zeta = F^{-1}(F(\xi)+\sigma)-\xi \to 0$. Therefore, these identities and the definitions (3) of $\mathcal{D},\mathcal{S}$ imply that

$$\mathcal{D}(G) = F(\Gamma_+), \quad \mathcal{S}(G) = F(\Gamma_-), \tag{18}$$

with

$$\Gamma_\pm = \bigcap_{\lambda>0} \operatorname{clos}\left\{\xi\in\mathbb{C}\colon \liminf_{|\zeta|\to 0} \frac{1-|L_H(\xi,\zeta)|^2}{|1\pm L_H(\xi,\zeta)|^2} \le \lambda\right\},$$

as in Theorem 1.6. Similar calculations (or the duality of Lemma 2.3) give $\mathcal{D}(G^*) = F_*(\Gamma_-)$ and $\mathcal{S}(G^*) = F_*(\Gamma_+)$. □

Remark 4.2. Reciprocally, if $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and strictly monotone, the Minty-Browder theorem ensures that $\psi\colon \mathbb{R}^2\to\mathbb{R}^2$ given by

$$\psi(\xi) = \frac{\xi+G(\xi)}{2},$$

is a homeomorphism, so is its pointwise conjugate $\phi = \overline{\psi}$, and then the map $H\colon \mathbb{C}\to\mathbb{C}$ given

$$H(\phi(\xi)) = \frac{\xi - G(\xi)}{2},$$

is strictly 1-Lipschitz. If it satisfies (13), then G can be recovered as in Proposition 4.1.

Thanks to (16) and (17), it becomes apparent that Theorem 1.6 is a consequence of Theorem 1.3. In fact, keeping track of the role of M, we obtain the following more precise version of Theorem 1.6.

Theorem 4.3. *Let $H\colon \mathbb{C}\to\mathbb{C}$ strictly 1-Lipschitz, and $M>0$ such that $F(\Gamma_+)\cap F(\Gamma_-)\cap \overline{B}_M(\xi_0/2)$ and $F_*(\Gamma_+)\cap F_*(\Gamma_-)\cap\overline{B}_M(\xi_0/2i)$ are finite, where $\xi_0 = H(0)$. Then any Lipschitz solution f of $f_{\bar z} = H(f_z)$ in B_1 with $|f_z|\le M$ is C^1.*

Proof. Replacing H by $H-\xi_0$ and f by $f-\xi_0\bar{z}$ we assume without loss of generality that $H(0)=0$. Then the property $|f_z|\leq M$ implies $|f_{\bar{z}}|\leq M$. Writing $f=u+iv$ we deduce $|u_{\bar{z}}|\leq M$ and $|v_{\bar{z}}|\leq M$, that is, $|\nabla u|\leq 2M$ and $|\nabla v|\leq 2M$.

Thanks to the implication (16), and since $F(\Gamma_+)\cap F(\Gamma_-)=\mathcal{D}(G)\cap\mathcal{S}(G)$, we can apply Theorem 1.3 to $u/2$ and the vector field G, so that u is C^1. Similarly we obtain that v is C^1 and conclude that f is C^1. □

Remark 4.4. It can be instructive to contemplate the correspondence (17) in the case of the p-Laplacian. For $G(\xi)=|\xi|^{p-2}\xi$, we have on the one hand $\mathcal{D}(G)=\{0\}$, $\mathcal{S}(G)=\emptyset$ if $p>2$ and $\mathcal{D}(G)=\emptyset$, $\mathcal{S}(G)=\{0\}$ for $p<2$. On the other hand, with the bijection $\phi(\xi)=\overline{(\xi+G(\xi))}/2$ as in Remark 4.2, we have

$$\frac{H(\phi(z))-H(\phi(0))}{\overline{\phi(z)-\phi(0)}}=\frac{1-|z|^{p-2}}{1+|z|^{p-2}}\in\mathbb{R}$$

As $z\to 0$, this quantity goes to ± 1 depending on the value of p. In this case, H acts like $\pm$ the conjugation around the origin, and $L_H\in\mathbb{R}$. Depending on the value of p we can check that $(\Gamma_+,\Gamma_-)=(\emptyset,\{0\})$ or $(\{0\},\emptyset)$, in accordance with (17).

5 Proof of Proposition 1.4

In this section we consider $G\colon\mathbb{R}^2\to\mathbb{R}^2$ continuous strictly monotone, and we prove the basic property stated in Proposition 1.4 :

$$\widetilde{\mathcal{S}}(G)\subset\mathcal{S}(G)\,, \tag{19}$$

with equality if $G=\nabla F$.

The inclusion (19) is a consequence of Cauchy-Schwarz' inequality : for all $\xi\in\mathbb{R}^2$ and $\zeta\neq 0$ we have

$$\frac{\langle G(\xi+\zeta)-G(\xi),\zeta\rangle}{|G(\xi+\zeta)-G(\xi)|^2}\leq\frac{|\zeta|^2}{\langle G(\xi+\zeta)-G(\xi),\zeta\rangle}\,,$$

and the conclusion follows by taking the liminf as $\zeta\to 0$ and recalling the definitions (3) and (4) of $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

Next, we assume that $G=\nabla F$, fix $\xi_0\in\mathbb{R}^2\setminus\widetilde{\mathcal{S}}(\nabla F)$, and prove that $\xi_0\notin\mathcal{S}(\nabla F)$, which implies equality in (19).

By definition (4) of $\widetilde{\mathcal{S}}$, there exist $\Lambda, r > 0$ such that

$$\limsup_{\zeta\to 0} \frac{\langle \nabla F(\xi+\zeta) - \nabla F(\xi), \zeta\rangle}{|\zeta|^2} \le \Lambda \quad \forall \xi \in B_{3r}(\xi_0).$$

Fix $\xi \in B_{2r}(\xi_0)$ and $\zeta \in B_r$, then the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) = \langle \nabla F(\xi + t\zeta), \zeta\rangle \quad \forall t \in [0,1],$$

is monotone nondecreasing, and the above property of F ensures that

$$\lim_{s\to 0^+} \frac{f(t+s) - f(t)}{s} \le \Lambda|\zeta|^2, \quad \forall t \in [0,1].$$

This implies that f is absolutely continuous and $0 \le f' \le \Lambda|\zeta|^2$. We infer that $f(1) - f(0) \le \Lambda|\zeta|^2$, that is,

$$\frac{\langle \nabla F(\xi+\zeta) - \nabla F(\xi), \zeta\rangle}{|\zeta|^2} \le \Lambda, \qquad \forall \xi \in B_{2r}(\xi_0),\ \forall \zeta \in B_r.$$

Consider the mollified function $F_\epsilon(\xi) = \int F(\xi + \epsilon z)\rho(z)\,dz$, for some smooth nonnegative kernel $\rho \in C_c^\infty(B_1)$. From the last inequality, we infer, for $0 < \epsilon < r/2$,

$$\frac{\langle \nabla F_\epsilon(\xi+\zeta) - \nabla F_\epsilon(\xi), \zeta\rangle}{|\zeta|^2} \le \Lambda, \qquad \forall \xi \in B_r(\xi_0),\ \forall \zeta \in B_r.$$

Letting $\zeta \to 0$, this implies

$$0 \le \nabla^2 F_\epsilon(\xi) \le \Lambda \qquad \forall \xi \in B_r(\xi_0).$$

Since $\nabla^2 F_\epsilon(\xi)$ is symmetric nonnegative, we infer

$$|\nabla^2 F_\epsilon(\xi)|^2 \le \Lambda \langle \nabla^2 F_\epsilon(\xi)\zeta, \zeta\rangle \qquad \forall \xi \in B_r(\xi_0),\ \forall \zeta \in \mathbb{R}^2.$$

Using also Jensen's inequality, this implies, for all $\xi \in B_{r/2}(\xi_0)$ and $\zeta \in B_{r/2}$,

$$\begin{aligned}
|\nabla F_\epsilon(\xi+\zeta) - \nabla F_\epsilon(\xi)|^2 &= \left| \int_0^1 \nabla^2 F_\epsilon(\xi + t\zeta)\zeta\,dt \right|^2 \\
&\le \int_0^1 |\nabla^2 F_\epsilon(\xi+t\zeta)\zeta|^2\,dt \\
&\le \Lambda \int_0^1 \langle \nabla^2 F_\epsilon(\xi + t\zeta)\zeta, \zeta\rangle\,dt \\
&= \Lambda \langle \nabla F_\epsilon(\xi+\zeta) - \nabla F_\epsilon(\xi), \zeta\rangle.
\end{aligned}$$

Letting $\epsilon \to 0$ we deduce

$$|\nabla F(\xi+\zeta) - \nabla F(\xi)|^2 \le \Lambda \langle \nabla F(\xi+\zeta) - \nabla F(\xi), \zeta \rangle,$$

for all $\xi \in B_{r/2}(\xi_0)$ and $\zeta \in B_{r/2}$. This shows that $\xi_0 \in V_\Lambda(\nabla F)$ and concludes the proof that $\mathbb{R}^2 \setminus \widetilde{\mathcal{S}}(\nabla F) \subset \mathbb{R}^2 \setminus \mathcal{S}(\nabla F)$. $\square$

6 Example

Proof of Theorem 1.5. We start from the observation that the Lipschitz function $f\colon B_1 \to \mathbb{C}$ given by

$$f(re^{i\theta}) = \frac{2}{3} r i e^{2i\theta},$$

is not C^1 at the origin and satisfies

$$f_z = ie^{i\theta}, \quad f_{\bar z} = -\frac{1}{3} i e^{3i\theta} = \frac{1}{3}(f_z)^3.$$

We claim that there exists a smooth function $H\colon \mathbb{C} \to \mathbb{C}$ with compact support and such that

$$H(z) = \frac{z^3}{3} \quad \forall z \in \mathbb{S}^1, \qquad \text{and } \|\nabla H(z)\| < 1 \quad \forall z \in \mathbb{C} \setminus \mathbb{S}^1. \tag{20}$$

Here $\|\cdot\|$ denotes the operator norm. Since $\mathbb{S}^1$ contains no segment, this implies that H is strictly 1-Lipschitz. And since $|f_z| = 1$, the function f is a solution of $f_{\bar z} = H(f_z)$.

The construction of H can be achieved e.g. by setting $H(re^{i\theta}) = g(r)e^{3i\theta}/3$ with $g\colon (0,\infty) \to \mathbb{R}$ smooth, compactly supported and satisfying

$$g(1) = 1, \quad \text{and } |g(r)| < r, \ |g'(r)| < 3, \quad \forall r \neq 1.$$

Note that this forces $g'(1) = 1$, since the quotient $(r - g(r))/(r-1)$ is positive for $r > 1$, negative for $r < 1$, and tends to $1 - g'(1)$ as $r \to 1$.

The differential of H is given by

$$\begin{aligned}\nabla H(re^{i\theta}) &= \partial_r H \otimes e^{i\theta} + \frac{1}{r}\partial_\theta H \otimes ie^{i\theta} \\ &= \frac{g'(r)}{3} e^{3i\theta} \otimes e^{i\theta} + \frac{g(r)}{r} ie^{3i\theta} \otimes ie^{i\theta},\end{aligned}$$

which implies

$$\|\nabla H(re^{i\theta})\| = \max\left(\frac{|g'(r)|}{3}, \frac{|g(r)|}{r}\right) < 1, \quad \forall r \neq 1,$$

and ensures that (20) is satisfied.

Since H is compactly supported, Proposition 4.1 provides a continuous strictly monotone vector field G such that $u = \frac{1}{2}\mathfrak{Re}\, f$ solves $\operatorname{div} G(\nabla u) = 0$ in B_1. This function $u(re^{i\theta}) = -(r/3)\sin(2\theta)$ is Lipschitz but not C^1.

The sets $\Gamma_\pm$ associated to H are easily calculated. We have $\Gamma_\pm \subset \mathbb{S}^1$, since outside $\mathbb{S}^1$ the function H is smooth with $\|\nabla H\| < 1$, and noting that

$$\lim_{t\to 0} L_H(e^{i\theta}, e^{i\theta}(e^{it}-1)) = -e^{4i\theta},$$

we see that $e^{i\theta} \in \Gamma_\pm$ for all $\theta \notin \frac{\pi}{4}\mathbb{Z}$. As this sets are closed we infer $\Gamma_+ = \Gamma_- = \mathbb{S}^1$. This implies that $\mathcal{D}(G) = \mathcal{S}(G) = F(\mathbb{S}^1)$, with $F(\xi) = (H(\xi)+\bar{\xi})/2$ as in Proposition 4.1.

Finally we show that

$$\widetilde{\mathcal{S}}(G) = F(\{\pm 1, \pm i\}). \tag{21}$$

First note that $\widetilde{\mathcal{S}}(G) \subset \mathcal{S}(G) = F(\mathbb{S}^1)$, so it suffices to consider the behavior of G around points $F(e^{i\theta})$. We have

$$\begin{aligned} 2\nabla F(e^{i\theta}) &= e^{-i\theta} \otimes e^{i\theta} - ie^{-i\theta} \otimes ie^{i\theta} + \nabla H(e^{i\theta}) \\ &= \left(e^{-i\theta} + \frac{1}{3}e^{3i\theta}\right) \otimes e^{i\theta} + \left(ie^{3i\theta} - ie^{-i\theta}\right) \otimes ie^{i\theta}, \end{aligned}$$

hence the matrix of $\nabla F(e^{i\theta})$ in the orthonormal basis $(e^{i\theta}, ie^{i\theta})$ is given by

$$[\nabla F(e^{i\theta})] = \begin{pmatrix} \frac{2}{3}\cos(2\theta) & -\sin(2\theta) \\ -\frac{1}{3}\sin(2\theta) & 0 \end{pmatrix}.$$

In particular we see that $\det \nabla F(e^{i\theta}) = -(1/3)\sin^2(2\theta)$. If $\theta \notin \frac{\pi}{2}\mathbb{Z}$, the inverse function theorem ensures that F is a local C^1 diffeomorphism in a neighborhood of $e^{i\theta}$, and so $G = -F_* \circ F^{-1}$ is C^1 in a neighborhood of $F(e^{i\theta})$. This implies already that $\widetilde{\mathcal{S}}(G) \subset F(\{\pm 1, \pm i\})$. To prove (21) we calculate $DG(F(e^{i\theta}))$ for $\theta \notin \frac{\pi}{2}\mathbb{Z}$.

Differentating the identity $G(F(z)) = -iF_*(z) = (\bar{z} - H(z))/2$, we obtain

$$\begin{aligned} 2\nabla G(F(e^{i\theta}))\nabla F(e^{i\theta}) &= e^{-i\theta} \otimes e^{i\theta} - ie^{-i\theta} \otimes ie^{i\theta} - \nabla H(e^{i\theta}) \\ &= \left(e^{-i\theta} - \frac{1}{3}e^{3i\theta}\right) \otimes e^{i\theta} - \left(ie^{3i\theta} + ie^{-i\theta}\right) \otimes ie^{i\theta}, \end{aligned}$$

or, in the orthonormal basis $(e^{i\theta}, ie^{i\theta})$,

$$[\nabla G(F(e^{i\theta}))\nabla F(e^{i\theta})] = \begin{pmatrix} \frac{1}{3}\cos(2\theta) & 0 \\ -\frac{2}{3}\sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Using the above expression of $\nabla F(e^{i\theta})$ we deduce

$$[\nabla G(F(e^{i\theta}))] = \frac{1}{\sin^2(2\theta)} \begin{pmatrix} 0 & -\cos(2\theta)\sin(2\theta) \\ \cos(2\theta)\sin(2\theta) & 2 \end{pmatrix},$$

so that the symmetric part is given by

$$\nabla^s G(F(e^{i\theta})) = \frac{2}{\sin^2(2\theta)} ie^{i\theta} \otimes ie^{i\theta}, \qquad \forall \theta \notin \frac{\pi}{2}\mathbb{Z},$$

and this implies that $F(\pm 1), F(\pm i) \in \widetilde{\mathcal{S}}(G)$, since otherwise $\nabla^s G$ would be bounded near these points. Indeed, if $\xi_0 \notin \widetilde{\mathcal{S}}(G)$ then there exist $\Lambda, r > 0$ such that $\limsup_{|\zeta| \to 0} |\zeta|^{-2} \langle D^\zeta G(\xi), \zeta \rangle \leq \Lambda$ for all $\xi \in B_r(\xi_0)$, and if G is differentiable in $B_\rho(\xi_0) \setminus \{\xi_0\}$ for some $0 < \rho \leq r$, this implies $\nabla^s G \leq \Lambda$ in $B_\rho(\xi_0)$. □

Appendix A Modification and approximation lemmas

In this appendix we prove various technical results needed for the approximation argument in § 3. First we show how to modify G at infinity so that we can assume the set $\mathcal{D}(G) \cap \mathcal{S}(G)$ finite in the whole plane, along with some other technical conditions.

Lemma A.1. *Let $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ a continuous strictly monotone vector field, and $M > 0$. Then there exists $\widetilde{G} \colon \mathbb{R}^2 \to \mathbb{R}^2$ a continuous strictly monotone vector field equal to G in $\overline{B}_M$ and smooth outside B_{4M}, such that*

$$\mathcal{D}(\widetilde{G}) \cap \mathcal{S}(\widetilde{G}) \subset \mathcal{D}(G) \cap \mathcal{S}(G) \cap \overline{B_M},$$

and

$$
\begin{aligned}
&c \le \nabla^s \widetilde{G}(\xi) \le |\nabla \widetilde{G}(\xi)| \le 4c \quad \forall \xi \in \mathbb{R}^2 \setminus B_{4M},\\
&|\widetilde{G}(\xi)| \le L(1+|\xi|) \quad \forall \xi \in \mathbb{R}^2,
\end{aligned}
$$

for some constants $L, c > 0$ depending on M and $\|G\|_{L^\infty(B_{4M})}$.

Proof. Fix a smooth cut-off function η such that

$$
\mathbf{1}_{B_{2M}} \le \eta \le \mathbf{1}_{B_{4M}} \text{ and } |\nabla \eta| \le \frac{1}{M}\mathbf{1}_{B_{4M}\setminus B_{2M}},
$$

and a convex function $F(x) = c\mathbf{1}_{|x|\ge M}(|x|-M)^2$, with $c > 0$ to be chosen later on. Note that

$$
2c\frac{|x|-M}{|x|}\mathbf{1}_{|x|\ge M} \le \nabla^2 F(x) \le 2c\mathbf{1}_{|x|\ge M},
$$

in the sense of distributions. Define

$$
\widetilde{G} = \eta G + \nabla F.
$$

The function $\widetilde{G}$ is continuous and equal to G in $\overline{B}_M$. Outside B_{4M}, it is smooth equal to ∇F and $c \le \nabla^s \widetilde{G} \le |\nabla \widetilde{G}| \le 4c$. And for all $\xi \in \mathbb{R}^2$ we have

$$
|\widetilde{G}(\xi)| \le \|G\|_{L^\infty(B_{4M})} + 2c(|\xi| + M) \le L(1+|\xi|),
$$

with $L = 2c + 2cM + \|G\|_{L^\infty(B_{4M})}$. It remains to check that $\widetilde{G}$ is strictly monotone and that $\mathcal{D}(\widetilde{G}) \cap \mathcal{S}(\widetilde{G}) \subset \mathcal{D}(G) \cap \mathcal{S}(G) \cap \overline{B_M}$.

The distributional symmetric gradient of $\widetilde{G}$ is given by

$$
\nabla^s \widetilde{G} = \eta \nabla^s G + \nabla^2 F + \nabla \eta \odot G,
$$

where $a \odot b = (a \otimes b)^s$ is the matrix with entries $(a_i b_j + a_j b_i)/2$. From the properties of η and F we have

$$
\begin{aligned}
\frac{1}{2}\nabla^2 F + \nabla \eta \odot G &\ge c\frac{|x|-M}{|x|}\mathbf{1}_{|x|\ge M} - \frac{\|G\|_{L^\infty(B_{4M})}}{M}\mathbf{1}_{2M\le|x|\le 4M}\\
&\ge \left(\frac{c}{2} - \frac{\|G\|_{L^\infty(B_{4M})}}{M}\right)\mathbf{1}_{|x|\ge 2M} \ge 0,
\end{aligned}
$$

provided we chose $c = 2\|G\|_{L^\infty(B_{4M})}/M$. Then we deduce

$$\nabla^s \widetilde{G} \geq \eta \nabla^s G + \frac{1}{2}\nabla^2 F \geq \frac{1}{2}\nabla^2 F. \tag{22}$$

In particular, the distributional symmetric gradient $\nabla^s \widetilde{G}$ is nonnegative, so $\widetilde{G}$ is monotone.

If $\widetilde{G}$ is not strictly monotone then there is a nontrivial segment $[\xi, \xi+\zeta]$ along which $\langle \zeta, \widetilde{G}\rangle$ is constant. This segment must intersect either B_M or $\mathbb{R}^2 \setminus B_M$. If $[\xi, \xi+\zeta]$ intersects B_M, then this is impossible because $\widetilde{G} = G$ in B_M and G is strictly monotone. If $[\xi, \xi+\zeta]$ intersects $\mathbb{R}^2 \setminus B_M$, then this is also impossible because there we have $\nabla^s \widetilde{G} \geq \frac{1}{2}\nabla^2 F > 0$. We infer that $\widetilde{G}$ is strictly monotone.

From the inequalities (22) we have $\mathcal{D}(\widetilde{G}) \subset \mathcal{D}(G) \cap \overline{B}_M$. To conclude, it suffices to show that $\mathcal{S}(\widetilde{G}) \cap \overline{B}_M \subset \mathcal{S}(G) \cup \mathcal{D}(G)$, which will imply that $\mathcal{D}(\widetilde{G}) \cap \mathcal{S}(\widetilde{G}) \subset \overline{B}_M \cap \mathcal{D}(G) \cap \mathcal{S}(G)$.

If $\xi \in \overline{B}_M \setminus \mathcal{S}(G) \cup \mathcal{D}(G)$, there exist $\Lambda, \lambda > 0$ such that $\xi \in O_{4\lambda}(G) \cap V_{\Lambda/4}(G)$. This implies the existence of a small $r \in (0, M)$ such that, for all $\zeta \in B_r$,

$$\begin{aligned} &\langle G(\xi+\zeta) - G(\xi), \zeta\rangle \geq 2\lambda|\zeta|^2, \\ \text{and}\quad &\langle G(\xi+\zeta) - G(\xi), \zeta\rangle \geq \frac{2}{\Lambda}|G(\xi+\zeta) - G(\xi)|^2. \end{aligned}$$

Setting $\alpha = \min(\lambda, 1/\Lambda) > 0$, we deduce

$$\langle G(\xi+\zeta) - G(\xi), \zeta\rangle \geq \alpha|\zeta|^2 + \alpha|G(\xi+\zeta) - G(\xi)|^2, \quad \forall \zeta \in B_r.$$

Since $\widetilde{G} = G + \nabla F$ in B_{2M}, we infer, for any $\beta \in (0, \alpha/2)$,

$$\begin{aligned} &\langle \widetilde{G}(\xi+\zeta) - \widetilde{G}(\xi), \zeta\rangle \\ &\geq \alpha|\zeta|^2 + \alpha|G(\xi+\zeta) - G(\xi)|^2 \\ &\geq \alpha|\zeta|^2 + \beta|\widetilde{G}(\xi+\zeta) - \widetilde{G}(\xi)|^2 - 2\beta|\nabla F(\xi+\zeta) - \nabla F(\xi)|^2 \\ &\geq (\alpha - 4c\beta)|\zeta|^2 + \beta|\widetilde{G}(\xi+\zeta) - \widetilde{G}(\xi)|^2. \end{aligned}$$

In the last inequality we have used that ∇F is $2c$-Lispchitz. Choosing $\beta \leq \alpha/4c$, we deduce that $\xi \notin \mathcal{S}(\widetilde{G})$. This shows that $\mathcal{S}(\widetilde{G}) \cap \overline{B}_M \subset \mathcal{S}(G) \cup \mathcal{D}(G)$ and concludes the proof. □

Next, we establish that G can be approximated by smooth strongly monotone vector fields, with control on the modulus of monotony and on the sets O_λ, V_Λ.

Lemma A.2. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ a continuous strictly monotone vector field. Assume that there exist $M, L \geq 1$, $c > 0$ such that G is smooth in $\mathbb{R}^2 \setminus B_{4M}$ and*

$$\begin{aligned}
&c \leq \nabla^s G(\xi) \leq |\nabla G(\xi)| \leq 4c \quad \forall \xi \in \mathbb{R}^2 \setminus B_{4M},\\
&|G(\xi)| \leq L(1+|\xi|) \quad \forall \xi \in \mathbb{R}^2.
\end{aligned}$$

Then there exists a sequence G_ϵ of smooth and strongly monotone (8) *vector fields such that $G_\epsilon \to G$ locally uniformly as $\epsilon \to 0$, and*

$$\begin{aligned}
&\nabla^s G_\epsilon(\xi) \geq c \qquad \forall \xi \in \mathbb{R}^2 \setminus B_{5M},\\
&|G_\epsilon(\xi)| \leq 2L(1+|\xi|) \qquad \forall \xi \in \mathbb{R}^2\,,\\
&\omega_{G_\epsilon} \geq \omega_G,\\
&B_{2\epsilon}(\xi) \subset O_\lambda(G) \;\Rightarrow\; \xi \in O_\lambda(G_\epsilon)\,,\\
&B_{2\epsilon}(\xi) \subset V_\Lambda(G) \;\Rightarrow\; \xi \in V_{\Lambda+\epsilon}(G_\epsilon)\,,
\end{aligned}$$

for all $\epsilon \in (0,1)$.

Proof. We fix a smooth kernel $\rho \in C_c^\infty(B_1)$, such that $\rho \geq 0$ and $\int_{B_1} \rho = 1$, define $\rho_\epsilon(\xi) = \epsilon^{-2}\rho(\xi/\epsilon)$, and

$$G_\epsilon(\xi) = G * \rho_\epsilon(\xi) + \epsilon\,\xi.$$

Then G_ϵ is smooth and converges locally uniformly to G.

It is globally Lipschitz because $|\nabla G_\epsilon| \leq \epsilon^{-1}\|G\|_{L^\infty(B_{6M})}$ on B_{5M} and $|\nabla G_\epsilon| \leq 4c$ outside B_{5M}. Global Lipschitzness combined with the inequality

$$\nabla^s G_\epsilon = \nabla^s G * \rho_\epsilon + \epsilon\, I \geq \epsilon,$$

implies that G_ϵ is strongly monotone.

Outside B_{5M} we have $\nabla^s G_\epsilon \geq \nabla^s G * \rho_\epsilon \geq c$. And for all $\xi \in \mathbb{R}^2$ we have $|G_\epsilon(\xi)| \leq L(1+|\xi|+\epsilon) + \epsilon\,|\xi| \leq 2L(1+|\xi|)$.

In the rest of the proof we use the notation D^ζ for the finite difference operator

$$D^\zeta G(\xi) = G(\xi+\zeta) - G(\xi).$$

For any $\xi, \zeta \in \mathbb{R}^2$ we have

$$\langle D^\zeta G_\epsilon(\xi), \zeta\rangle = \int_{B_1} \langle D^\zeta G(\xi + \epsilon\eta), \zeta\rangle \rho(\eta) d\eta + \epsilon\, |\zeta|^2 \geq \omega_G(|\zeta|),$$

so that $\omega_{G_\epsilon} \geq \omega_G$.

Let $\lambda > 0$ and assume that $B_{2\epsilon}(\xi_0) \subset O_\lambda(G)$. Let $\xi \in B_\epsilon(\xi_0)$, so that $B_\epsilon(\xi) \subset O_\lambda(G)$. By definition (6) of O_λ, for all $\eta \in B_1$ there exists $\varphi(\eta, r)$ such that $0 < \varphi \leq \lambda$, $\varphi(\eta, r) \to 0$ as $r \to 0$ and

$$\langle D^\zeta G(\xi + \epsilon\eta), \zeta\rangle \geq (\lambda - \varphi(\eta, |\zeta|))|\zeta|^2.$$

Then we have

$$\begin{aligned} \langle D^\zeta G_\epsilon(\xi), \zeta\rangle &= \int_{B_1} \langle D^\zeta G(\xi + \epsilon\eta), \zeta\rangle \rho(\eta) d\eta + \epsilon\, |\zeta|^2 \\ &\geq (\lambda - \psi(|\zeta|))\, |\zeta|^2, \qquad \psi(r) = \int_{B_1} \varphi(\eta, r)\, \rho(\eta)\, d\eta, \end{aligned}$$

and $\psi(r) \to 0$ as $r \to 0$ by dominated convergence, so

$$\liminf_{|\zeta| \to 0} \frac{\langle D^\zeta G_\epsilon(\xi), \zeta\rangle}{|\zeta|^2} \geq \lambda \qquad \forall \xi \in B_\epsilon(\xi_0),$$

and we deduce that $\xi_0 \in O_\lambda(G_\epsilon)$.

Now let $\Lambda > 0$ and assume that $B_{2\epsilon}(\xi_0) \subset V_\Lambda(G)$. In order to show that $\xi_0 \in V_{\Lambda+\epsilon}(G_\epsilon)$ we argue slightly differently than for O_λ.

First we observe that G is Lipschitz in $B_{2\epsilon}(\xi_0)$ (see Lemma A.3 below), hence differentiable almost everywhere. Then the inclusion $B_{2\epsilon}(\xi_0) \subset V_\Lambda(G)$ implies

$$|\nabla G(\xi)\zeta|^2 \leq \Lambda \langle \nabla G(\xi)\zeta, \zeta\rangle \qquad \text{for a.e. } \xi \in B_{2\epsilon}(\xi_0) \text{ and all } \zeta \in \mathbb{R}^2.$$

Thus we find, for $\xi \in B_\epsilon(\xi_0)$ and $\zeta \in \mathbb{R}^2$,

$$\begin{aligned} |\nabla G_\epsilon(\xi)\zeta|^2 &= \left| \int_{B_1} \nabla G(\xi + \epsilon\eta)\zeta\, \rho(\eta)\, d\eta + \epsilon\zeta \right|^2 \\ &\leq \left(1 + \frac{\epsilon}{\Lambda}\right) \int_{B_1} |\nabla G(\xi + \epsilon\eta)\zeta|^2 \rho(\eta)\, d\eta + (\epsilon + \Lambda)\epsilon|\zeta|^2 \\ &\leq (\Lambda + \epsilon) \int_{B_1} \langle \nabla G(\xi + \epsilon\eta)\zeta, \zeta\rangle\, \rho(\eta) d\eta + (\epsilon + \Lambda)\epsilon|\zeta|^2 \\ &= (\Lambda + \epsilon)\langle \nabla G_\epsilon(\xi)\zeta, \zeta\rangle\, . \end{aligned}$$

This implies that $\xi_0 \in V_{\Lambda+\epsilon}(G_\epsilon)$. □

In the proof of Lemma A.2 we used the following elementary property of the set V_Λ.

Lemma A.3. *If $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous strictly monotone vector field and $B_r(\xi_0) \subset V_\Lambda(G)$ for some $\xi \in \mathbb{R}^2$ and $\Lambda, r > 0$, then G is Λ-Lipschitz in $B_r(\xi_0)$.*

Proof of Lemma A.3. By definition (7) of V_Λ, for any fixed $\xi \in B_r(\xi_0)$ and $\delta > 0$, if $|\zeta|$ is small enough we have

$$|D^\zeta G(\xi)|^2 \leq (1+\delta)\Lambda\langle D^\zeta G(\xi), \zeta\rangle \leq \frac{1}{2}|D^\zeta G(\xi)|^2 + \frac{1}{2}(1+\delta)^2\Lambda^2|\zeta|^2,$$

so that, letting $|\zeta| \to 0$ and then $\delta \to 0$ we deduce

$$\limsup_{\zeta \to 0} \frac{|D^\zeta G(\xi)|}{|\zeta|} \leq \Lambda \quad \forall \xi \in B_r(\xi_0)\,.$$

This infinitesimal Lipschitz property implies that G is Λ-Lipschitz in the convex set $B_r(\xi_0)$. Indeed, for $[\xi, \xi+\zeta] \subset B_r(\xi_0)$ and $\delta > 0$, by compactness and infinitesimal Lipschitzness we can find $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $|G(\xi + t_{j+1}\zeta) - G(\xi + t_j\zeta)| \leq (1+\delta)\Lambda(t_{j+1} - t_j)|\zeta|$, and concatenating these inequalities gives $|D^\zeta G(\xi)| \leq (1+\delta)\Lambda|\zeta|$. □

Finally, we check that solutions of the equation given by the smooth approximating vector fields G_ϵ are locally uniformly Lipschitz, thanks to the results of [20].

Lemma A.4. *Let $G, G_\epsilon\colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in Lemma A.2, and u a solution of $\operatorname{div} G(\nabla u) = 0$ in B_1 with $|\nabla u| \leq M$. For $\epsilon \in (0,1)$, let u_ϵ be the unique smooth solution of the boudary value problem*

$$\operatorname{div} G_\epsilon(\nabla u_\epsilon) = 0 \quad \textit{in } B_1, \qquad u_\varepsilon = u \quad \textit{in } \partial B_1.$$

Then we have

$$\sup_{\epsilon \in (0,1)} \|\nabla u_\epsilon\|_{L^\infty(K)} < \infty \qquad \textit{for all compact } K \subset B_1,$$

and $u_\epsilon \to u$ locally uniformly in B_1, and strongly in $H^1(B_1)$.

Proof of Lemma A.4. The existence of a unique solution $u_\epsilon \in H^1(B_1)$ follows from the strict monotony of G_ϵ and its behavior at infinity, see e.g. [21, § 9.1]. Moreover, this solution u_ϵ is Lipschitz thanks to [20, Theorem 4.1] (applied with $g(t) = t^2$), and therefore smooth since G_ϵ is smooth. On each compact $K \subset B_1$, the Lipschitz constant of u_ϵ provided by [20, Theorem 4.1] depends on the constants c, L, M such that

$$\nabla^s G_\epsilon(\xi) \geq c \quad \forall \xi \in \mathbb{R}^2 \setminus B_{5M}, \quad |G_\epsilon(\xi)| \leq 2L(1+|\xi|) \quad \forall \xi \in \mathbb{R}^2, \quad (23)$$

and on the L^2 norm of ∇u_ϵ in B_1. Therefore, the locally uniform Lipschitz bound will follow from

$$\sup_{\epsilon\in(0,1)} \int_{B_1} |\nabla u_\epsilon|^2 \, dx < \infty.$$

From (23) we infer the existence of $D > 0$ depending on M and L such that

$$\langle G_\epsilon(\xi) - G_\epsilon(\zeta), \xi - \zeta \rangle \geq c|\xi - \zeta|^2 - D \qquad \forall \xi, \zeta \in \mathbb{R}^2.$$

Testing the equation $\operatorname{div}(G_\epsilon(\nabla u_\epsilon) - G_\epsilon(\nabla u)) = \operatorname{div}(G(\nabla u) - G_\epsilon(\nabla u))$ against $u_\epsilon - u$, we infer

$$\begin{aligned} c\int_{B_1} |\nabla u_\epsilon - \nabla u|^2 \, dx &\leq D + \int_{B_1} \|G_\epsilon - G\|_{L^\infty(B_M)} |\nabla u - \nabla u_\epsilon| \, dx \\ &\leq D + \frac{\pi M^2}{2c} \|G_\epsilon - G\|^2_{L^\infty(B_M)} + \frac{c}{2}\int_{B_1} |\nabla u_\epsilon - \nabla u|^2 \, dx, \end{aligned}$$

which implies the uniform boundedness of $\int_{B_1} |\nabla u_\epsilon - \nabla u|^2 \, dx$, and therefore of $\int_{B_1} |\nabla u_\epsilon|^2 \, dx$ since u is M-Lipschitz.

We may extract a subsequence $\epsilon_k \to 0$ such that u_{ϵ_k} converges locally uniformly in B_1 and weakly in $H^1(B_1)$. Passing to the limit in the identity

$$\int_{B_1} \langle G_\epsilon(\nabla u_\epsilon) - G(\nabla u), \nabla u_\epsilon - \nabla u \rangle \, dx = 0,$$

using the locally uniform convergence $G_\epsilon \to G$ and the strict monotony of G, we obtain that any Young measure generated by a subsequence of (∇u_ϵ) is concentrated at ∇u, see e.g. [30, Lemma 2.8 & Remark 2.9] for details. We conclude that $u_\epsilon \to u$ locally uniformly, and strongly in $H^1(B_1)$. □

Appendix B Degenerate linear Beltrami equations

In this appendix we prove the assertion claimed in Remark 1.7.

Proposition B.1. *Let* $\mu, \nu \in \mathbb{C}$ *such that* $|\mu| + |\nu| = 1$. *Then, for any open set* $\Omega \subset \mathbb{C}$, *the implication*

$$f_{\bar z} = \mu f_z + \nu \overline{f_z} \quad \Rightarrow \quad \mathfrak{Re}\, f \text{ or } \mathfrak{Im}\, f \text{ is constant},$$

is true for all differentiable $f \colon \Omega \to \mathbb{C}$ *if and only if* $\mu = 0$ *and* $\nu = \pm 1$.

Proof. With $f = u + iv$, the equation $f_{\bar z} = \mu f_z + \nu \overline{f_z}$ is equivalent to

$$\begin{cases} (1 - \mu - \nu)\partial_x u - (1 + \mu + \nu)\partial_y v = 0 \\ (1 + \mu - \nu)\partial_y u + (1 - \mu + \nu)\partial_x v = 0. \end{cases}$$

If $(\mu, \nu) = (0, -1)$ or $(0, +1)$, one directly deduces $\partial_x u = \partial_y u = 0$ or $\partial_x v = \partial_y v = 0$, so that either u or v is constant.

Conversely, assume that $(\mu, \nu) \notin \{(0, \pm 1)\}$, and let us construct an affine solution f such that both u and v are not constant. We rewrite the system as

$$\begin{cases} a\partial_x u = b\partial_y v, \\ c\partial_y u = d\partial_x v, \end{cases} \qquad (a, c) \neq (0, 0) \text{ and } (b, d) \neq (0, 0).$$

If $a \neq 0$ and $b \neq 0$, we set $u(x, y) = \frac{b}{a}x$ and $v(x, y) = y$. Then $f = u + iv$ satisfies the equation and both u and v are not constant.

If $a \neq 0$ and $b = 0$, then $d \neq 0$. We have two cases in this situation depending on the value of c. If $c = 0$, we can take $u(x, y) = v(x, y) = y$ as a solution. If $c \neq 0$ then, $u(x, y) = \frac{d}{c}y$ and $v(x, y) = x$ is a solution.

The cases where $a = 0$ can be dealt with similarly. $\square$

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